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Useful topologies and separable systems

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Abstract. Let X be an arbitrary set. A topology t on X is said to be *useful* if every continuous linear preorder on X is representable by a continuous real–valued order preserving function. Continuous linear preorders on X are induced by certain families of open subsets of X that are called (linear) separable systems on X. Therefore, in a first step useful topologies on X will be characterized by means of (linear) separable systems on X. Then, in a second step particular topologies on X are studied that do not allow the construction of (linear) separable systems on X that correspond to non–representable continuous linear preorders. In this way generalizations of the Eilenberg–Debreu theorems which state that second countable or separable and connected topologies on X are useful and of the theorem of Estévez and Hervés which states that a metrizable topology on X is useful, if and only if it is second countable can be proved.

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1. Introduction

A topology t on an arbitrary set X is said to be *useful*, if every continuous linear (total) preorder $\precsim$ on X has a continuous utility representation, i.e. can be represented by a continuous real–valued order preserving function (utility function) (see [15]). Continuity of $\precsim$ means that the order topology $t^{\precsim}$ induced by $\precsim$ is coarser than t. Sufficient conditions for a topology t on X to be useful are, for instance, given by the classical Eilenberg–Debreu Theorems (EDT) and (DT) ([9, 10, 11]). Necessary and sufficient conditions for a topology t on X to be useful have been presented by the theorem of Estévez and Hervés (EHT) in case that t is a metrizable topology on X ([13], see also [6, 7]) Using the concept of a useful topology t on X these theorems can be restated as follows:

(EDT) Every connected and separable topology t on X is useful.

(DT) Every second countable topology t on X is useful.

(EHT) A metrizable topology t on X is useful, if and only if t is second countable.

The aim of this paper is the characterization of all useful topologies t on X. A theorem which solves this problem, in particular, would generalize the Eilenberg-Debreu Theorems and the Theorem of Estévez and Hervés. Meanwhile Banach-spaces or, more generally, convex spaces are frequently studied in mathematical utility theory. In the infinite dimensional case these spaces may fail to be second countable or separable. This means that continuous representation of linear orderings (preference orderings) in these spaces is not guaranteed by the classical Eilenberg-Debreu Theorems. Therefore, a characterization of useful topologies is of particular interest in mathematical utility theory (cf. also Remark 6.8).

2. A first approach

Let throughout this section X be a fixed given set and let t be some topology on X. For every subset A of X we denote by $\overline{A}$ its topological closure. The most fundamental result that is known on useful topologies is *DT*. *DT* easily implies *EDT* (cf. [14, 16]) and also generalizes the sufficiency part of the Theorem of Estévez and Hervés. On the other hand, it is well known that second countability, in general, is not necessary for t to be useful (cf., for instance, the Niemitzki plane that is extensively discussed in [32]). Hence, in order to at least approximate second countability we consider *linearly ordered* subtopologies t^l of t. t^l is linearly ordered if it is linearly ordered by set inclusion.

It is easily to be seen that second countability of t implies second countability of all its linearly ordered subtopologies t^l. Indeed, let t^l be a linearly ordered subtopology of the second countable topology t. Then we choose a countable base $\mathcal{B}$ of t and consider the countable subset

$$\begin{aligned}\mathcal{B}^l \ := \ & \{O \in t^l \mid \exists B \in \mathcal{B}(B \subset O \wedge \forall O' \in t^l(O' \subsetneqq O \Longrightarrow B \not\subset O'))\} \\ & \cup \{O \in t^l \mid \exists B \in \mathcal{B}(O = \cup_{B \not\subset O' \in t^l} O')\} \\ & \cup \{\varnothing, X\}\end{aligned}$$

of t^l in order to immediately verify that $\mathcal{B}^l$ is a base of t^l. Let us now assume that all linearly ordered subtopologies t^l of t are second countable and let $\precsim$ be a continuous linear (total) preorder on X. Then we consider the family

$$\mathcal{L} := \{L(x)\}_{x \in X} := \{\{y \in X \mid y \prec x\}\}_{x \in X}$$

of open decreasing subsets of X. The linearly ordered subtopology t^l of t that is induced by $\mathcal{L}$ is second countable which means that there exists a countable subset $\mathcal{L}_{\mathcal{B}}$ of $\mathcal{L} \cup \{\varnothing, X\}$ that is a base of t^l. The countability of $\mathcal{L}_{\mathcal{B}}$ implies that the corresponding chain $(\mathcal{L}_{\mathcal{B}}, \subset)$ only has countably many jumps. The reader may recall that a jump of $(\mathcal{L}_{\mathcal{B}}, \subset)$ is a pair of sets $E \subsetneqq E' \in \mathcal{L}_{\mathcal{B}}$ such that there exists no set $E'' \in \mathcal{L}_{\mathcal{B}}$ such that $E \subsetneqq E'' \subsetneqq E'$. By interposing the rationals into the jumps of $(\mathcal{L}_{\mathcal{B}}, \subset)$ we, thus, obtain some chain $(\mathcal{L}^e_{\mathcal{B}}, \leq)$ that extends $(\mathcal{L}_{\mathcal{B}}, \subset)$ and may, without loss of generality, be assumed to be order-isomorphic to the chain $([0,1]_{\mathbb{Q}}, \leq)$ of all rationals in the real interval $[0,1]$. Let g: $(\mathcal{L}^e_{\mathcal{B}}, \leq) \longrightarrow ([0,1]_{\mathbb{Q}}, \leq)$ be some order-isomorphism. Then one verifies that f: $(X, \precsim) \longrightarrow ([0,1]_{\mathbb{R}}, \leq)$ defined for all points $x \in X$ by $f(x) := \sup\{g(L(y)) \mid L(y) \in \mathcal{L}_{\mathcal{B}}, y \precsim x\}$ is a continuous utility representation of $\precsim$.

Clearly, t is not necessarily second countable if all its linearly ordered subtopologies t^l are second countable. In order to obtain a counterexample one only has to choose some topology on the natural numbers that is not second countable. Hence, the above result does not only provide an alternative proof of DT but also generalizes DT.

In order to also generalize EDT we consider the first infinite ordinal ω. Then we consider the family $\mathcal{T}_{\mathcal{C}}$ of all linearly ordered subtopologies t^l of t that are induced by some linearly (totally) ordered set $(\mathcal{O}, \subset)$ of open subsets of X that satisfy the following conditions:

(LO1): $\forall O' \in \mathcal{O}(\overline{O'} \subset \bigcap_{O' \subsetneqq O \in \mathcal{O}} O)$ or, equivalently, $\forall O' \in \mathcal{O}\ \forall O \in \mathcal{O}(O' \subsetneqq O \Longrightarrow \overline{O'} \subset O)$, and

(LO2): $\left|\left\{O \in \mathcal{O} \mid \bigcup_{\mathcal{O} \ni O' \subsetneqq O} \overline{O'} \subsetneqq O \wedge \overline{\bigcup_{\mathcal{O} \ni O' \subsetneqq O} O'} \cap X \setminus O \neq \varnothing\right\}\right| \leq \omega$.

Now it follows that in case that t is a separable and connected topology on X every linearly ordered subtopology $t^l \in \mathcal{T}_{\mathcal{C}}$ of t must be second countable. Indeed, let some topology $t^l \in \mathcal{T}_{\mathcal{C}}$ be arbitrarily chosen. Then the separability of t implies that no chain $(\mathcal{O}, \subset)$ or $(\mathcal{O}, \supset)$ of open subsets of X which satisfies the conditions (LO1) and (LO2) and induces t^l contains some uncountable well-ordered subchain, i.e. $(\mathcal{O}, \subset)$ or $(\mathcal{O}, \supset)$ is *short* which, in particular, means that t^l is first countable (cf. [1]). In addition, the connectedness of t implies that none of the sets $O \setminus \overline{\bigcup_{\mathcal{O} \ni O' \subsetneqq O} O'}$ such that $\varnothing \subsetneqq \overline{\bigcup_{\mathcal{O} \ni O' \subsetneqq O} O'} \subset O \subsetneqq X$ is empty. Let, therefore, S be a countable dense subset of X. Then we choose for every point $x \in S$ some countable base of t^l-neighborhoods of x. The union of the collection of these t^l-neighborhoods with the countable set $\left\{O \in \mathcal{O} \mid \bigcup_{\mathcal{O} \ni O' \subsetneqq O} \overline{O'} \subsetneqq O \wedge \overline{\bigcup_{\mathcal{O} \ni O' \subsetneqq O} O'} \cap X \setminus O \neq \varnothing\right\}$ is a countable base of t^l. Let us now assume that all linearly ordered subtopologies $t^l \in \mathcal{T}_{\mathcal{C}}$ are second countable. Then the same arguments that already have been applied in order to generalize DT allow us to conclude that every continuous linear (total) preorder $\precsim$ on X has a continuous utility representation.

On the other hand, it also cannot be expected that second countability of the linearly ordered subtopologies $t^l \in \mathcal{T}_{\mathcal{C}}$ of t is necessary in order to guarantee usefulness of t. Indeed, let $\precsim$ be some arbitrary continuous linear preorder on X. Then the linearly ordered set $(\mathcal{L}, \subset) := (\{L(x)\}_{x \in X}, \subset)$ satisfies the following additional condition that strengthens condition (LO2).

$$\text{(LO3)}: \quad \forall O \in \mathcal{L}\Big(\bigcup_{\mathcal{L} \ni O' \subsetneqq O} \overline{O'} \subsetneqq O \Longrightarrow \overline{\bigcup_{\mathcal{L} \ni O' \subsetneqq O} O'} \subset O\Big).$$

Therefore, it is somewhat surprising that in case that we concentrate on normal topologies t on X (cf. Definition 3.3) the conditions (LO1) and (LO2) completed by two straightforward conditions that are necessary in order to also include the case that t is not necessarily connected already characterize useful topologies. This characterization provides a generalization of EDT in the just described way. In particular, it can be shown that our results are generalizations of the theorem of Estévez and Hervés. The reader may still notice that the afore-discussed generalizations of DT and EDT provide a possibility of how to apply our results on useful topologies that will be proved in the following sections.

