

Sheaf Cohomology on network codings: Maxflow-Mincut Theorem

MIRADAIN ATON TSA NGUEMO^a AND CALVIN TCHEKA^b

^a IRMP, Université Catholique de Louvain, Chemin du Cyclotron 2 , Louvain-La-Neuve, Belgium (miradain.atontsa@uclouvain.be)

^b Department of Mathematics, University of Dschang, PO Box: 67, Dschang, Cameroon (jtcheka@yahoo.fr)

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ABSTRACT

Surveying briefly a novel algebraic topological application sheaf theory into directed network coding problems, we obtain the weak duality in multiple source scenario by using the idea of modified graph. Furthermore, we establish the maxflow-mincut theorem with network coding sheaves in the case of multiple source.

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KEYWORDS: *network information flow; network coding sheaves; topological cut; relative sheaf cohomology.*

1. INTRODUCTION

A sheaf is a mathematical tool for storing local information over a global structure. In our case, it assigns vector spaces to each open set. Sheaf theory was invented in the mid 1940s as a branch of algebraic topology to deal with the collation of local data on topological spaces. However, in spite of its generality dealing with local to global transitions, applications to other sciences have not well been established so far except for logic and semantics in computer science. The purpose of this paper is to show the usefulness of sheaf theory and sheaf cohomology on a network, which as a tool, might also be extendable on higher dimensional space such as simplicial complexes [6], [2]. The first

step in this direction had been taken by Robert Ghrist and Yasuaki Hiraoka [4] who introduced in 2011 a class of sheaves designed to model the flow of information over graphs. They proved the maxflow bound inequality using a lot of homological algebra tools such as relative homology and exact sequences. In this paper, we extend this approach proved on the single source context to the multi source case, and we solve the optimization problem with this class through the Maxflow-Mincut theorem.

From a network coding X with multiple sources, we associate a modified graph \overline{X} by duplicating the edges which conduct information from multiple sources. The information flow of \overline{X} is very related to the original flow as we use it to construct an upper bound for the maxflow.

Theorem A: If D is an open of the graph X which include some receiver nodes, and not the sources, then D defines a cut and we have this inequality:

$$\mathbf{maxflow}(\mathbf{X}) = \max_F \dim H^0(X; F) \leq \mathbf{mincut}(\mathbf{X}) = \min_D \dim H^0(D; F)$$

The proof uses Lemma 1 and Lemma 2 which give a concrete description between $H^0(X; F)$ and $H^0(\overline{X}; \overline{F})$.

To complete the proof of our theorem we prove that the maxflow is an upper bound of the mincut.

Theorem B: If X is a graph with multiple source nodes and some receivers,

$$\mathbf{maxflow}(\mathbf{X}) := \max_F \dim H^0(X; F) \geq \mathbf{mincut}(\mathbf{X}) =: \min_D \dim H^0(D; F)$$

The technical idea of the proof is this result uses a dual version of theorem A. Namely, from a network X , we associate a new one X_1 by adding nodes to edges outcoming from sources in $X - S$. The information on the two graphs are equivalent(Lemma 3). Moreover the rank-nullity theorem on the differential δ^0 tells that $H^0(X; F) \cong H^1(X; F)$, so we rather use H^1 , still unused, which correlates more the two graphs.

This paper has 2 sections. We start with the preliminaries which remind the basics in network coding theory, and sheaf theory on graphs. The second section is more technical and contains all our constructions to prove the maxflow-mincut theorem. The first three subsections aim to prove Theorem A, and the last subsection is based on the proof of Theorem B.

Throughout the paper, we refer to [3] and [1] for general discussion on sheaf theory.

2. PRELIMINARIES

This section is a survey of the work of Ghrist-Hiraoka who pioneered this application of algebraic topology in network coding theory. Therefore we refer to [4] for additional comprehension to the topic.

We also fix in the notation to understand our main results in the next section. Throughout the paper, \mathbb{K} if a fix field, and the networks viewed as directed graphs are not allowed to have loops.

2.1. Network Coding. A network coding is a directed graph $G = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} and \mathcal{E} are finite sets of nodes and edges¹, respectively, satisfying the following properties:

