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On the topology of the chain recurrent set of a dynamical system

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Abstract

In this paper we associate a pseudo-metric to a dynamical system on a compact metric space. We show that this pseudo-metric is identically zero if and only if the system is chain transitive. If we associate this pseudo-metric to the identity map, then we can present a characterization for connected and totally disconnected metric spaces.

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1. Introduction

One of the main problems in dynamical systems is the description of the orbit structure of a system from a topological point of view [1, 3, 4]. Recurrence behavior is one of the most important concepts in topological dynamics. Various notions of recurrence have been considered in dynamics such as recurrent points, chain recurrent points and non-wandering points [7].

In this paper (X, d) is a compact metric space and $f : X \to X$ is a continuous map. An ϵ-pseudo-orbit (or ϵ-chain) of f from x to y is a sequence $\{x_i\}_{i=0}^n$ with $x_0 = x$, $x_n = y$ and

$$d(f(x_k), x_{k+1}) < \epsilon \quad \text{for } k = 0, 1, ..., n-1.$$

A point x in X is called chain recurrent if there is an ϵ-chain from x to itself. We can define an equivalence relation on the set of chain recurrent points in such

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a way that two points x and y are said to be equivalent if for every $\epsilon > 0$ there exist an ϵ-chain from x to y and an ϵ-chain from y to x. The equivalence classes of this relation are called chain components. These are compact invariant sets and cannot be decomposed into two disjoint compact invariant sets, hence serve as building blocks of the dynamics. The topology of chain recurrent set and chain components have been always in particular interest [2, 5, 6, 8].

We use the symbol $\mathcal{O}_\delta(f, x, y)$ for the set of δ-pseudo-orbits $\{x_i\}_{i=0}^n$ of f with $x_0 = x$ and $x_n = y$. For given points $x, y \in X$ we write $x \overset{\epsilon}{\rightsquigarrow} y$ if $\mathcal{O}_\epsilon(f, x, y) \neq \varnothing$ and we write $x \rightsquigarrow y$ if $\mathcal{O}_\epsilon(f, x, y) \neq \varnothing$ for each $\epsilon > 0$. We write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The set $\{x \in X : x \leftrightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{CR}(f)$. If we define a relation R on $X \times X$ with $x\ R\ y \Leftrightarrow x \leftrightsquigarrow y$, then R is an equivalence relation on $\mathcal{CR}(f)$.

A dynamical system f is called *chain recurrent* if $\mathcal{CR}(f) = X$. A dynamical system f is called *chain transitive* if for each $x, y \in X$ we deduce $x \leftrightsquigarrow y$.

We say that a dynamical system f has the *pseudo-orbit tracing property* (POTP) on X if for each $\epsilon > 0$ there is $\delta > 0$ so that for a given sequence $\xi = \{x_k\}_{n\in\mathbb{Z}}$ with

$$d(f(x_k), x_{k+1}) < \delta \quad \text{for } k \in \mathbb{N}$$

there exists a point $x \in X$ such that

$$d(f^k(x), x_k) < \epsilon \quad \text{for } k \in \mathbb{N}$$

(in this case we say that $p \in X$ ϵ-shadowed ξ).

Let X_0 be a nonempty set. Then a map $d_0 : X_0 \times X_0 \to \mathbb{R}$ is called *pseudo-metric* if for all $x, y \in X_0$ the following hold

(1) $d_0(x, x) = 0$;
(2) $d_0(x, y) = d(y, x) \geq 0$;
(3) $d_0(x, y) \leq d_0(x, z) + d_0(z, y)$.

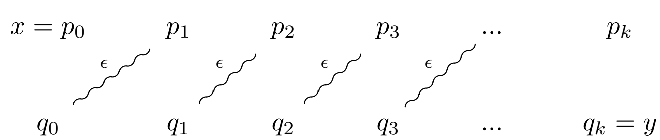
The pair (X_0, d_0) is called a pseudo-metric space. Let (X_0, d_0) be a pseudo-metric space. Then, the open balls in X_0 together with the empty set form a basis for a topology on X_0. This topology is first countable and in it closed balls are closed. Moreover, this topology is a Hausdorff topology if and only if X_0 is a metric space.

Now we are going to present a pseudo-metric on $\mathcal{CR}(f)$.