3. R-separable systems

It is well known that continuous linear preorders are closely related to R-separable systems (see [16]). Therefore, we shall approach the characterization of useful topologies in a first step with help of R-separable systems.

Suppose that R is an arbitrary binary relation on some fixed given topological space (X, t) (briefly we speak of an *R-space*). Then the reader may recall at first the following notation: A subset A of X is said to be *R-decreasing* (or simply *decreasing*, if the relation is clear from the context), if $a \in A$ and bRa imply that $b \in A$. An *increasing* set is defined in an analogous manner. Each subset F of X gives rise to the smallest decreasing (respectively, increasing) subset $d(F)$ (respectively, $i(F)$) containing F. If $F = \{x\}$ for some point $x \in X$, then we write $d(x)$ (respectively, $i(x)$) instead of $d(\{x\})$ (respectively, of $i(\{x\})$). For each subset F of X there is a smallest closed decreasing subset $D(F)$ (respectively, smallest closed increasing subset $I(F)$) containing F. If $F = \{x\}$ for some point $x \in X$, then we write $D(x)$ (respectively, $I(x)$) instead of $D(\{x\})$ (respectively, of $I(\{x\})$. Notice that for each subset F of X we have $\overline{F} \subset D(F)$. In general, this inequality is strict as is seen from the following simple example. Let $X := \{1, 2\}, t := \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$ and $R := \{(i, j) \in X \times X \mid i \leq j\}$. Then $\overline{\{2\}} = \{2\} \subsetneqq D(2) = \{1, 2\} = X$. With these preliminaries we are fully prepared for the following definition.

Definition 3.1. *A family $\mathcal{E}$ of open R–decreasing subsets of X is said to be an R-separable system on X, if it satisfies the following conditions:*

(RS1) *There exist sets $E_1, E_2 \in \mathcal{E}$ such that $\overline{E_1} \subset E_2$.*

(RS2) *For all sets $E_1, E_2 \in \mathcal{E}$ such that $\overline{E_1} \subset E_2$ there exists some set $E_3 \in \mathcal{E}$ such that $\overline{E_1} \subset E_3 \subset \overline{E_3} \subset E_2$.*

Moreover, if R is the equality relation "$=$" on X, i.e. the discrete order on X, we say that $\mathcal{E}$ is a separable system on X.

Remark 3.2. In mathematical utility theory R-separable systems on X were constructed for the first time by Peleg [30] in order to prove his utility representation theorem. In Peleg‘s Theorem R is a strict partial order or briefly an order on X. In 1977 Burgess and Fitzpatrick [4] studied decreasing scales in X. We recall that a family $\mathcal{S} := \{F_r\}_{r \in D}$ of open decreasing subsets of X is said to be a *decreasing scale in X*, if the following two conditions are satisfied:

(DS1) D is a dense subset of the real interval $[0, 1]$ such that $1 \in D$ and $F_1 = X$.

(DS2) For every pair of real numbers $r_1 < r_2 \in D$ the inclusion $\overline{F_{r_1}} \subset F_{r_2}$ holds.

One immediately verifies that decreasing scales in X are particular cases of R-separable systems on X.

The concept of an R-separable system on X is closely related to the concept of a *normally preordered space* (cf. [29]) or more generally *normal R-space.*

Definition 3.3. *An R-space $(X,\, R,\, t)$ is said to be a normal R-space, if for any pair A, B of disjoint closed decreasing (respectively, increasing) subsets of X there exist disjoint open decreasing, (respectively, increasing) subsets U, V of X such that $A \subset U$ and $B \subset V$.*

Notice that, if R coincides with the equality-relation " $=$ " on X, then $(X,\, R,\, t)$ is a normal space.

The connections between the concept of an R–separable system and of a normal R–space is described in the following lemma ([16, Lemma 2.1]).

Lemma 3.4. *Let (X, R, t) be an arbitrary R-space. Then in order for (X, R, t) to be a normal R-space it is necessary and sufficient that for every pair C_1, C_2 of disjoint closed subsets of X, C_1 being decreasing and C_2 increasing, there exists an R-separable system $\mathcal{E}$ on X such that $C_1 \subset E$ and $C_2 \subset X \backslash E$ for every set $E \in \mathcal{E}$.*

Now we turn our attention to *linear R-separable systems.* Given an arbitrary R-space $(X,\, R,\, t)$, an R-separable system $\mathcal{E}$ on X is said to be *linear* if, for every pair of sets E, $E' \in \mathcal{E}$ such that $E \neq E'$ at least one of the inclusions $\overline{E} \subset E'$ or $\overline{E'} \subset E$ holds. Linear R-separable systems $\mathcal{E}$ on X easily can be characterized ([17, Proposition 1.4.1]).

Proposition 3.5. *Let $\mathcal{E}$ be a family of open decreasing subsets of X, that is linearly ordered by set inclusion, and let $\mathcal{B}$ be the family of all sets $E \in \mathcal{E}$ such that $E \subsetneqq \overline{E}$ and for which there exists some set $B \in \mathcal{E}$ such that $E \subsetneqq B$. Then the following assertions are equivalent:*

(i) *$\mathcal{E}$ is a linear R-separable system on X,*

(ii) $\forall E \in \mathcal{B}(\bigcap_{E \subsetneqq B \in \mathcal{E}} B = \bigcap_{E \subsetneqq B \in \mathcal{E}} \overline{B})$,

(iii) $\forall E \in \mathcal{B}((\overline{E} \subset \bigcap_{E \subsetneqq B \in \mathcal{E}} B) \wedge (\bigcap_{E \subsetneqq B \in \mathcal{E}} B \in \mathcal{E} \Longrightarrow \bigcap_{E \subsetneqq B \in \mathcal{E}} B = \overline{\bigcap_{E \subsetneqq B \in \mathcal{E}} B}))$.

Every R-separable system $\mathcal{E}$ on X contains some linear R-separable system. Indeed, let $\mathbb{Q}$ denote the rationals. Then this result is an immediate consequence of the following lemma ([16, Lemma 2.2]).

Lemma 3.6. *Let $\mathcal{E}$ be an R-separable system on X. Then there exists a function $f : \mathbb{Q} \longrightarrow \mathcal{E}$ such that $f(p) \subset \overline{f(p)} \subset f(q)$ for all $p < q \in \mathbb{Q}$.*

The reader may recall that a real-valued function f on X is said to be *increasing* if, for all pairs $(x, y) \in R$, the inequality $f(x) \leq f(y)$ holds. With help of this notation we are already able to present the general separation theorem *GST* of Nachbin-Urysohn-type, which corresponds to *GURT* in [16] (see also [29], [34]).

Theorem 3.7. *Let (X, R, t) be an R-space. Then in order for (X, R, t) to be a normal R-space it is necessary and sufficient that for any two disjoint closed subsets C_1, C_2 of X such that C_1 is decreasing and C_2 increasing, there exists some continuous increasing real-valued function f on X such that $0 \leq f \leq 1$, $f(x) = 0$ for all $x \in C_1$ and $f(x) = 1$ for all $x \in C_2$.*

Now we present the most important result of this section (see [16, Lemma 2.3]).

Theorem 3.8. *Let (X, R, t) be an R-space. Then every linear R-separable system $\mathcal{E}$ on X induces a linear preorder $\precsim$ on X which satisfies the following properties:*

(L1): $R \subset\precsim$,,

(L2): *The order topology $t^{\precsim}$ is coarser than $t_{\mathcal{E}}$.*

For later use we recall the definition of $\precsim$. Let $\mathcal{E}$ be a linear R-separable system on X and let points $x, y \in X$ be arbitrarily chosen. Set $\mathcal{E}_y := \{E \in \mathcal{E} | y \in E\}$ and define $\precsim$ by setting

$$x \precsim y \Longleftrightarrow \mathcal{E}_y = \varnothing \vee \forall E \in \mathcal{E}_y \forall B \in \mathcal{E}_y (\overline{E} \not\subset B \vee x \in B).$$

It is easily seen that $\precsim$ can be divided into the following two less complicated subrelations:

(i) $x \prec y \Longleftrightarrow$ there exists some R-separable system$\mathcal{B} \subset \mathcal{E}$ on X such that $x \in B$ and $y \in X \backslash \overline{B}$ for all sets $B \in \mathcal{B}$,

(ii) $x \sim y \Longleftrightarrow \neg(x \prec y)$ and $\neg(y \prec x)$.