- (i) there exist a subset $\mathcal{S} = \{s_1, s_2, \dots, s_k\} \subset \mathcal{V}$ of nodes called **sources** which transmit elements of $\mathbb{K}^{n_{s_i}}$ for each $s_i, i \in \{1, 2, \dots, k\}$
- (ii) there exists a subset $\mathcal{R} = \{r_1, r_2, \dots, r_l\} \subset \mathcal{V}$ of nodes called **receivers** which require each information from some sources.
- (iii) there exists a **capacity map** $\text{cap}: \mathcal{E} \rightarrow \mathbb{N}$ which assign any edge e with a capacity $\text{cap}(e)$.
- (iv) For each edge $e = |vw|$, there exists **local coding maps** ϕ_{wv} ², which are linear maps given by

$$\phi_{wv} : \mathbb{K}^{n_v} \oplus \mathbb{K}^{l_v} \rightarrow \mathbb{K}^{\text{cap}(e)}$$

with $\begin{cases} l_v = \sum_{e \in \text{In}(v)} \text{cap}(e) \\ n_v = 0 \end{cases}$ if $v \notin \mathcal{S}$

and $\text{In}(v)$ (resp. $\text{Out}(v)$) the subset of edges having v as a head (resp. tail).

Given a network coding $G = (\mathcal{V}, \mathcal{E})$, one build an augmented network $X = (\mathcal{V}, \tilde{\mathcal{E}})$, where $\tilde{\mathcal{E}}$ is obtained from the set \mathcal{E} by adding the edges $e = |r_j s_i|$, whenever the node r_j receives information from s_i . One assign to these edge the capacities $\text{cap}(e) = n_{s_i}$. In the next sections we will rather work with augmented graphs since they encode decoding on the network.

2.2. Network Coding Sheaves. To see X as a topological space, we consider its geometric realization. The following definition is not as general as possible and will be restricted to the setting of sheaves over a graph. We will first assign sections to special open sets.

Definition 2.1 (Local sections).

- 1.:** For a connected open set U contained in an edge $e \in X, F(U) := \mathbb{K}^{\text{cap}(e)}$.
- 2.:** For a connected open set U which contains only one node $v, F(U) := \mathbb{K}^{n_v} \oplus \mathbb{K}^{l_v}$

Under these definitions of local sections, we define the following local maps.

Definition 2.2 (Local restriction maps).

- 1.:** For connected open sets $V \subset U \subset e$, for some edge $e \in X, \varrho_{VU} := \text{Id}_{F(U)} : F(U) \rightarrow F(V)$.
- 2.:** For connected open sets $V \subset U$, where U contains only one node v , and V is located in one edge $e \in \text{In}(v)$, $\varrho_{VU} : F(U) \rightarrow F(V)$ is a natural projection of $\mathbb{K}^{n_v} \oplus \mathbb{K}^{l_v}$ on $\mathbb{K}^{\text{cap}(e)}$.

¹If $e = |vw| \in \mathcal{E}$, *head* and *tail* are two applications defined as: $\text{head}(e) := w$ and $\text{tail}(e) := v$

² ϕ_{wv} is also denoted ϕ_e

3.: For connected open sets $V \subset U$, where U contains only one node v , and V is located in one edge $e = |vw| \in \text{Out}(v)$, $\varrho_{VU} := \phi_{wv} : F(U) \rightarrow F(V)$

These local sections and local restriction maps are used to construct $F(U)$ for arbitrary open sets $U \subset X$. The process used is called *sheafification* which is an universal tool for tuning any pre-sheaf into a sheaf.

Definition 2.3. For an open set $U \subset X$, $F(U)$ is the set of equivalent classes $\sigma = [(\sigma_i, U_i)_{i \in I}]$, where a representative $(\sigma_i, U_i)_{i \in I}$ with a covering $U = \bigcup_{i \in I} U_i$ is given by a family of local sections $\sigma_i \in F(U_i)$ satisfying $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$, and the equivalent relation is defined by:

$$(\sigma_i, U_i)_{i \in I} \sim (\tau_j, V_j)_{j \in J} \iff \sigma_i|_{U_i \cap V_j} = \tau_j|_{U_i \cap V_j} \text{ for } i \in I, j \in J.$$

For arbitrary open sets $V \subset U \subset X$ The restriction map $\varrho_{VU} : F(U) \rightarrow F(V)$ is induced by the local restriction maps on a representative. The sheaf defined by the sheafification process from the local sections and local coding maps is called **network coding sheaf** of the network X .

2.3. Čech cohomology. There are six operations on sheaves that are important in the general theory, but only one of them (Cohomology) plays an important role in this note. Suppose F is a sheaf on X , and that $\mathcal{U} = \{U_1, U_2, \dots\}$ is a cover of X . We define the Čech cochain spaces $\check{C}^n(\mathcal{U}; F)$ to be the direct sum of sections over each n -wise intersection of elements in \mathcal{U} . i.e $\check{C}^n(\mathcal{U}; F) = \prod_{\{i_0, i_1, \dots, i_n\} \subset \{1, 2, \dots\}} F(U_{i_0} \cap \dots \cap U_{i_n})$.