Definition 1.1. Let $x, y \in \mathcal{CR}(f)$, we define

$$d_{f,\epsilon}(x, y) = \inf\{\sum_{i=0}^{k} d(p_i, q_i) : p_0 = x, q_k = y, k \in \mathbb{N}\}$$

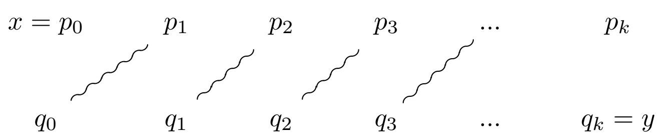
where the infimum is taken over all choices of p_i and q_i so that $q_i \overset{\epsilon}{\leftrightsquigarrow} p_{i+1}$ for all $i = 0, 1, ..., k-1$.



We also define

$$d_f(x,y) = \inf\{\sum_{i=0}^{k} d(p_i,q_i) : p_0 = x, q_k = y, k \in \mathbb{N}\}$$

where the infimum is taken over all choices of p_i and q_i so that $q_i \leftrightsquigarrow p_{i+1}$ for all $i = 0, 1, ..., k-1$.



The straightforward calculations imply that for $\epsilon_1 \leq \epsilon_2$ we deduce $d_{f,\epsilon_2}(x,y) \leq d_{f,\epsilon_1}(x,y) \leq d_f(x,y) \leq d(x,y)$ and $(\mathcal{CR}(f), d_f)$ is a pseudo-metric space. If we define $B_r^f(x) = \{y \in X; d_f(x,y) < r\}$, then the collection $\tau_f = \{B_r^f(x) : x \in X, r > 0\} \cup \{\varnothing\}$ is a basis of a topology on $\mathcal{CR}(f)$ which is finer than τ_d. So $(\mathcal{CR}(f), d_f)$ is a compact space. Obviously $\leftrightsquigarrow$ is an equivalence relation on $\mathcal{CR}(f)$. Let $\widetilde{\mathcal{CR}}(f) = \mathcal{CR}(f)/R$, and $\pi : \mathcal{CR}(f) \to \widetilde{\mathcal{CR}}(f)$ be the quotient map, i.e.

$$\pi(x) = \{y \in \mathcal{CR}(f) : x \leftrightsquigarrow y\}.$$

Then we can define a metric

$$\tilde{d}_f(\pi(x), \pi(y)) = d_f(x,y) \quad \text{for } x, y \in \mathcal{CR}(f)$$

on $\widetilde{\mathcal{CR}}(f)$. With this metric π is a distance preserving map. The topology induced by $\tilde{d}_f$ is denoted by $\widetilde{\tau}_f$.
The induced map $\tilde{f} : \widetilde{\mathcal{CR}}(f) \to \widetilde{\mathcal{CR}}(f)$ with $\tilde{f}(\pi(x)) = \pi(f(x))$ is the identity map.
In this paper we are going to prove the following theorems.

Theorem 1.2. *Let $f : X \to X$ be a chain recurrent continuous map. Then f is chain transitive if and only if $d_f(x,y) = 0$ for all $x, y \in X$.*

Theorem 1.3. *Let $f : X \to X$ be a chain recurrent continuous map. Then $d_f(x,y) = d(x,y)$ for all $x, y \in X$ if and only if f is the identity map and X is totally disconnected.*

Theorem 1.4. *Let (X,d) be a compact metric space. Then the following conditions are mutually equivalent:*

1. *X is connected;*
2. *The identity map $\iota : X \to X$ is chain transitive;*
3. *For each $x, y \in X$, $d_\iota(x,y) = 0$.*

2. Proof of theorems

For the proof of theorem 1.2 we first prove the following lemma.

Lemma 2.1. *Let x, y in $\mathcal{CR}(f)$ and $\epsilon > 0$ be given. Then $d_{f,\epsilon}(x,y) = 0$ if and only if $x \overset{\epsilon}{\leftrightsquigarrow} y$.*