It seems that Theorem 3.8 is closely related to the famous Szpilrajn's Theorem [33] which states that every partially order can be refined to a linear order. But $\precsim$ is not necessarily a refinement of A, since we did not require that $\prec$ is contained in the set of all pairs $(x, y) \in R$ such that $(y, x) \notin R$. For later use we abbreviate this set by R_S. Hence, Szpilrajn's Theorem is not a consequence of Theorem 3.8. As far as the authors know the continuous analogous of Szpilrajn's Theorem never has been discussed in the literature. In order to be more precise, the reader may recall that a linear preorder $\precsim$ on X is continuous if and only if for each pair of points $x \prec y \in X$ there exists some continuous increasing real-valued function f_{xy} on X such that $f_{xy}(x) < f_{xy}(y)$ (the reader may apply [22, Lemma 1] and *GST* to verify this result). Obviously, this characterization of continuous linear preorders can be generalized to arbitrary preorders. Therefore, a continuous preorder $\precsim$ on X is said to satisfy the *Szpilrajn-property*, if there exists a continuous linear preorder $\precsim^*$ on X such that $\precsim^*$ is a refinement of $\precsim$. The Szpilrajn-property will be discussed in [17, Chapter 6].

4. Real order-embeddings

Let throughout this section X be some fixed given set and R some relation on X. We recall some definitions. (X, R) is said to be *Jaffray-separable* if there exists a countable subset Z of X such that, if $x, y \in X$ and $(x, y) \in R_S$, then there exist points $z, z' \in Z$ such that $xRxR_Sz'Ry$. (X, R) is said to be *Birkhoff-separable* if there exists a countable subset Z of X such that, for every pair $(x,y) \in R_S \cap (X \setminus Z) \times (X \setminus Z)$, there exists some $z \in Z$ such that xR_SzR_Sy. The space (X, R) is called *Debreu-separable* if there exists a countable subset Z of X such that for every pair $(x,y) \in R_S$ there exists some $z \in Z$ such that $xRzRy$, and it is called *Cantor-separable* if there exists a countable subset Z of X such that, for every pair $(x,y) \in R_S$, there exists some $z \in Z$ such that xR_SzR_Sy.

Now we are fully prepared for presenting the following Representation Theorem (see, for example, [3, Proposition 1.6.11] and [13, Lemma 3.1]).

Theorem 4.1. *Let $(X, \precsim)$ be a linearly preordered set. Then the following assertions are equivalent:*

(i) *There exists an order preserving function $f : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$,*
(ii) *$(X, \precsim)$ is Jaffray-separable,*
(iii) *$(X_{|\sim}, \precsim_{|\sim})$ is Birkhoff-separable,*
(iv) *$(X, \precsim)$ is Debreu-separable,*
(v) *$(X, t^{\precsim})$ is separable and $(X_{|\sim}, \precsim_{|\sim})$ has only countably many jumps,*
(vi) *$(X, t^{\precsim})$ is second countable.*

The reader may notice that in contrast to Proposition 1.6.11 in Bridges and Mehta [3] the assertion concerning Birkhoff-separability has been modified somewhat. Indeed, the concepts of Birkhoff-separability and Debreu-separability are not equivalent in the context of preorders but only in the context of orders. This hint is due to Mehta [24, November 1999, oral communication].

5. The structure of useful topologies

Let X be a fixed given set and let t be some topology on X. It is the aim of this section to characterize all useful topologies on X with help of linear separable systems on X. Because of Proposition 3.5 the characterization of useful topologies with help of linear separable systems is a quite satisfactory approximation of the desired results that have been announced in the second section. In order to also include the non-connected case we need at first the following notation. A topological space (X, t) is said to satisfy the *open-closed countable chain condition* (OCCC), if every family $\mathcal{F}$ of non-empty open and closed subsets F of X that satisfies the following two conditions is countable:

(OC1): $\forall F \in \mathcal{F} (F \subset F' \vee F' \subset F)$,
(OC2): $\forall F \in \mathcal{F} (\cup \{F' \in \mathcal{F} \mid F' \subsetneqq F\} \subsetneqq F \subsetneqq \cap \{F'' \in \mathcal{F} \mid F \subsetneqq F''\})$.

Let now $\mathcal{E}$ be a linear separable system on X. We consider the set $\mathcal{Z}(\mathcal{E})$ of all pairs $B \subsetneqq E \in \mathcal{E}$ for which there exists some set $C \in \mathcal{E}$ such that $\overline{B} \subsetneqq C \subset \overline{C} \subsetneqq E$. Then $\mathcal{E}$ is said to have a *countable refinement* if there exists a countable family $\mathcal{O}$ of non–empty open subsets of X such that for every pair $(B, E) \in \mathcal{Z}(\mathcal{E})$ there exists some set $O \in \mathcal{O}$ such that $O \subset E \cap X \setminus \overline{B}$. $\mathcal{E}$ is said to be *second countable*, if there exists a countable subset $\mathcal{H}$ of $\mathcal{E}$ such that for every pair of sets $(B, E) \in \mathcal{Z}(\mathcal{E})$ there exists some set $E^+ \in \mathcal{H}$ such that $\overline{B} \subset E^+ \subset \overline{E^+} \subset E$. In addition, if $\mathcal{G}(\mathcal{E})$ is the set of all (open) sets $E \in \mathcal{E}$ such that $\bigcup_{\mathcal{E} \ni B \subsetneqq E} \overline{B} \subsetneqq E$, let $\mathbf{G}_{\mathcal{G}}$ denote the family of all linear separable systems $\mathcal{E}$

on X for which $\mathcal{G}(\mathcal{E})$ is a countable set. With help of this notation the following proposition characterizes all useful topologies t on X.

Proposition 5.1. *For a topology t on a set X, the following assertions are equivalent:*

(i) *t is useful,*

(ii) *t satisfies* OCCC and every linear separable system $\mathcal{E}$ on X has a countable refinement,

(iii) t satisfies OCCC and every linear separable system $\mathcal{E}$ on X is second countable,

(iv) t satisfies OCCC and every linearly ordered subtopology t^l of t that is induced by some linear separable system $\mathcal{E} \in \mathbf{G}_{\mathcal{G}}$ is second countable.

Proof. (i) $\Longrightarrow$ (ii) At first we assume, in contrast, that t does not satisfy OCCC. Then there exists an uncountable family $\mathcal{F}$ of non-empty open and closed subsets F of X that satisfies the conditions (OC1) and (OC2). Since every set $F \in \mathcal{F}$ is open and closed, condition (OC1) implies that the preorder $\precsim$ defined for every pair of points $x, y \in X$ by

$$x \precsim y \Longleftrightarrow \forall F \in \mathcal{F}(y \in F \Longrightarrow x \in F)$$

is linear and continuous. In addition, the uncountability of $\mathcal{F}$ allows us to conclude with help of condition (OC2) that $\precsim$ has uncountably many jumps. Indeed, for every set $F \in \mathcal{F}$ any pair of points $x \in F \setminus \cup \{B \in \mathcal{F} \mid B \subsetneqq F\}$, $y \in \cap \{C \in \mathcal{F} \mid F \subsetneqq C\} \setminus F$ defines a jump of $\precsim$. Hence, $\precsim$ is not representable. This contradiction implies that t satisfies OCCC. Let now $\mathcal{E}$ be a linear separable system on X. It remains to show that there exists a countable family $\mathcal{O}$ of open subsets of X such that for every pair $(B, E) \in \mathcal{Z}(\mathcal{E})$ there exists some set $O \in \mathcal{O}$ such that $O \subset E \cap X \setminus \overline{B}$. As remarked after Theorem 3.8 we may define a continuous linear preorder $\precsim$ on X induced by $\mathcal{E}$ such that for every pair $(B, E) \in \mathcal{Z}(\mathcal{E})$ and every pair of points $x \in C \setminus \overline{B}, y \in E \setminus \overline{C}$ the strict inequality $x \prec y$ holds. This means that we may choose for every pair $(B, E) \in \mathcal{Z}(\mathcal{E})$ points $x \prec y \in X$ such that $]x, y[\subset E \cap X \setminus \overline{B}$. Because of assertion (i), the linear preorder $\precsim$ is representable. This means, in particular, that $(X_{|\sim}, \precsim_{|\sim})$ only has countably many jumps and that $t^{\precsim}$ is second countable (cf. Theorem 4.1, assertions (v) and (vi)). The existence of the desired family $\mathcal{O}$ of open subsets of X, thus, follows immediately, which finishes the proof of assertion *(ii)*.

(ii) $\Longrightarrow$ (i) Let $\precsim$ be a continuous linear preorder on X. Because of the Open Gap Lemma ([9, 10]) it suffices to prove that $\precsim$ is representable. Therefore, we consider the linear separable system $\mathcal{L} := \{L(x)\}_{x \in X} := \{\{y \in X \mid y \prec x\}\}_{x \in X}$ on X. Since (X, t) satisfies OCCC it follows that $(X_{|\sim}, \precsim_{|\sim})$ only has countably many jumps. Indeed, let $\{([x_i], [y_i])\}_{i \in I}$ be the family of all jumps of $(X_{|\sim}, \precsim_{|\sim}$). Then we may choose for every index $i \in I$ the open and closed subset $F_i := \{z \in X \mid z \precsim x_i\} = \{z \in X \mid z \prec y_i\}$ of X. Let $\mathcal{F}$ be the family of these subsets. Because $(X, \precsim)$ is a chain, we may conclude that $\mathcal{F}$ satisfies condition (OC1). In addition, the definition of $\mathcal{F}$ implies that $\mathcal{F}$ also satisfies condition

(OC2). Hence, it follows from OCCC that $\mathcal{F}$ is countable, and this means that the family $\{([x_i],[y_i])\}_{i\in I}$ of all jumps of $(X_{|\sim}, \precsim_{|\sim})$ actually is countable. In order to now finish the proof of the representability of $\precsim$ it remains to verify that $(X, t^{\precsim})$ is separable (cf. Theorem 4.1 (v)). $\mathcal{L}$ has a countable refinement. Hence, there exists a countable family $\mathcal{O}$ of non–empty open subsets of X such that for every pair $x \prec y \in X$ for which $]x,y[$ is neither empty nor contains a jump of $(X_{|\sim}, \precsim_{|\sim})$, there exists some set $O \in \mathcal{O}$ such that $O \subset]x,y[$. Choosing in every set $O \in \mathcal{O}$ some point $x \in O$ and considering, in addition, for every jump $([x],[y])$ of $(X_{|\sim}, \precsim_{|\sim})$ points $x \in [x]$ and $y \in [y]$ respectively, we may conclude that $(X, t^{\precsim})$ must be separable, and assertion (i) follows.