We define a sequence of linear maps $\delta^n : \check{C}^n(\mathcal{U}; F) \rightarrow \check{C}^{n+1}(\mathcal{U}; F)$ by

$$\delta^n(\sigma)(U_{i_1}, \dots, U_{i_{n+1}}) = \sum_{j=0}^{n+1} (-1)^j \sigma(U_{i_1} \cap \dots \cap \hat{U}_{i_j} \cap \dots \cap U_{i_{n+1}})|_{U_{i_1} \cap \dots \cap U_{i_j} \cap \dots \cap U_{i_{n+1}}}$$

where the hat means that an element has been omitted from the list. The standard computation shows that $\delta^{n+1} \circ \delta^n = 0$. We obtain then the following Čech complex: $(\check{C}^*(\mathcal{U}; F), \delta^*)$, and the Čech cohomology $\check{H}^*(\mathcal{U}; F)$. We define the n -cohomology group of F on X to be the direct limit of the $\check{H}^n(\mathcal{U}; F)$ as \mathcal{U} becomes finer.

$$\check{H}^n(X; F) = \varinjlim \check{H}^n(\mathcal{U}; F).$$

In practice, this direct limit is not easy to work with. Instead, what is used is the fact that under some condition on the cover \mathcal{U} , the equalities $\check{H}^n(X; F) = \check{H}^n(\mathcal{U}; F)$ hold for all n . This condition is that \mathcal{U} be acyclic for F , in the sense that $\check{H}^n(U_{i_1} \cap \dots \cap U_{i_l}) = 0$, for $n > 0$ and any i_1, i_2, \dots, i_l .

2.4. Sheaf cohomology. We basically define cohomology of sheaves using the derived functors of the global section functor (see [9]), but Robin has proven in [7] that sheaf cohomology and Čech cohomology coincides on graphs.

Let consider the open covering $X = \bigcup_{v \in \mathcal{V}} U_v$, where U_v is the largest connected open of X which contains only the node v . For any edge $e = |vw|$,

the intersection $U_v \cap U_w := U_e$ is the biggest open contained in e . Then $\check{C}^0(X; F) = \check{C}^0(\mathcal{U}; F) = \prod_{v \in \mathcal{V}} F(U_v)$, $\check{C}^1(X; F) = \check{C}^1(\mathcal{U}; F) = \prod_{e \in \mathcal{E}} F(U_e)$ and the Čech complex is reduced to be :

$$0 \longrightarrow \check{C}^0(X; F) \xrightarrow{\delta^0} \check{C}^1(X; F) \longrightarrow 0.$$

The sheaf cohomology is the cohomology associated to the Čech complex defined as follow:

$$H^0(X; F) := \ker(\delta^0), \text{ and } H^1(X; F) := \check{C}^1(X; F)/\text{Im}(\delta^0).$$

2.5. Information flow.

Definition 2.4. The **information flow** on a network X for a family of transmitted data

$$Z = (Z_{s_1}, Z_{s_2}, \dots, Z_{s_k}),$$

where $Z_{s_i} \in \mathbb{K}^{n_{s_i}}$, is an assignment $\psi(e) \in \mathbb{K}^{\text{cap}(e)}$ for each edge e , which satisfies the following so-called **flow conditions**:

For $e = |vw|$ and assuming that $\text{In}(v) = \{e_1, e_2, \dots, e_n\}$,

- (i) $\phi_{wv}(\psi(e_1), \psi(e_2), \dots, \psi(e_n)) = \psi(e)$, for $v \notin (\mathcal{S} \cup \mathcal{R})$.
- (ii) $\phi_{wv}(\psi(e_1), \psi(e_2), \dots, \psi(e_n)) = \psi(e)$, for $v = s_i \in \mathcal{S} \setminus \mathcal{R}$.
- (iii) $\phi_{wv}(\psi(e_1), \psi(e_2), \dots, \psi(e_n)) = Z_{s_i}$, for $v = r_j \in \mathcal{R} \setminus \mathcal{S}$ and $w = s_i$.
- (iv) $\phi_{wv}(\psi(e_1), \psi(e_2), \dots, \psi(e_n)) = \psi(e)$, for $v = r_j \in \mathcal{R} \setminus \mathcal{S}$ and $w \notin \mathcal{S}(r_j)$

Let $\sigma = \{\sigma_v\} \in H^0(X; F)$, where for each $i, \sigma_{s_i} = (Z_{s_i}, \tilde{\sigma}_{s_i}) \in \mathbb{K}^{n_{s_i}} \oplus \mathbb{K}^{l_{s_i}}$. One defines the map $\psi : \mathcal{E} \longrightarrow \bigoplus_{e \in \mathcal{E}} \mathbb{K}^{\text{cap}(e)}$ as follows: $\forall (e = |vw|) \in \mathcal{E}, \psi(e) := \sigma_v|_{U_e} - \sigma_w|_{U_e}$.