Proof. Clearly if $x \overset{\epsilon}{\leftrightsquigarrow} y$ then $d_{f,\epsilon}(x,y) = 0$. Now let $d_{f,\epsilon}(x,y) = 0$. We choose $0 < \delta < \epsilon$ so that the inequality $d(t,s) < \delta$ implies $d(f(t), f(s)) < \epsilon/2$. Thus there exist sequences $\{p_i\}_{i=0}^k$ and $\{q_i\}_{i=0}^k$ with $p_0 = x$ and $q_k = y$ so that $q_i \overset{\epsilon}{\leftrightsquigarrow} p_{i+1}$ for $i = 0, 1, ..., k-1$ and

$$\sum_{i=0}^{k} d(p_i, q_i) < \delta.$$

So $d(f(p_i), f(q_i)) < \epsilon/2$ for $i = 0, 1, ..., k$. If $\{q_{i,n}\}_{n=0}^{l_i} \in \mathcal{O}_{\epsilon/2}(q_i, p_{i+1})$, then $d(q_{i,1}, f(q_i)) < \epsilon/2$. Hence

$$d(q_{i,1}, f(p_i)) < \epsilon.$$

Let $\{y_i\}_{i=0}^m \in \mathcal{O}_{\epsilon/2}(y,y)$. Then

$$d(y_1, f(p_k)) \leq d(y_1, f(y)) + d(f(y), f(p_k)) < \epsilon.$$

Therefore the sequence

$$\{p_0, q_{0,1}, q_{0,2}, ..., q_{0,l_0}, q_{1,1}, q_{1,2}, \dots \quad ..., q_{k-1,1}, q_{k-1,2}, ..., q_{k-1,l_{k-1}} = p_k, y_1, ..., y_m\}$$

is belong to $\mathcal{O}_\epsilon(f, x, y)$. So $x \overset{\epsilon}{\rightsquigarrow} y$. Since $d_{f,\epsilon}(y,x) = 0$, then $y \overset{\epsilon}{\rightsquigarrow} x$. □

Corollary 2.2. *Let $x, y \in \mathcal{CR}(f)$. Then $d_f(x,y) = 0$ if and only if $x \leftrightsquigarrow y$.*

Corollary 2.3. *Let $f : X \to X$ be a chain recurrent continuous map. Then f is chain transitive if and only if $d_f(x,y) = 0$ for all $x, y \in X$.*

Corollary 2.4. *If $x \leftrightsquigarrow x'$ and $y \leftrightsquigarrow y'$ for $x, x', y, y' \in X$, then $d_f(x,y) = d_f(x', y')$*

Corollary 2.5. *If $x \in \mathcal{CR}(f)$ then $d_f \equiv 0$ on $\mathcal{O}(f,x) \times \mathcal{O}(f,x)$, where $\mathcal{O}(f,x) = \{f^n(x);\ n \in \mathbb{N}\} \cup \{x\}$.*

Proof. It is enough to show that $x \leftrightsquigarrow f(x)$. Given $\epsilon > 0$ clearly $\{x, f(x)\} \in \mathcal{O}_\epsilon(f, x, f(x))$, i.e. $x \overset{\epsilon}{\rightsquigarrow} f(x)$. We can choose $0 < \delta < \epsilon/2$ so that $d(x,y) < \delta$ implies to $d(f(x), f(y)) < \epsilon/2$. Now let $\{x_0, ..., x_n\} \in \mathcal{O}_\delta(f, x, x)$, then $d(x_1, f(x)) < \delta$ implies that $d(f(x_1), f^2(x)) < \epsilon/2$. So $d(x_2, f^2(x)) < \epsilon$. Thus $\{f(x), x_2, ..., x_n\} \in \mathcal{O}_\epsilon(f, f(x), x)$. Therefore $x \overset{\epsilon}{\leftrightsquigarrow} f(x)$. □

Corollary 2.6. *The map $f : (\mathcal{CR}(f), d_f) \to (\mathcal{CR}(f), d_f)$ is an isometry.*

Proof of theorem 1.3. First suppose that for each $x, y \in X$, $d_f(x,y) = d(x,y)$. Hence $d(x, f(x)) = d_f(x, f(x)) = 0$ for each $x \in X$. Let α be a connected component of X and α contains x. Given $\epsilon > 0$ we consider the sets

$$\pi_\epsilon(x) = \{y \in \alpha :\ x \overset{\epsilon}{\leftrightsquigarrow} y\} \quad \text{and} \quad \pi(x) = \{y \in \alpha :\ x \leftrightsquigarrow y\}.$$