(i) $\wedge$ (ii) $\Longrightarrow$ (iii) Let $\mathcal{E}$ be a linear separable system on X. It suffices to show that $\mathcal{E}$ is second countable. Let, therefore, $\mathcal{O}$ be a countable family of non–empty open subsets of X such that for every pair of sets $(B,E) \in \mathcal{Z}(\mathcal{E})$ there exists some set $O \in \mathcal{O}$ such that $O \subset E \cap X\backslash\overline{B}$. By eliminating redundant sets we may assume without loss of generality that for every set $O \in \mathcal{O}$ there exist sets $\overline{B} \subset E \in \mathcal{E}$ such that $O \subset E \cap X\backslash\overline{B}$. Hence, we may choose for every set $O \in \mathcal{O}$ the non–empty linear separable systems $\mathcal{W}_1 := \{B \in \mathcal{E} \mid O\backslash B \neq \varnothing\}$ and $\mathcal{W}_2 := \{E \in \mathcal{E} \mid O\backslash E = \varnothing\}$. It follows that there exist countable sets $\mathcal{O}_1 \subset \mathcal{W}_1$ and $\mathcal{O}_2 \subset \mathcal{W}_2$ such that $\bigcup_{E\in\mathcal{O}_1} E = \bigcup_{E\in\mathcal{W}_1} E$ and $\bigcap_{E\in\mathcal{O}_2} E = \bigcap_{E\in\mathcal{W}_2} E$. Indeed, otherwise the construction described after Theorem 3.8 implies that both continuous linear preorders $\precsim_1$ and $\precsim_2$ on X which are induced by $\mathcal{W}_1$ and by $\mathcal{W}_2$, respectively, are not short and, thus, not representable in contrast to assertion (i). Since $\mathcal{O}$ is countable we may conclude that $\mathcal{B} := \bigcup_{O\in\mathcal{O}} \mathcal{O}_1 \cup \bigcup_{O\in\mathcal{O}} \mathcal{O}_2$ is a countable set. The construction of $\mathcal{B}$ implies that for every pair of sets $(B,E) \in \mathcal{Z}(\mathcal{E})$ there exists some set $E^+ \in \mathcal{B}$ such that $\overline{B} \subset E^+ \subset \overline{E}^+ \subset E$, as desired.

(iii) $\Longrightarrow$ (iv) Trivial.

(iv) $\Longrightarrow$ (i) Let $\precsim$ be some continuous linear (total) preorder on X. Then we consider the linear separable system $\mathcal{L} := \{L(x)\}_{x\in X}$ on X. In the proof of the implication (ii) $\Longrightarrow$ (i) it already has been shown that OCCC implies that $(X_{|\sim}, \precsim_{|\sim})$ only has countably many jumps. Since $\mathcal{L}$ satisfies condition (LO3) it follows that $L \in \mathbf{G}_{\mathcal{G}}$. The reader may recall that condition (LO3) implies condition (LO2). Assertion (iv), thus, implies that the linearly ordered subtopology t^l of t that is induced by $\mathcal{L}$ is second countable which allows us to conclude with help of the considerations in the second section that $\precsim$ has a continuous utility representation. Therefore, the proof of the proposition is complete. $\square$

Clearly, in case that t is connected OCCC may be omitted. Hence, the characterization of t to be useful simplifies somewhat.

Corollary 5.2. *Let t be connected. Then the following assertions are equivalent:*

(i) *t is useful,*
(ii) *every linear separable system $\mathcal{E}$ on X has a countable refinement,*
(iii) *every linear separable system $\mathcal{E}$ on X is second countable,*

(iv) *every linear ordered subtopology t^l of t that is induced by some linear separable system $\mathcal{E} \in \mathbf{G}_{\mathcal{G}}$ is second countable.*

6. A different approach

Let (X, t) be an arbitrary topological space and let $\mathbf{G} := \{\mathcal{E}_i\}_{i\in I}$ be a family of separable systems on X. Then $\mathbf{G}$ is said to be *well-separated*, if it satisfies the following conditions:

(WS1): $\forall i \in I\, \forall j \in I\, \forall E \in \mathcal{E}_i \forall B \in \mathcal{E}_j (i \neq j \Longrightarrow E \cap B = \varnothing)$.

(WS2): $\forall \{E_i\}_{i\in I}\, (E_i \in \mathcal{E}_i \Longrightarrow \bigcup_{i\in I} \overline{E_i} = \overline{\bigcup_{i\in I} E_i})$.

Now the following lemma holds:

Lemma 6.1. *Let t be a useful topology on X. Then every well-separated family $\mathbf{G} := \{\mathcal{E}_i\}_{i\in I}$ of separable systems on X is countable.*

Proof. Let $\mathbf{G} := \{\mathcal{E}_i\}_{i\in I}$ be some well-separated family of separable systems on X. Then we may assume without loss of generality that every subset J of I for which there exists for every $j \in J$ some non–empty open and closed set $E_j \in \mathcal{E}_j$ is countable. Indeed, otherwise we consider some well ordering $\leq$ on J, choose for every $j \in J$ some fixed non–empty open and closed set $E_j \in \mathcal{E}_j$ in order to consider for every $j \in J$ the set $F_j := \bigcup_{i\leq j} E_i$. Then the conditions (WS1) and (WS2) imply that $\mathcal{F} := \{F_j\}_{j\in J}$ is an uncountable family of non–empty open and closed subsets of X that satisfies the conditions (OC1) and (OC2) and, thus, contradicts the usefulness of t.

Let us now assume, in contrast, that I is uncountable. Then we consider some well-ordering $\leq$ on I, choose the first uncountable ordinal ω_1 and consider some subfamily $\{E_\alpha\}_{\alpha<\omega_1}$ of $\mathbf{G}$. The above considerations allow us to assume that $E_\alpha \subsetneqq \overline{E}_\alpha$ for every (open) set $E_\alpha \in \mathcal{E}_\alpha$ and every ordinal number $\alpha < \omega_1$. Now we choose for every ordinal number $\alpha < \omega_1$ non-empty (open) sets $\overline{E}_{\alpha_1} \subsetneqq E_{\alpha_2} \subsetneqq \overline{E}_{\alpha_2} \subsetneqq E_{\alpha_3}$ and fixed points $x_\alpha \in \overline{E}_{\alpha_1}$. Lemma 3.6 allows us to consider for every ordinal number $\alpha < \omega_1$ some countable linear separable system $\mathcal{B}_\alpha$ on X such that $\overline{E}_{\alpha_1} \subsetneqq B_\alpha \subsetneqq \overline{B}_\alpha \subsetneqq E_{\alpha_2}$ for every (open) set $B_\alpha \in \mathcal{B}_\alpha$. For every ordinal number $\alpha < \omega_1$ we then set $O_\alpha := \cup\mathcal{B}_\alpha$. Let now $\{C_\alpha\}_{\alpha<\omega_1}$ be a family of closed subsets $C_\alpha \subset O_\alpha$. Condition (WS2) implies that $\bigcup_{\alpha<\omega_1} \overline{E}_{\alpha_2}$ is closed. Hence, it follows that also $\bigcup_{\alpha<\omega_1} C_\alpha$ is closed. Obviously, the same argument also implies that $\bigcup_{\alpha<\omega_1} \overline{O}_\alpha$ is closed. We abbreviate these observations by (*). With help of Theorem 3.7, we may conclude that for every ordinal number $\alpha < \omega_1$ there exists some continuous function $f_\alpha : X \longrightarrow [0,1]$ such that $f_\alpha(x_\alpha) = 0$ and $f_\alpha(X\backslash O_\alpha) = \{1\}$. For every ordinal number $\alpha < \omega_1$ there exists some order preserving function $g_\alpha : \{0\,, \dots,\, \alpha\} \longrightarrow [0,1]$ such that $g_\alpha(0) = 0$ and $g_\alpha(\alpha) = 1$. We, thus, may conclude that for every ordinal number $\alpha < \omega_1$ there exists some order-isomorphism $\phi_\alpha : [0,1] \longrightarrow [0,\alpha] \subset \mathbf{L}^*$, where $\mathbf{L}^*$ is the Long Line (see, for example [32]). ϕ_α is the canonical order-isomorphism that is induced by g_α^{-1}. Since the sets $O_\alpha(\alpha < \omega_1)$ are pairwise disjoint there exists for every point $y \in \bigcup_{\alpha<\omega_1} O_\alpha$ some uniquely determined ordinal number

$\alpha_y < \omega_1$ such that $y \in O_{\alpha_y}$. Hence, we may define a total preorder $\precsim$ on X by setting

$$\precsim := \{(y,z) \in X \times X \mid y \notin \cup_{\alpha<\omega_1} O_\alpha\} \cup$$
$$\left\{(y,z) \in X \times X \mid \phi_{\alpha_y}(1 - f_{\alpha_y}(y)) \leq \phi_{\alpha_z}(1 - f_{\alpha_z}(y))\right\}.$$

With help of observations (*) it follows that $\precsim$ is a continuous linear (total) preorder on X. In addition, the definition of $\precsim$ implies that $(X_{|\sim}, \precsim_{|\sim})$ is not a short chain. This contradiction finishes the proof. $\square$

Lemma 6.1 is based upon a generalization and completion of the central idea of the proof of the already quoted result of Estévez and Hervés on metrizable topologies. It implies that in case that t is a useful completely regular topology on X, it follows that t must satisfy some particular countable chain condition. Let, therefore, t said to satisfy the *countable chain condition for locally finite families of open sets (CLF)* if every locally finite family $\{O_i\}_{i\in I}$ of pairwise disjoint open subsets of X is countable. Then the following corollary holds.