It is clear that the family $\{\psi(e)\}$ defines an information flow on the network for the data $Z = (Z_{s_1}, Z_{s_2}, \dots, Z_{s_k})$.

This construction makes it possible to apply homological algebra tools to network coding problems as we state in the next theorem.

Theorem 2.5 (Information theoretical meaning; [4]). *For any network coding sheaf F of a graph X fitted with ψ , $H^0(X; F)$ is equivalent to the information flows on the network.*

Namely this theorem tells that $H^0(X; F)$ carries all the data that can be transmitted on the network.

2.6. Topological Cut. $\alpha : \mathcal{S} \rightarrow 2^{\mathcal{R}}$ is the function which assigns to each source s_i the set $\alpha(s_i)$ of all receiver nodes receiving information from s_i ,

Definition 2.6 (Cut-set and cut value in network theory).

- (1) A cut-set C on a graph X is a set of edges whose removal breaks the connection between a source and some receivers. Namely, if s is a source and D is an open set of a graph X which includes some receivers $r_1, r_2, \dots, r_j \in \alpha(s)$ but the node s , then the set of incoming edges into D define a cut-set C between s and r_1, r_2, \dots, r_j .

- (2) The value of a cut-set C , denoted $\text{Val}(C)$, is the sum of the capacities of its edges.

Remark 2.7. The topological cut values of the graph X which satisfy the sheaf F are presented by the cohomology classes of $H^0(D; F)$.

The following theorem from which we recover the physical definition of cut value shows that $H^0(D; F)$ is independent of the sheaf F .

Proposition 2.8 ([4]). *The value of the cut-set is equivalent to topological cut values. More precisely*

$$\text{Val}(C) = \dim(H^0(D; F)).$$

3. OPTIMIZATION OF THE NETWORK: MAXFLOW-MINCUT

3.1. Duplication of networks. In this subsection, we construct from a network X , a modified network \overline{X} which still keep some information from the original one. Let $\gamma(v, w)$ be the reunion of all paths in X connecting v to w , and $\gamma(s_i) := \bigcup_{r \in \alpha(s_i)} \gamma(s_i, r)$. The modified graph \overline{X} is obtained by duplicating edges which conduct information from multiple sources. More precisely, $\overline{X} := \{(x, i), x \in \gamma(s_i), i = 1, 2, \dots, k\}$, and hence \overline{X} is the disjoint union of the subgraphs $\gamma(s_i)$. Heuristically for us, the purpose for this construction is to apply the results of the single source case [4] to each subgraphs $\gamma(s_i)$.

As an illustration, Fig.1. and Fig.2 show how we construct the graph \overline{X} from X . The adding edges $e = |r_j s_i|$ are not represented on purpose to simplify the construction.