If $y \in \pi_\epsilon(x)$ then $B_\epsilon(y) \subseteq \pi_\epsilon(x)$. So $\pi_\epsilon(x)$ is an open set. Now let $y \in \overline{\pi_\epsilon(x)}$ then there is a sequence $\{y_n\} \subseteq \pi_\epsilon(x)$ such that $y_n \to y$. So $y \in B_\epsilon(y_n)$ for some $n \in \mathbb{N}$. Thus $y \in \pi_\epsilon(x)$. Hence $\pi_\epsilon(x)$ is both open and closed. Since $\pi_\epsilon(x) \neq \varnothing$ then $\pi_\epsilon(x) = \alpha$. Therefore $\alpha = \cap_{\epsilon>0}\pi_\epsilon(x) = \pi(x) = \{x\}$.
Now let X be totally disconnected and let there exist $x, y \in X$ so that $d_\iota(x, y) \neq d(x, y)$ where $\iota : X \to X$ is the identity map. Then there exist points $p_0, ..., p_n$, $q_0, ..., q_n \in X$ so that $q_i \leftrightsquigarrow p_{i+1}$ for all $i = 0, 1, ..., k-1$, $p_0 = x$, $q_k = y$ and $\sum_{i=0}^{k} d(p_i, q_i) < d(x, y)$. Hence there is at least one index i such that $q_i \neq p_{i+1}$. So if α is a connected component contains q_i, as the same as the first part we deduce $\pi(q_i) = \alpha$. Hence $q_i, p_{i+1} \in \alpha$ which contradicts the totally disconnectedness of X. $\square$

Let X be a compact metric space. Topological dimension of the space X is said to be less than n if for all $\epsilon > 0$ there exists a cover α of X by open sets with diameter less than ϵ such that each point belongs to at most $n + 1$ sets of α. We know that X is 0-dimensional if and only if it is totally disconnected [1].

Corollary 2.7. *Let $\iota : X \to X$ be the identity map. Then $d_\iota(x, y) = d(x, y)$ for all $x, y \in X$ if and only if X has dimension zero.*

Proof of theorem 1.4. Clearly 2 is equivalent to 3. Suppose that X is connected and $x \in X$. It is enough to show that for each $\epsilon > 0$ the set $\pi_\epsilon = \{y \in X : x \overset{\epsilon}{\leftrightsquigarrow} y\}$ is both open and closed. If $y \in \pi_\epsilon(x)$ then $B_\epsilon(y) \subseteq \pi_\epsilon(x)$. So $\pi_\epsilon(x)$ is open. If $\{y_n\}$ is a sequence with $y_n \to y$ then $y \in B_\epsilon(y_N)$ for some $N \in \mathbb{N}$. Hence $y \in \pi_\epsilon(x)$. If X is not connected then there is a nonempty proper subset A of X such that it is both open and closed. Therefore A and A^c are disjoint nonempty compact subsets of X. So $\epsilon = d(A, A^c)/2 > 0$. By assumption there is an ϵ-pseudo orbit $x_0, x_1, ..., x_n$ so that $x_0 = x$ and $x_n = y$. Thus there is an index $0 \leq i \leq n-1$ such that $x_i \in A$ and $x_{i+1} \in A^c$, which is a contradiction. $\square$

Proposition 2.8. *If $f : (\mathcal{CR}(f), d_f) \to (\mathcal{CR}(f), d_f)$ has the POTP with respect to d_f, then $(\mathcal{CR}(f), d_f)$ is complete.*

Proof. Given $\epsilon > 0$, by assumption there exists $\delta > 0$ so that any δ-pseudo-orbit in $\mathcal{CR}(f)$ can be ϵ-shadowed with a point in $\mathcal{CR}(f)$. Let $\{x_n\}$ be a Cauchy sequence. So there is $N \in \mathbb{N}$ such that $d_f(f(x_n), x_{n+1}) = d_f(x_n, x_{n+1}) < \delta$ for $n \geq N$. Thus there exists $x \in \mathcal{CR}(f)$ so that $d_f(x_n, x) = d_f(x_n, f^n(x)) < \epsilon$ for $n \geq N$. $\square$

Let $f : X \to X$ be a continuous surjection. Then if

$$X_f = \lim_{\leftarrow}(X, f) = \{(x_i) : \ x_i \in X \text{ and } f(x_{i+1}) = x_i, \ i \geq 0\}$$

and

$$\bar{d}((x_i), (y_i)) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i}$$

then $(X_f, \bar{d})$ is a metric space called inverse limit space. The homeomorphism $\sigma : X_f \to X_f$ with $\sigma((x_i)_{i=0}^{\infty}) = (f(x_i))_{i=0}^{\infty}$ is called the shift map. We know that $\mathcal{CR}(\sigma) = \lim\limits_{\leftarrow}(\mathcal{CR}(f), f)$ [1].