Corollary 6.2. *Let t be a useful completely regular topology on a set X. Then t satisfies CLF.*

Proof. Let $\{O_i\}_{i\in I}$ be a locally finite family of pairwise disjoint open subsets of X. We may assume without loss of generality that $O_i \neq \varnothing$ for all $i \in I$. Hence, we may choose in every (open) set $O_i \in \{O_i\}_{i\in I}$ some fixed point x_i in order to consider some continuous function $h_i : X \longrightarrow [0,1]$ such that $h_i(x_i) = 0$ and $h_i(X \setminus O_i) = \{1\}$. For every $i \in I$ we finally set $\mathcal{E}_i := \left\{h_i^{-1}([0,q[)\right\}_{q\in[0,1]}$. Since $\{O_i\}_{i\in I}$ is a locally finite family of open sets it follows that $\mathbf{G} := \{\mathcal{E}_i\}_{i\in I}$ is a well-separated family of separable systems on X. Now the desired conclusion is implied by Lemma 6.1. $\square$

Before formulating and proving the main result of this section we still want to discuss some consequences of the above considerations. For every topological space (Y, t), we denote by $\sigma(Y, C(Y))$ the *weak* topology on Y that is induced by the family of all continuous real-valued functions on Y, i.e. $\sigma(Y, C(Y))$ is the coarsest topology on Y for which every continuous real-valued function on Y is continuous. It is well known that $\sigma(Y, C(Y))$ is completely regular or, equivalently, uniformizable (see, for instance [8]). The next lemma is at least implicitly well known. Its proof is based upon the fact that a continuous linear preorder $\precsim$ on a topological space (Y, t) is continuous if and only if for every pair of points $x \prec y \in Y$ there exists some continuous and increasing real-valued function $f_{x,y}$ on Y such that $f_{x,y}(x) < f_{x,y}(y)$ (cf. the corresponding remark on Theorem 3.8). The proof of the next lemma, therefore, may be omitted for the sake of brevity.

Lemma 6.3. *The following assertions are equivalent:*

(i) *t is useful,*
(ii) *$\sigma(Y, C(Y))$ is useful.*

The reader may recall that a pseudometric δ on a set X is a function $\delta : X \times X \longrightarrow [0, +\infty[$ that satisfies for all points $x, y, z \in X$ the following conditions.

(PM1): $\delta(x,x) = 0$,
(PM2): $\delta(x,y) = \delta(y,x)$,
(PM3): $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$.

The concept of a pseudometric on X, thus, differs from the concept of a metric on X only in condition (PM1). Indeed, a metric δ on X is a function $\delta : X \times X \longrightarrow [0, +\infty[$ that satisfies the conditions (PM2) and (PM3) and the stronger *definiteness condition*

(DM1): $\delta(x,y) = 0 \Longleftrightarrow x = y$.

In case that $t = t_\delta$ for some pseudometric δ on X, then the topology t is said to be *pseudometrizable.*

The following lemma is implicitly well-known (see [12] or [13]).

Lemma 6.4. *Let δ be some pseudometric on X and let the topology t_δ that is induced by δ be coarser than t. Then in order for t_δ to satisfy CLF it is necessary and sufficient that t_δ is second countable.*

Proof. Since the necessity part of the lemma is trivial it suffices to verify the suffiency part. Therefore, we assume that t_δ satisfies *CLF*. Let $t_{\delta_{|\sim}}$ be the quotient topology on $X_{|\sim}$ that is induced by the canonical equivalence relation $x \sim y \Longleftrightarrow \delta(x,y) = 0$. Then it is well known that $t_{\delta_{|\sim}}$ is induced by the metric $\overline{\delta}$ on $X_{|\sim}$ that is induced by δ and that t_δ is second countable if and only if $t_{\delta_{|\sim}}$ is second countable or, equivalently, $t_{\delta_{|\sim}}$ is separable. In addition, it is easily to be seen that t_δ satisfies *CLF* if and only if $t_{\delta_{|\sim}}$ satisfies *CLF*. Summarizing these considerations we may assume that $t_{\delta_{|\sim}}$ satisfies *CLF* and that it is sufficient to prove that $t_{\delta_{|\sim}}$ is separable. Let us assume, in contrast, that $t_{\delta_{|\sim}}$ is not separable. We shall show that in this case there exists some real $\epsilon > 0$ and some uncountable subset S of $X_{|\sim}$ such that $\overline{\delta}(x,y) \geq \epsilon$ for all points $x \neq y \in S$. Indeed, otherwise, for every natural number $n > 0$, every subset Z_n of $X_{|\sim}$ such that $\overline{\delta}(x,y) \geq \frac{1}{n}$ for all points $x \neq y \in Z_n$ is countable. The Lemma of Zorn allows us to choose, for every natural number $n > 0$, some maximal subset Y_n of $X_{|\sim}$ such that $\overline{\delta}(x,y) \geq \frac{1}{n}$ for all points $x \neq y \in Y_n$. Then $Y := \bigcup_{n \in \mathbb{N}\setminus\{0\}} Y_n$ is a countable subset of $X_{|\sim}$ such that $\overline{Y} = X_{|\sim}$, a contradiction. Thus, the existence of S follows. The inclusion $t_\delta \subset t$ implies that $t_{\delta_{|\sim}} \subset t_{|\sim}$. Hence, we may conclude that the family $\left\{\left\{y \in X_{|\sim} \,|\, \overline{\delta}(x,y) < \frac{\epsilon}{3}\right\}\right\}_{x \in S}$ is an uncountable family of pairwise disjoint non–empty open subsets of $X_{|\sim}$ that, obviously, is locally finite. This contradiction finishes the proof. □

Now we are ready to summarize our considerations for some interesting results.

Proposition 6.5. *Let $\sigma(X, C(X))$ be induced by some uniformity that has a countable base. Then the following assertions are equivalent:*

(i) *t is useful,*
(ii) *$\sigma(X, C(X))$ is second countable.*

Proof. (i) $\Longrightarrow$ (ii) Since $\sigma(X, C(X))$ is induced by some uniformity which has a countable base we may conclude that $\sigma(X, C(X))$ is induced by some pseudometric δ on X. With help of the inclusion $\sigma(X, C(X)) \subset t$ and Lemma 6.3 the desired conclusion now follows from Lemma 6.4.

(ii) $\Longrightarrow$ (i) Let $\precsim$ be a continuous linear preorder on X. Then $t^{\precsim} \subset \sigma(X, C(X))$ and assertion (i) follows from DT. □

Corollary 6.6. *Let $(G, \circ, t)$ be a first countable topological group. Then the following assertions are equivalent:*

(i) *t is useful,*
(ii) *t is second countable.*

Proof. Every topology t of a first countable topological group $(G, \circ, t)$ is induced by some uniformity which has a countable base (see, for example, [18]). □

Corollary 6.7. [13] *Let t be induced by some metric δ. Then the following assertions are equivalent:*

(i) *t is useful,*
(ii) *t is second countable.*

Remark 6.8. Corollary 6.7 has an important consequence. It implies, in particular, that for a metric space (X, δ) the assumptions of Debreu's Theorem are not only sufficient but also necessary for a continuous linear preorder $\precsim$ on (X, δ) to be representable by a continuous utility function. On the other hand, metric spaces (in particular Hilbert spaces or more generally Banach spaces) which are not second countable are meanwhile commonly encountered in economic theory (see [23] or our remark in the introduction). This is the case, for example, if the commodity space is $L^\infty(\mu)$, the space of μ-essentially bounded μ-measurable functions on a σ-finite measure space, which arises in the analysis of allocation of resources over time or states of nature ([2]), or $ca(K)$, the space of countably additive signed measures on a compact metric space which has been exploited for the analysis of commodity differentiation ([21] and [19]). Linear preorders defined on these spaces principally must satisfy more properties than just being continuous in order to have a continuous utility representation. Hence, the approaches of Shafer [31], Mas-Colell [21], Monteiro [28], Mehta and Monteiro [25] and others gain additional importance. The problem which arises is to look for useful natural additional conditions which a linear preorder on these spaces should satisfy and which also guarantee its representability by a continuous utility function. Such a useful condition could be *countably boundedness.* The reader may recall that a linear preorder $\precsim$ on X is *countably bounded* if there exists a countable subset Y of X such that for every point $x \in X$ there exist points $y, y' \in Y$ such that $y \precsim x \precsim y'$. For example, since every convex subset of the space $L^\infty(\mu)$ and $ca(K)$ respectively is path connected, it follows from Monteiro [28] that every continuous countably bounded linear preorder on a convex subset of $L^\infty(\mu)$ and $ca(K)$ respectively has a continuous utility representation. Another useful condition could be *convexity* (see [5, Theorem 3]).