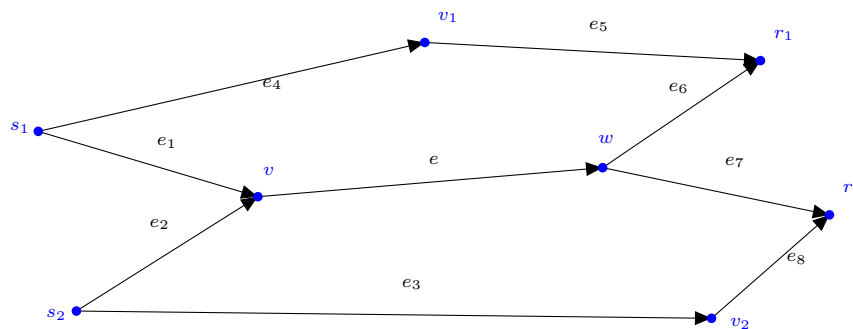


Fig.1. Graph X

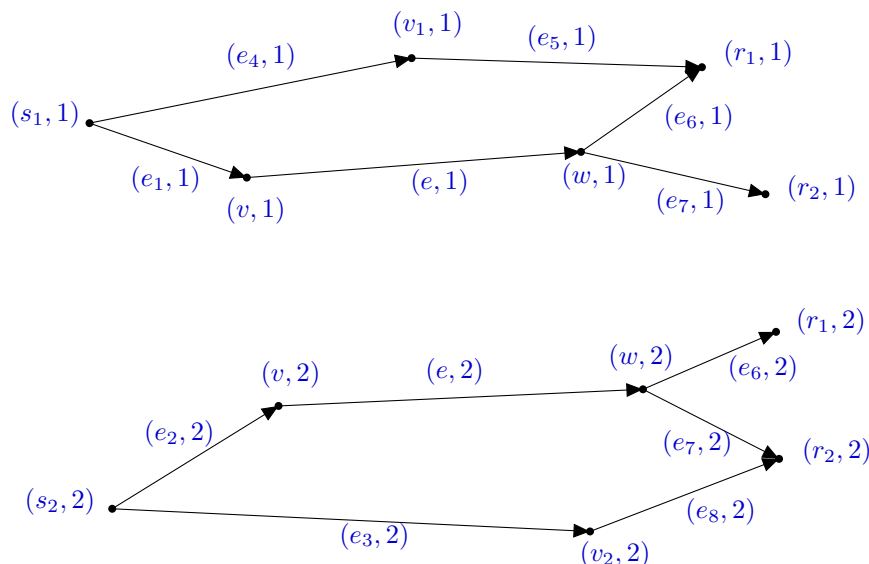


Fig.2. Graph $\overline{X} = \gamma(s_1) \cup \gamma(s_2)$

To make the difference with the sheaf F of X , we denote by $\overline{F} : \overline{X} \rightarrow Vect$, the network coding sheaf on $\overline{X} = \bigcup_{i \in \{1, \dots, k\}} \bigcup_{v \in \mathcal{V} \cap \gamma(s_i)} (U_v, i)$.

The natural projection $j : \overline{X} \rightarrow X, (x, i) \mapsto x$, induces an injective map: $j^* : H^0(X; F) \rightarrow H^0(\overline{X}; \overline{F})$. Therefore $\dim H^0(X; F) \leq \dim H^0(\overline{X}; \overline{F})$, and in terms of information theory, any information flow on the network X can be extended to an information flow on \overline{X} .

3.2. Relative Sheaf Cohomology. let $D \subset X$ be an open of a graph X . That inclusion induces a surjective map $p^* : C^*(X; F) \rightarrow C^*(D; F)$. Therefore we have the short exact sequence defined as follow:

$$0 \rightarrow C^*(X, D; F) \xrightarrow{i^*} C^*(X; F) \xrightarrow{p^*} C^*(D; F) \rightarrow 0$$

where $C^*(X, D; F) \cong \frac{C^*(D; F)}{C^*(X; F)}$

The short exact sequence induces the long exact sequence below:

$$0 \rightarrow H^0(X, D; F) \xrightarrow{i^0} H^0(X; F) \xrightarrow{p^0} H^0(D; F) \xrightarrow{\delta^0} H^1(X, D; F) \xrightarrow{i^1} \dots$$

It has been proven in [4] that if the graph X contains only one source s and the open D includes some receivers, but does not include the source node s , then for any network coding sheaf $F, H^0(X, D; F) = 0$. Thus the long exact sequence theorem applied to the above short exact sequence shows that

$$\dim H^0(X; F) \leq \dim H^0(D; F).$$

In the sequel we consider that the graph X has multiple sources along with multiple receivers. Using the preliminary result on the single source case to the graph \bar{X} introduced in Section 6, we obtain the following inequalities on its subgraphs $\gamma(s_i) = \bigcup_{r \in \alpha(s_i)} \gamma(s_i, r)$, for all $i \in \{1, 2, \dots, k\}$:

$$\dim H_i^0(\bar{X}; \bar{F}) \leq \dim H^0(D_i; \bar{F}),$$

for any open D_i of the subgraph $\gamma(s_i) = \bigcup_{r \in \alpha(s_i)} \gamma(s_i, r)$ which contains some receiver nodes and does not include the source node s_i , and $H_i^*(\bar{X}; \bar{F}) := H^0(\gamma(s_i); \bar{F})$. If $\mathcal{B} := \bigcup_i D_i$, this leads to the following inequality:

$$\begin{aligned} \sum_{i \in \{1, 2, \dots, k\}} \dim H_i^0(\bar{X}; \bar{F}) &\leq \sum_{i \in \{1, 2, \dots, k\}} \dim H^0(D_i; \bar{F}) \\ \parallel & \parallel \\ \dim H^0(\bar{X}; \bar{F}) &\leq \dim H^0(\mathcal{B}; \bar{F}) \end{aligned}$$

3.3. Maxflow bound. The following theorem generally known as the weak duality generalizes the upper bound theorem proved in [4] for the single-source scenario.

Theorem A: If D is an open of the graph X which include some receiver nodes, and not the sources, then D defines a cut and we have this inequality:

$$\mathbf{maxflow}(\mathbf{X}) = \max_F \dim H^0(X; F) \leq \mathbf{mincut}(\mathbf{X}) = \min_D \dim H^0(D; F)$$

To prove this theorem, we will need the next two lemmas.