Proposition 2.9. *Let f be a continuous surjection on a compact metric space X to itself and $\mathcal{CR}(\sigma)$ be the chain recurrent set for the shift map $\sigma : X_f \to X_f$. Then we deduce*

$$2d_f(x_0, y_0) \leq \bar{d}_\sigma((x_i), (y_i))$$

Proof. Suppose that $(x_i), (y_i) \in \mathcal{CR}(\sigma)$ and $(p_i^j), (q_i^j) \in \mathcal{CR}(\sigma)$, $j = 0, 1, ..., k$ so that $(p_i^0) = (x_i)$, $(q_i^k) = (y_i)$ and $(q_i^j) \rightsquigarrow (p_i^{j+1})$ for $j = 0, 1, ..., k-1$. We show that for each $m \geq 0$ and $j = 0, 1, ...k-1$, $q_m^j \rightsquigarrow p_m^{j+1}$. Fixed $m, j \geq 0$, for given $\epsilon > 0$ $\mathcal{O}_{\epsilon/2^m}(\sigma, (q_i^j), (p_i^{j+1})) \neq \varnothing$. Let $\{(r_i^0), (r_i^1), ..., (r_i^n)\} \in \mathcal{O}_{\epsilon/2^m}(\sigma, (q_i^j), (p_i^{j+1}))$. Then

$$\frac{d(f(r_m^l), r_m^{l+1})}{2^m} = \frac{d(r_{m-1}^l, r_m^{l+1})}{2^m} \leq \bar{d}(\sigma(r_i^l), (r_i^{l+1})) < \epsilon/2^m$$

for $l = 0, 1, ..., n-1$. Thus $\{r_m^0, r_m^1, ..., r_m^n\} \in \mathcal{O}_\epsilon(f, q_m^j, p_m^{j+1})$ so $q_m^j \overset{\epsilon}{\rightsquigarrow} p_m^{j+1}$. Since $\epsilon > 0$ is arbitrary then $q_m^j \rightsquigarrow p_m^{j+1}$. Hence $d_f(x_i, y_i) \leq \sum_{j=0}^k d(p_i^j, q_i^j)$. Therefore

$$\sum_{i=0}^{\infty} \frac{d_f(x_i, y_i)}{2^i} \leq \sum_{j=0}^{k} \sum_{i=0}^{\infty} \frac{d(p_i^j, q_i^j)}{2^i} = \sum_{j=0}^{k} \bar{d}((p_i^j), (q_i^j)).$$

So

$$\sum_{i=0}^{\infty} \frac{d_f(x_i, y_i)}{2^i} \leq \bar{d}_\sigma((x_i), (y_i)).$$

Corollary 2.5 implies

$$2d_f(x_0, y_0) = \sum_{i=0}^{\infty} \frac{d_f(x_i, y_i)}{2^i} \leq \bar{d}_\sigma((x_i), (y_i)).$$

□

Corollary 2.10. *Let X be a compact metric space and f be a chain recurrent continuous surjection from X to itself. Then if the shift map $\sigma : X_f \to X_f$ is chain transitive then $f : X \to X$ is so.*

Theorem 2.11. *The topology $\tilde{\tau}_f$ coincide with quotient topology on $\widetilde{\mathcal{CR}}(f)$*

Proof. Every continuous bijection from a compact topological space to a Hausdorff space is a homeomorphism. Since X is Hausdorff, any two elements $\pi(x), \pi(y) \in \widetilde{\mathcal{CR}}(f)$ as compact subsets posses disjoint saturated neighborhood, so $\widetilde{\mathcal{CR}}(f)$ is a Hausdorff space with the quotient topology. Also $(\widetilde{\mathcal{CR}}(f), \tilde{\tau}_f)$ is compact. Thus the identity map is a homeomorphism. □

Recall that the Hausdorff metric on the compact subsets A, B of X is defined as follows

$$d_{\mathcal{H}}(A,B) = \max\{\max_{a\in A} d(a,B), \max_{b\in B} d(A,b)\}.$$

Let $\tau_{\mathcal{H}}$ be the topology induced by Hausdorff metric $d_{\mathcal{H}}$ on $\widetilde{\mathcal{CR}}(f)$. Then we deduce the following proposition.