In order to now prove the main result of this section let t be an arbitrary but fixed given topology on X. In case that t' is a topology on X that cannot be excluded to be different from t we denote for every subset A of X by $c'(A)$ the t'-closure of A. In case that we are sure that $t' = t$ the t-closure of A is abbreviated as usual by $\overline{A}$. Then a topology t' on X is said to be *well-compatible*, if for every t'-open subset O of X and every point $x \in O$ the t'-closure $c'(\{x\})$ of $\{x\}$ is contained in O. The reader may recall that t' is well-compatible, if and only if for each pair of points $x, y \in X$ the equivalence $c'(\{x\}) = c'(\{y\}) \Longleftrightarrow c'(\{x\}) \cap c'(\{y\}) \neq \varnothing$ holds. Now $\mathcal{L}_t$ is the family of all well-compatible topologies t' on X for which there exists some linear separable system $\mathcal{E} \in \mathbf{G}_{\mathcal{G}}$ (cf. section 5) such that $\mathcal{E} \subset t' \subset t_{\mathcal{E}}$ and $c'(E') \subset E$ for every pair of sets $E' \subset \overline{E'} \subset E \in \mathcal{E}$. $t_{\mathcal{E}}$ is the topology on X that is induced by $\mathcal{E}$ (cf. section 3). The reader may prove as an easy exercise that $c'(E') \subset E$ for every pair of sets $E' \subset \overline{E'} \subsetneqq E \in \mathcal{E}$, if and only if $(c'(E) \setminus E) \cap c'(E') = \varnothing$ for every pair of sets $E' \subset \overline{E'} \subsetneqq E \in \mathcal{E}$. In case that t is connected it follows that $\mathcal{L}_t$ is the set of all well-compatible topologies t' on X for which there exists some linear separable system $\mathcal{E}$ on X such that $\mathcal{E} \subset t' \subset t_{\mathcal{E}}$ and $c'(E') \subset E$ for every pair of sets $E' \subset \overline{E'} \subsetneqq E \in \mathcal{E}$. Let finally $\mathcal{E}$ be some arbitrarily chosen linear separable system on X. Then the reader may verify, in addition, that $t_{\mathcal{E}} \in \mathcal{L}_t$, if and only if for every pair of sets $E' \subset \overline{E'} \subset E \in \mathcal{E}$ the equation $\overline{E'} = \cap \left\{ E'' \in \mathcal{E} \mid \overline{E'} \subset E'' \subset E \right\}$ holds and, furthermore, a possible first element or a possible last element of the chain $(\mathcal{E}, \subset)$ is open and closed.

Now we are fully prepared for proving the main result of this section.

Proposition 6.9. *The following assertions are equivalent:*

(i) t *is useful,*

(ii) t *satisfies* OCCC, every well-separated family $\mathcal{G} := \{\mathcal{E}_i\}_{i \in I}$ of separable systems on X is countable and every topology $t' \in \mathcal{L}_t$ is pseudometrizable.

(iii) t satisfies OCCC, $\sigma(X, C(X))$ satisfies *CLF* and every topology $t' \in \mathcal{L}_t$ is pseudometrizable.

Proof. (i) $\Longrightarrow$ (ii) Because of Proposition 5.1 and Lemma 6.1 it is sufficient to prove that every topology $t' \in \mathcal{L}_t$ is pseudometrizable. Let, therefore, some topology $t' \in \mathcal{L}_t$ be arbitrarily chosen. Then there exists some linear separable system $\mathcal{E}$ on X, whose associated set $\mathcal{G}(\mathcal{E})$ is countable, such that $\mathcal{E} \subset t' \subset t_{\mathcal{E}}$ and $c'(E') \subset E$ for every pair of sets $E' \subset \overline{E'} \subset E \in \mathcal{E}$. Let now $t'_{|\sim}$ be the quotient topology that is induced by the equivalence relation "$x \sim y \Longleftrightarrow c'(\{x\}) = c'(\{y\})$". The well-compatibility of t' implies that the equivalence relation $\sim$ is *open*, i.e. the canonical projection $p : X \longrightarrow X_{|\sim}$ is open. This means, in particular, that t' is pseudometrizable, if and only if $t'_{|\sim}$ is metrizable. In order to verify that $t'_{|\sim}$ is metrizable we show at first that $t'_{|\sim}$ is second countable. Then we prove that $t'_{|\sim}$ is normal which finally allows us to conclude with help of the Alexandroff-Urysohn Metrization Theorem that $t'_{|\sim}$ is metrizable. One more application of the well compatibility of t' implies that $t'_{|\sim}$ is second countable, if and only if t' is second countable. Since $\mathcal{G}(\mathcal{E})$ is countable it follows with help of Proposition 5.1 (iii) that $t_{\mathcal{E}}$ is second countable. This means that there exists a countable set $\mathcal{C}(\mathcal{E})$ of pairs of sets $E' \subset \overline{E'} \subset E \in \mathcal{E} \cup \{\varnothing, X\}$ such that the family of corresponding open sets $E \cap X \backslash \overline{E'}$ is a base of $t_{\mathcal{E}}$. We want to show that $\mathcal{B} := \{E \cap X \backslash c'(E') \,|\, E' \subset \overline{E'} \subset E \in \mathcal{C}(\mathcal{E})\}$ is a base of t'. Then the second countability of t' follows. Let, therefore, O be some (non-empty) t'-open subset of X and let $x \in O$ be some arbitrary point. We must show that there exists some set $E^+ \cap X \backslash c'(E^{++}) \in \mathcal{B}$ such that $x \in E^+ \cap X \backslash c'(E^{++}) \subset O$. The inclusion $t' \subset t_{\mathcal{E}}$ implies that there exists some pair of sets $E' \subset \overline{E'} \subsetneqq E \in \mathcal{C}(\mathcal{E})$ such that $x \in E \cap X \backslash \overline{E'} \subset O$. We, thus, distinguish between the following two cases:

Case 1. $x \in E \backslash c'(E')$. In this case the inclusion $\overline{E'} \subset c'(E')$ implies that $x \in E \cap X \backslash c'(E') \subset E \cap X \backslash \overline{E'} \subset O$ and we are done.

Case 2. $x \notin E \backslash c'(E')$, i.e. $x \in c'(E') \backslash \overline{E'} \subset c'(E') \backslash E'$. In this situation we show at first that $\overline{E'} \backslash E' \cap O \neq \varnothing$. In order to verify this inequality the validity of the equation $c'(\overline{E'} \backslash E') = c'(E') \backslash E'$ is needed. Since $\overline{E'} \backslash E' \subset c'(E') \backslash E'$ and $c'(E') \backslash E'$ is t'-closed the inclusion $c'(\overline{E'} \backslash E') \subset c'(E') \backslash E'$ follows immediately. Hence, the desired equation will be proved if we are able to show that also the inclusion $c'(E') \backslash E' \subset c'(\overline{E'} \backslash E')$ holds. Let, therefore, some point $y \in c'(E') \backslash E'$ and some t'-neighborhood U of y be arbitrarily chosen. We must show that $U \cap \overline{E'} \backslash E' \neq \varnothing$. Then $y \in c'(\overline{E'} \backslash E')$ and the inclusion follows. The inclusion $t' \subset t_{\mathcal{E}}$ implies that there exist some pair of sets $E^{**} \subset \overline{E^{**}} \subsetneqq E^* \in \mathcal{C}(\mathcal{E})$ such that $y \in E^* \cap X \backslash \overline{E^{**}} \subset U$. Since $y \in E^* \backslash E'$ the linearity of $\mathcal{E}$ implies that $\overline{E'} \subset E^*$. On the other hand, it follows that $E^{**} \subset E'$. Indeed, otherwise we may conclude that $E' \subsetneqq E^{**}$, which means that $c'(E') \subset E^{**}$. Since $y \in c'(E')$ this inclusion contradicts the relation $y \in E \cap X \backslash \overline{E^{**}}$. A combination of these considerations implies that $\overline{E'} \backslash E' \subset E \cap X \backslash \overline{E^{**}}$. Hence, $U \cap \overline{E'} \backslash E' \neq \varnothing$, which finishes the proof of the desired equation. With help of the equation $c'(\overline{E'} \backslash E') = c'(E') \backslash E'$ the inequality $\overline{E'} \backslash E' \cap O \neq \varnothing$ easily can be verified. Indeed, otherwise we may conclude that $\overline{E'} \backslash E' \subset X \backslash O$. Since $X \backslash O$ is t'-closed it, thus, follows that $c'(E') \backslash E' = c'(\overline{E'} \backslash E') \subset X \backslash O$ which contradicts the relation $x \in c'(E') \backslash E'$ and $x \in O$. Now we proceed by choosing some arbitrary point $z \in \overline{E'} \backslash E' \cap O$ in order to then consider some pair of sets $E^{++} \subset \overline{E^{++}} \subsetneqq E^+ \in \mathcal{C}(\mathcal{E})$ such that $z \in E^+ \cap X \backslash \overline{E^{++}} \subset O$. Since $z \in \overline{E'} \backslash E'$ and, thus, $z \in E^+ \backslash E'$ the linearity of $\mathcal{E}$ implies that $E' \subset \overline{E'} \subset E^+$ which, in particular, means that $x \in c'(E') \subset E^+$. Because of the relations $z \notin \overline{E^{++}}$ and $z \in \overline{E'}$ it follows, on the other hand, that $E^{++} \subset \overline{E^{++}} \subsetneqq E'$. Hence, $c'(E^{++}) \subset E'$. We, thus, may summarize our considerations for the conclusions $x \in E^+ \cap X \backslash c'(E^{++}) \subset O$ and $E^+ \cap X \backslash c'(E^{++}) \in \mathcal{B}$ which completes the second case and, therefore, shows that t' is second countable. The particular construction of $\mathcal{B}$ finally allows us to apply the arguments of the proof of Proposition 1.4.2 in [17] in order to verify that t' is normal. Then the well-compatibility of t' implies that also $t'_{|\sim}$

is normal and the Alexandroff-Urysohn Metrization Theorem can be applied. This last conclusion finishes the proof of assertion (ii).