Let $p : \bar{X} \rightarrow X$ the natural surjection, $I = \bigcup_{s_i, s_j \in S} p(\gamma(s_i)) \cap p(\gamma(s_j))$, and if D denotes a cut on the network X , let $J = \{e = |vw| \subset I, e \text{ cutting edge}\}$.

Lemma 3.1. *There exists a finite vector space M of dimension $\dim M = \sum_{e=|vw| \subset I} \text{cap}(e)$ such that*

$$H^0(\bar{X}; \bar{F}) \cong H^0(X; F) \oplus M,$$

Proof. The surjection $p : \bar{X} \rightarrow X$ induces the injection: $p^* : H^0(X; F) \rightarrow H^0(\bar{X}; \bar{F})$. Therefore we have $H^0(\bar{X}; \bar{F}) = p(H^0(X; F)) \oplus M$, where $\dim M \leq \dim H^0(\bar{X}; \bar{F})$. Let $\sigma = \{\sigma_{v,i}\} \in H^0(\bar{X}; \bar{F})$.

$\sigma = \sigma_1 + \sigma_2$, with $\sigma_1 \in H^0(X; F)$ and $\sigma_2 \in M$. $\sigma \notin p(H^0(X; F)) \Leftrightarrow \sigma_2 \neq 0$. However, $\sigma \notin p(H^0(X; F)) \Leftrightarrow \exists (i, j) \in \{1, 2, \dots, k\}^2, i \neq j$, and $e = |vw| \in p(\gamma(s_i)) \cap p(\gamma(s_j))$ such that $\sigma_{w,i}|_{(U_e,i)} \neq \sigma_{w,j}|_{(U_e,j)} \Leftrightarrow \sigma_{w,i}|_{(U_e,i)} - \sigma_{w,j}|_{(U_e,j)} \neq 0$.

i.e $(\sigma_{w,i}|_{(U_e,i)}, \sigma_{w,j}|_{(U_e,j)}) \in \text{Im} f$, where $f : \mathbb{K}^{\text{cap}(e)} \times \mathbb{K}^{\text{cap}(e)} \rightarrow \mathbb{K}^{\text{cap}(e)}$
 $(x, y) \mapsto x - y$.

It is clear that σ_2 consists of such couple $(\sigma_{w,i}|_{(U_e,i)}, \sigma_{w,j}|_{(U_e,j)}) \in \text{Im} f$ and $\dim \text{Im} f = \text{cap}(e)$.

Hence $\dim M = \sum_{e=|vw| \subset I} \text{cap}(e)$ □

Lemma 3.2. *There exists a finite vector space N of dimension $\dim N = \sum_{e=|vw| \subset J} \text{cap}(e)$ such that*

$$H^0(p^{-1}(D); \overline{F}) \cong H^0(D; F) \oplus N$$

Proof. The surjection $p : p^{-1}(D) \rightarrow D$ implies the injection: $p^* : H^0(D; F) \rightarrow H^0(p^{-1}(D); \overline{F})$. Therefore $H^0(p^{-1}(D); \overline{F}) = p^*(H^0(D; F)) \oplus N$, where $\dim N \leq \dim H^0(p^{-1}(D); \overline{F})$. Let $p^{-1}(D) = \bigcup_{i \in \{1, 2, \dots, k\}} D_i$. $H^0(p^{-1}(D); \overline{F}) = H^0(\bigcup_{i \in \{1, 2, \dots, k\}} D_i; \overline{F}) = \bigoplus_{i \in \{1, 2, \dots, k\}} H^0(D_i; \overline{F})$.

$$\dim H^0(p^{-1}(D); \overline{F}) = \sum_i \dim H^0(D_i; \overline{F}) = \sum_i \text{val}(C_i),$$

where C_i denotes the cut created by D_i on the subgraph $\gamma(s_i)$.

On the other hand, $\dim H^0(D; F) = \text{val}(C)$, where C is the cut created by the open D on the graph X . $\text{val}(C) = \sum_{e \text{ is cutting edge}} \text{cap}(e)$. It is however clear that,

$$\sum_i \text{val}(C_i) = \sum_{e \text{ is cutting edge on } X \text{ by } D} \text{cap}(e) + \sum_{e \subset p(\gamma(s_i)) \cap p(\gamma(s_j)), e \text{ is cutting edge}} \text{cap}(e)$$

this leads to:

$$\dim H^0(p^{-1}(D); \overline{F}) = \dim H^0(D; F) + \sum_{e \subset p(\gamma(s_i)) \cap p(\gamma(s_j)), e \text{ is cutting edge}} \text{cap}(e).$$

Hence

$$\dim N = \sum_{e \subset J} \text{cap}(e).$$

□

Proof of Theorem A.