Proposition 2.12. *The topology $\tau_{\mathcal{H}}$ is finer than $\tilde{\tau}_f$.*

Proof. Suppose that $\pi(x), \pi(y) \in \widetilde{\mathcal{CR}}(f)$. We can choose $y' \in \pi(y)$ so that $d(x,y') = d(x,\pi(y))$. Thus

$$\tilde{d}_f(\pi(x),\pi(y)) = d_f(x,y') \leq d(x,y') \leq d_{\mathcal{H}}(\pi(x),\pi(y)).$$

Therefore $\tilde{\tau}_f \subset \tau_{\mathcal{H}}$. □

The next example shows that $\tilde{d}_f$ and $d_{\mathcal{H}}$ are not equal in general.

Example 2.13. Let $f : [0,1] \to [0,1]$ be a strictly increasing continuous map so that

- $f(x) = x$ for each $x \in [2^{-(2i+1)}, 2^{-(2i)}]$, $i = 0,1,...;$
- $f(x) > x$ for each $x \in [2^{-(2i+2)}, 2^{-(2i+1)}]$, $i = 0,1,...;$
- $f(0) = 0$.

Then we deduce

$$\mathcal{CR}(f) = \{0\} \cup \bigcup_{i=0}^{\infty} [2^{-(2i+1)}, 2^{-2i}]$$

and

$$\widetilde{\mathcal{CR}}(f) = \{[2^{-(2i+1)}, 2^{-2i}];\ i = 0,1,...\} \cup \{0\}.$$

If $x \in [2^{-(2i+1)}, 2^{-2i}]$ and $y \in [2^{-(2j+1)}, 2^{-2j}]$ for some$j < i$, then

$$d_f(x,y) = 2^{-(2j+1)} - \sum_{k=j+1}^{i} 2^{-(2k+1)}.$$

Thus we deduce $\tilde{d}_f(\pi(0),\pi(1)) = 1/3$, but $d_{\mathcal{H}}(\pi(0),\pi(1)) = 1$.

Proposition 2.14. *Let (X,d) and (Y,d') be two compact metric spaces and $f : X \to X$ and $g : Y \to Y$ be continuous maps. Then if f and g are topologically conjugate then $(\widetilde{\mathcal{CR}}(f), \tilde{d}_f)$ and $(\widetilde{\mathcal{CR}}(g), \tilde{d}_g)$ are isometric.*

Proof. Suppose that $h : X \to Y$ is a homeomorphism so that $h \circ f = g \circ h$. Given $\epsilon > 0$ there is $\delta > 0$ so that for every $x, y \in X$, the inequality $d(x,y) < \delta$ implies to $d'(h(x),h(y)) < \epsilon$ and the inequality $d'(x,y) < \delta$ implies to $d(h^{-1}(x), h^{-1}(y)) < \epsilon$. If $\{x_i\}_{i=0}^n \in \mathcal{O}_\delta(f,p,q)$ for some $p,q \in X$, then $\{h(x_i)\}_{i=0}^n \in \mathcal{O}_\epsilon(g,h(p),h(q))$. This implies that if $p \leftrightsquigarrow q$ then $h(p) \leftrightsquigarrow h(q)$. Hence for every $x, y \in \mathcal{CR}(f)$ we deduce $d_f(x,y) \geq d'_g(h(x),h(y))$. If

$\{x_i\}_{i=0}^n \in \mathcal{O}_\delta(g, h(p), h(q))$, then $\{h^{-1}(x_i)\}_{i=0}^n \in \mathcal{O}_\delta(f, p, q)$. Thus $d_f(x, y) \leq d'_g(h(x), h(y))$. So

$$\begin{aligned}\tilde{d}_f(\pi(x), \pi(y)) = d_f(x, y) &= d'_g(h(x), h(y)) \\ &= \tilde{d}'_g(\pi(h(x)), \pi(h(y))) = \tilde{d}'_g(\tilde{h}(\pi(x), \tilde{h}(\pi(y)).\end{aligned}$$

Therefore $\tilde{h} : \widetilde{\mathcal{CR}}(f) \to \widetilde{\mathcal{CR}}(g)$ is an isometry. □

3. Conclusions

In this paper we introduce a pseudo-metric d_f associated to the dynamical system f. We show that the topology induced by d_f has a significant relation to some dynamical properties of f, such as transitivity and shadowing. By considering the identity map we obtained some equivalence conditions for connectedness of space. Investigate the relation between this topology and topological entropy of f will be a topic for future research.

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