(ii) $\Longrightarrow$ (iii) This implication follows immediately with help of Corollary 6.2.

(iii) $\Longrightarrow$ (i) Let $\precsim$ be some arbitrary continuous linear (total) preorder on X. Then we consider the linear separable system $\mathcal{L} := \{L(X)\}_{x\in X} := \{\{y \in X \mid y \prec x\}\}_{x\in X}$ on X. Since t satisfies OCCC it follows that $\precsim$ only has countably many jumps which means that $\mathcal{G}(\mathcal{L})$ is countable (cf. the corresponding argument in the proof of the implication (ii) $\Longrightarrow$ (i) of Proposition 5.1. Clearly, $\precsim$ coincides with the linear preorder on X that is induced by $\mathcal{L}$ (cf. Theorem 3.8). Hence, we may apply Theorem 3.8 in order to conclude that the order topology $t^{\precsim}$ that is induced by $\precsim$ is coarser than $t_{\mathcal{L}}$. For every point $x \in X$ its $t^{\precsim}$-closure $c'(\{x\})$ coincides with the equivalence class $[x]$ that is defined by $\sim$. Hence, it follows that $t^{\precsim}$ is well–compatible. Since, in addition, $\mathcal{L} \subset t^{\precsim}$ and $d(x) = \{y \in X \mid y \precsim x\} \subset L(z) = \{u \in X \mid u \prec z\}$ for every pair of points $y \prec z \in X$ we may conclude that $t^{\precsim} \in \mathcal{L}_t$. Hence, $t^{\precsim}$ is pseudometrizable. The underlying argument which the proof of Lemma 6.3 is based upon implies that $t^{\precsim} \subset \sigma(X, C(X))$. This means, in particular, that $t^{\precsim}$ satisfies *CLF*. Therefore, it follows from Lemma 6.4 that $t^{\precsim}$ is second countable which implies that $\precsim$ has a continuous utility representation. This last conclusion settles the implication (iii) $\Longrightarrow$ (i) and nothing remains to be shown. $\square$

Corollary 6.10. *Let t be connected. Then the following assertions are equivalent:*

(i) *t is useful,*

(ii) *Every well-separated family $\mathcal{G} := \{\mathcal{E}_i\}_{i\in I}$ of separable systems on X is countable and every topology $t' \in \mathcal{L}_t$ is pseudometrizable,*

(iii) *$\sigma(X, C(X))$ satisfies CLF and every topology $t' \in \mathcal{L}_t$ is pseudometrizable.*

Remark 6.11. The condition that every topology $t' \in \mathcal{L}_t$ is pseudometrizable seems to be a bit artificial. On the other hand, Proposition 6.9 means that t is useful if and only if t satisfies OCCC, $\sigma(X, C(X))$ satisfies *CLF* and t allows the definition of enough (continuous) pseudometrics on X. Hence, Proposition 6.9 which, in particular, generalizes the nice result of Estévez and Hervés completes Proposition 5.1 and may at least serve as basis for finally obtaining still more satisfactory results.

7. Useful normal topologies

In the second section we already have announced some optimal result on the usefulness of normal topologies. In order to prove this result let t be a fixed given normal topology on X. For every subset A of X the interior of A is denoted by A°. Then we choose the family $\mathcal{O}$ of all sets $\mathbf{O}$ of open subsets O of X that are linearly ordered by set inclusion and satisfy condition (LO1) (cf. section 5).

Let some set $\mathbf{O} \in \mathcal{O}$ be arbitrarily chosen. Then the sets $\mathbf{Z}(\mathbf{O})$, $\mathbf{G}(\mathbf{O})$ and $\mathcal{O}_{\mathbf{G}}$ and the concepts of $\mathbf{O}$ to have a *countable refinement* or to be *second countable* are defined in the same way as the corresponding sets $\mathcal{Z}(O)$, $\mathcal{G}(O)$ and $\mathbf{G}_{\mathcal{G}}$, and the similar concepts in section 5. In addition, we consider the family $\mathcal{F}$ of all sets $\mathbf{O} \in \mathcal{O}$ which also satisfy the following condition which completes condition (LO2) in order to also include the case that t is not necessarily connected (cf. section 2).

$$\text{LO2}^+ : \quad \left| \left\{ O \in \mathbf{O} \,|\, \bigcup_{\mathbf{O} \ni O' \subsetneqq O} \overline{O'} \subsetneqq O \wedge \overline{\bigcup_{\mathbf{O} \ni O' \subsetneqq O} O'} \cap X \backslash O \neq \varnothing \right\} \right| +$$

$$+ \left| \left\{ O \in \mathbf{O} \,|\, \overline{\bigcup_{\mathbf{O} \ni O' \subsetneqq O} O'} \subset O \wedge \overline{\bigcup_{\mathbf{O} \ni O' \subsetneqq O} O'} = (\overline{\bigcup_{\mathbf{O} \ni O' \subsetneqq O} O'})^{\circ} \vee O = \overline{O} \right\} \right| \leq$$

$$\leq \omega \,.$$

The reader may verify that in case that t is connected the conditions (LO2) and (LO2$^+$) coincide.

Proposition 7.1. *Let t be a normal topology on X. Then the following assertions are equivalent:*

(i) *t is useful,*
(ii) *t satisfies* OCCC *and every set $\mathbf{O} \in \mathcal{O}$ has a countable refinement,*
(iii) *t satisfies* OCCC *and every set $\mathbf{O} \in \mathcal{O}$ is second countable,*
(iv) *t satisfies* OCCC *and every linearly ordered subtopology t^l of t that is induced by some set $\mathbf{O} \in \mathcal{O}_{\mathbf{G}}$ is second countable,*
(v) *t satisfies* OCCC *and every linearly ordered subtopology t^l of t that is induced by some set $\mathbf{O} \in \mathcal{F}$ is second countable.*

Proof. (i) $\Longrightarrow$ (ii) Let some set $\mathbf{O} \in \mathcal{O}$ be arbitrarily chosen. Then we consider the set $\mathbf{M}(\mathbf{O})$ of all sets $O \in \mathbf{O}$ for which there exists some maximal set $\mathbf{O} \ni O' \subsetneqq O$. Since t is a normal topology on X it follows from Lemma 3.4 and Lemma 3.6 with help of condition (LO1) that for every pair of sets $\mathbf{O} \ni O' \subsetneqq O \in \mathbf{M}(\mathbf{O})$ there exists a linear separable system $\mathcal{E}(O)$ on X such that $O' \subset E \subset O$ for every set $E \in \mathcal{E}(O)$. We, thus, set $\mathcal{E}(\mathbf{O}) := \mathbf{O} \cup (\bigcup_{O \in \mathbf{M}(\mathbf{O})} \mathcal{E}(O))$, and show that $\mathcal{E}(\mathbf{O})$ is a linear separable system on X. Let, therefore, $\mathcal{E}'(\mathbf{O}) \subset \mathcal{E}(\mathbf{O})$ be the subset of all sets $E \in \mathcal{E}(\mathbf{O})$ such that $E \subsetneqq \overline{E}$ and for which $\mathbf{U}(E) := \{E' \in \mathcal{E}(\mathbf{O}) \,|\, E \subsetneqq E'\} \neq \varnothing$. Then we choose some arbitrary set $E \in \mathcal{E}'(\mathbf{O})$ and distinguish between the following two cases:

Case 1: $(\mathbf{U}(E), \subset)$ does not contain a minimal element. In this case the construction of $\mathcal{E}(\mathbf{O})$ allows us to conclude with help of condition (LO1) that $\bigcap_{E' \in \mathbf{U}(E)} E' = \bigcap_{E' \in \mathbf{U}(E)} \overline{E'}$.

Case 2: $(\mathbf{U}(E), \subset)$ contains a minimal element. Let E' be this minimal element of $(\mathbf{U}(E), \subset)$. Then the definition of $\mathbf{M}(\mathbf{O})$ implies with help of the construction of $\mathcal{E}(\mathbf{O})$ that E' is closed.

Summarizing both cases it follows with help of Proposition 3.5 (ii) that $\mathcal{E}(\mathbf{O})$, actually, is a linear separable system on X. Assertion (ii) now is an immediate consequence of the corresponding assertion of Proposition 5.1.

(ii) $\Longrightarrow$ (i) Since every linear separable system $\mathcal{E}$ on X satisfies condition (LO1) the desired implication follows with help of the implication (ii) $\Longrightarrow$ (i) in the proof of Proposition 5.1.