By using the inequality: $\dim H^0(\overline{X}; \overline{F}) \leq \dim H^0(p^{-1}(D); \overline{F})$ with **lemma 3.1.** and **lemma 3.2.**, we have this:

$$\dim H^0(X; F) + \dim M \leq \dim H^0(D; F) + \dim N \iff \dim H^0(X; F) + \dim M - \dim N \leq \dim H^0(D; F)$$

However, it is clear that $\dim M - \dim N \geq 0$, hence $\dim H^0(X; F) \leq \dim H^0(D; F)$, for all open D and network coding sheaf F on the graph X . Therefore we have the following result:

$$\max_F \dim H^0(X; F) \leq \min_D \dim H^0(D; F)$$

3.4. Mincut bound. As we consider in this paper that the graphs wear the decoding maps, each node is expressed by the incoming edges and one have naturally the isomorphism $C^0(X; F) \cong C^1(X; F)$. It turns out using the rank-nullity theorem on the differential δ^0 , which is linear in our case, that $H^0(X; F) \cong H^1(X; F)$. We will use this identification in proving the next result.

Theorem B: If X is a graph with multiple source nodes and some receivers,

$$\mathbf{maxflow}(\mathbf{X}) := \max_F \dim H^0(X; F) \geq \mathbf{mincut}(\mathbf{X}) =: \min_D \dim H^0(D; F)$$

The proof of this salient theorem uses the following construction on the network.

Let us consider now a graph X and \mathcal{U} an open which contains all the nodes but the source nodes s_i . We admit to add some virtual nodes on the cutting edges such that they are include in the open set \mathcal{U} , and we then obtain the modified graph X_1 . The following figures are an illustration of that concept where the nodes G and H have been added.

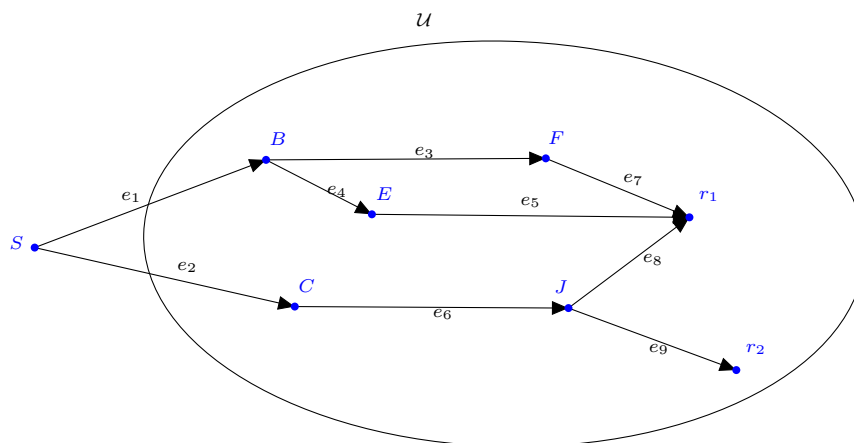


Fig.4. Graph X with an open \mathcal{U}

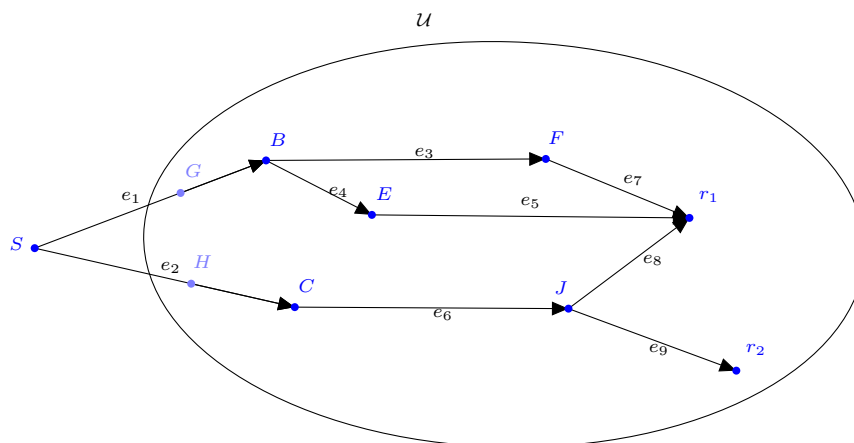


Fig.5. Graph X_1

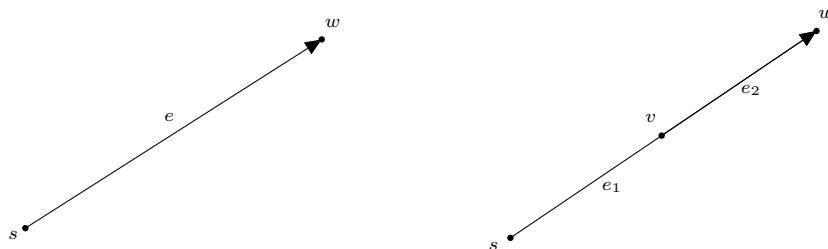
We consider in particular that the network X_1 have the following characteristics: for a cutting edge $e = |sw|$ of X , where we have added a node v , U_v will still be the largest connected open of the graph X_1 which contains only the node v . Moreover we define the local coding maps $\phi_{vs} := \phi_{ws}$ and $\phi_{vw} := id_{\mathbb{K}^{cap(e)}} 1_{\text{Im}\phi_{ws}}$, where $1_{\text{Im}\phi_{ws}}$ means the characteristic function on the vectorial space $\text{Im}\phi_{ws}$. The network coding sheaf on the network $(X_1, (\phi_{ba})_{e=|ab|\in X_1})$ defined above is denoted F_1 .

Heuristically for us, the purpose of this modification is to understand more the properties of the cut created by the open set \mathcal{U} without changing the value of the cut-set. To make sure that we have not significantly changed something, we have the following lemma:

Lemma 3.3. *For any graph X fitted with ψ , $H^0(X_1, F_1)$ is equivalent to the information on the graph X .*

Proof. We denote $X_1 = (\mathcal{V}_1, \mathcal{E}_1)$.

Let now $\sigma = \{\sigma_v\}_{v \in \mathcal{V}_1} \in H^0(X_1, F_1)$. If $e = |sw| \in \mathcal{E}$ is a cutting edge by the open set \mathcal{U} which have been added a node v as follows:



We now have the following proposition which first stresses a link between the two graphs.

Proposition 3.4. *For any open set \mathcal{U} satisfying the above conditions, we have the following equality:*

$$H^1(\mathcal{U}, F_1) \cong H^1(X, F).$$

Proof. Using the above notation, if $e = |sw|$ is a cutting edge and v is the node which is added and include in the cut \mathcal{U} , we denote $e_2 = |vw|$ and $e_1 = |sv|$. It is clear from the definition that $F_1(U_{e_2}) = F(U_e)$, and a direct computation shows that $\text{Im}\phi_{ws} \cong \text{Im}\phi_{vw}$. Therefore $F_1(U_{e_2})/\text{Im}\phi_{vw} \cong F(U_e)/\text{Im}\phi_{sw}$, and we obtain the isomorphism. \square

Proof of Theorem B.

Let $\mathcal{U} \subset X$ an open set which satisfies the condition stated earlier in Proposition 3.4.

We have the following square:

$$\begin{array}{ccc}
 H^1(\mathcal{U}; F_1) & \xrightarrow[\cong]{(1)} & H^1(X; F) \\
 \cong \downarrow (2) & \circlearrowleft & \cong \downarrow (3) \\
 H^0(\mathcal{U}; F_1) & \xrightarrow[\cong]{(4)} & H^0(X; F) \\
 \parallel & & \\
 H^0(\mathcal{U}; F) & &
 \end{array}$$

where (1) is the isomorphism of Proposition 3.4, (2) and (3) are isomorphisms that results from comments at the beginning of this subsection. Then (4) is the unique morphism which makes the diagram to commute. We obtain from the square the equality:

$$\dim H^0(\mathcal{U}; F) = \dim H^0(X; F)$$

Thus, $\min_D \dim H^0(D; F) \leq \dim H^0(\mathcal{U}, F) = \dim H^0(X; F) \leq \max_F \dim H^0(X; F)$.

$$\text{Then, } \begin{cases} \sigma_s|_{U_{e_1}} = \sigma_v|_{U_{e_1}} \\ \sigma_v|_{U_{e_2}} = \sigma_w|_{U_{e_2}} \end{cases} \iff \begin{cases} \phi_{vs}(\sigma_s) = \sigma_v \\ \phi_{wv}(\sigma_v) = \sigma_w|_{U_{e_2}} \end{cases} \quad (1).$$

But $\phi_{vs} := \phi_{ws}$ on the network X , and $\phi_{wv}(\sigma_v) = \phi_{wv}(\phi_{vs}(\sigma_s)) := \phi_{vs}(\sigma_s)$. Therefore

$$(1) \iff \phi_{vs}(\sigma_s) = \sigma_w|_{U_{e_2}}.$$

Hence any cocycle $\sigma = \{\sigma_v\}_{v \in \mathcal{V}_1} \in H^0(X_1, F_1)$ is equivalent to the induced element $\tilde{\sigma} = \{\sigma_v\}_{v \in \mathcal{V}} \in H^0(X, F)$. We then have $H^0(X_1, F_1) \cong H^0(X, F)$. \square

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