(i) $\Longrightarrow$ (iii) Let $\mathbf{O} \in \mathcal{O}$ be some arbitrarily chosen set. As in the proof of the implication (i) $\Longrightarrow$ (ii) we consider the set $\mathbf{M(O)}$ and construct the linear separable system $\mathcal{E}(\mathbf{O})$ on X. Of course, we may assume without loss of generality that for every pair of sets $\mathbf{O} \ni O' \subsetneqq O \in \mathbf{M(O)}$ such that O' or O is closed the corresponding linear separable system $\mathcal{E}(O)$ on X consists of O' and O. Let us abbreviate this assumption by (*). Because of Proposition 5.1 (iii) there exists some countable subset $\mathbf{E}'$ of $\mathcal{E}'(\mathbf{O})$ such that for every pair of sets $(E', E) \in \mathbf{Z}(\mathbf{E}(\mathbf{O}))$ there exists some set $E^+ \in \mathbf{E}'$ such that $\overline{E'} \subset E^+ \subset \overline{E^+} \subset E$. Now we consider the situation $\overline{O'} \subsetneqq O'' \subset \overline{O''} \subsetneqq O$ for some pair of sets $(O', O) \in \mathbf{Z(O)}$ and some set $O'' \in \mathbf{O}$. There exists some set $E \in \mathbf{E}'$ such that $\overline{O'} \subset E \subset \overline{E} \subset O$. If $E \notin \mathbf{O}$, then there exists because of (*) and the construction of $\mathcal{E}(\mathbf{O})$ some pair of sets $\mathbf{O} \ni O^+ \subsetneqq O^{++} \in \mathbf{M(O)}$ such that $\overline{O^+} \subsetneqq E \subset \overline{E} \subsetneqq O^{++}$. Since $\mathbf{O}$ is linearly ordered by set inclusion it follows with help of condition (LO1) and the chain $\overline{O'} \subsetneqq O'' \subset \overline{O''} \subsetneqq O$ that $\overline{O'} \subsetneqq O^+ \subset \overline{O^+} \subset O^{++} \subsetneqq O$. Hence, one immediately verifies that assertion (iii) will follow with help of assertion (iii) of Proposition 5.1, if we are able to show that the set of all pairs $\mathbf{O} \ni O^+ \subsetneqq O^{++} \in \mathbf{M(O)}$ for which there exists some set $E \in \mathbf{E}' \setminus \mathbf{O}$ such that $\overline{O^+} \subset E \subset \overline{E} \subset O^{++} \in \mathbf{M(O)}$ is countable. But this is easily seen since the corresponding sets $O^{++} \cap X \setminus \overline{O^+}$ are pairwise disjoint and $\mathbf{E}'$ is countable.

(iii) $\Longrightarrow$ (iv) Trivial.

(iv) $\Longrightarrow$ (i) In the same way as the implication (ii) $\Longrightarrow$ (i) also this implication follows with help of the proof of the corresponding implication (iv) $\Longrightarrow$ (i) of Proposition 5.1.

(i) $\wedge$ (iv) $\Longrightarrow$ (v) Let some set $\mathbf{O} \in \mathcal{F}$ be arbitrarily chosen. Then we consider the linear separable system $\mathcal{E}(\mathbf{O})$ on X that already has been constructed in the proof of the implication (i) $\Longrightarrow$ (ii). Since the linearly ordered subtopology t^l of t that is induced by $\mathbf{O}$ is coarser than the linearly ordered subtopology t'^l of t that is induced by $\mathcal{E}(\mathbf{O})$ it suffices to verify that t'^l is second countable (cf. the argument of the generalization of DT in the second section). In order to show that t'^l is second countable it is because of assertion (iv) and condition (LO2$^+$) sufficient to prove that the set $\mathbf{K(O)}$ of all (open) sets $O \in \mathbf{O}$ such that $(\overline{\bigcup_{\mathbf{O} \ni O' \subsetneqq O} O'})^\circ \subsetneqq \overline{\bigcup_{\mathbf{O} \ni O' \subsetneqq O} O'} \subsetneqq O \subsetneqq \overline{O}$ is countable. In order to show the countability of $\mathbf{K(O)}$ we apply the normality of t in order to construct for every (open) set $O \in \mathbf{K(O)}$ some linear separable system $\mathcal{E}'(O)$ on X such that $\overline{\bigcup_{\mathbf{O} \ni O' \subsetneqq O} O'} \subset E' \subset \overline{E'} \subset O$ for every set $E' \in \mathcal{E}'(\mathbf{O})$ (cf. corresponding argument in the proof of the implication (i) $\Longrightarrow$ (ii)). Since $(\overline{\bigcup_{\mathbf{O} \ni O' \subsetneqq O} O'})^\circ \subsetneqq \overline{\bigcup_{\mathbf{O} \ni O' \subsetneqq O} O'} \subsetneqq O \subsetneqq \overline{O}$ we may conclude that $\mathcal{E}'(\mathbf{O}) \neq \varnothing$. In the same way as in the corresponding part of the proof of the implication (i) $\Longrightarrow$ (ii) it follows that $\mathcal{E}'(\mathbf{O}) := \mathcal{E}(\mathbf{O}) \cup (\bigcup_{O \in \mathbf{K(O)}} \mathcal{E}'(O))$ also is a linear separable system on X. Let us now assume, in contrast, that $\mathbf{K(O)}$ is not countable. Then, since $\mathcal{E}'(O) \neq \varnothing$ for every (open) set $O \in \mathbf{K}(\mathbf{O})$ we may conclude that the continuous linear (total) preorder $\precsim$ on X that is induced by $\mathcal{E}'(\mathbf{O})$ has uncountably many jumps or contains an uncountable family of pairwise disjoint open (non-degenerate) intervals. This means, in particular, that $\precsim$ has no continuous utility representation, which contradicts assertion (i). Hence, assertion (v) follows.

(v) $\Longrightarrow$ (iv) Since $\mathcal{O}_{\mathbf{G}} \subset \mathcal{F}$ assertion (iv) is an immediate consequence of assertion (v). □

In case that t is connected, Proposition 7.1 is the generalization of *EDT* to normal topologies (cf. the corresponding remark in section 2).

Corollary 7.2. *Let t be a normal and connected topology on X. Then the following assertions are equivalent:*

(i) *t is useful,*
(ii) *every set $\mathbf{O} \in \mathcal{O}$ has a countable refinement,*
(iii) *every set $\mathbf{O} \in \mathcal{O}$ is second countable,*
(iv) *every linearly ordered subtopology t^l of t that is induced by some set $\mathbf{O} \in \mathcal{O}_{\mathbf{G}}$ is second countable,*
(v) *every linearly ordered subtopology t^l of t that is induced by some set $\mathbf{O} \in \mathcal{F}$ is second countable.*

Let, for the moment, a normal topology t on X said to be *short*, if every set $\mathbf{O} \in \mathcal{O}$ that is well-ordered by set inclusion is countable. Then the following interesting proposition holds which, in particular, shows that condition (LO1) is a generalization of *CLF*. This means that Proposition 7.1 and Corollary 7.2 are generalizations of the theorem of Estévez and Hervés.

Proposition 7.3. *In order for a normal topology t on X to be short it is necessary that t satisfies CLF.*

Proof. Let t be short. We assume, in contrast, that t does not satisfy *CLF*. Then there exists an uncountable locally finite family $\mathbf{O} := \{O_i\}_{i\in I}$ of pairwise disjoint (non-empty) open subsets of X. In analogy to the proof of Lemma 6.1 we may assume that none of the sets O_i $(i \in I)$ contains some non-empty open and closed subset. Let us abbreviate this assumption by (*). In addition, the proof of Lemma 6.1 allows us to assume that I coincides with the first uncountable ordinal ω_1, i.e., $\mathbf{O} = \{O_i\}_{i\in I} = \{O_\alpha\}_{\alpha<\omega_1}$. Now we proceed by choosing in every set $O_\alpha(\alpha < \omega_1)$ some fixed point x_α. The normality of t implies with help of the Long-Line-argument in the proof of Lemma 6.1 that for every ordinal number $\alpha < \omega_1$ there exists some set $\mathbf{U}_\alpha := \{U_\beta\}_{\beta\leq\alpha} \in \mathcal{O}$ such that $\overline{U}_\tau \subset U_\beta$, if $\tau < \beta \leq \alpha$, $\overline{U}_\alpha \subset O_\alpha$ and $x_\alpha \in U_\alpha \setminus \overline{\bigcup_{\beta<\alpha} U_\beta}$. Hence, we may construct by transfinite induction on all countable ordinals, i.e. on all ordinals $\gamma < \omega_1$ some uncountable set $\mathbf{O}' \in \mathcal{O}$ that is well-ordered by set inclusion. If $\gamma = 0$ we choose in every set $\mathbf{U}_\alpha$ $(\alpha < \omega_1)$ the set U_0. Then O'_α, is the union of these sets U_0 with $X \setminus \bigcup_{\alpha<\omega_1} U_\alpha$. On the other hand, if $0 < \gamma < \omega_1$, we choose in every set $\mathbf{U}_\alpha$ $(\alpha < \omega_1)$ all sets U_β for which $\beta \leq \gamma$. Then O'_γ is the union of these sets U_β with $X \setminus \bigcup_{\alpha<\omega_1} U_\alpha$.

Since the family $\mathbf{O}$ has been assumed to be locally finite it follows with help of assumption (*) that $\mathbf{O}' := \{O'_\gamma\}_{\gamma<\omega_1} \in \mathcal{O}$ which means that t cannot be short. This contradiction proves the proposition. $\square$

The reader may notice that the proof of Proposition 7.3 also clarifies the topological structure of the central idea of the proof of Estévez and Hervés on the usefulness of metrizable topologies.

References

[1] A.F. Beardon, J.C. Candeal, G. Herden, E. Indurain and G.B. Mehta, *The non-existence of a utility function and the structure of non-representable chains*, Preprint, Universidad de Zaragoza (J.C. Candeal), 1999.

[2] T. Bewley, *Existence of equilibria in economies with infinitely many commodities*, Journal of Economic Theory **4** (1972), 514-540.

[3] D. Bridges and G.B. Mehta, Representation of preference orderings, Springer, New York, 1995

[4] D.C.J. Burgess and M. Fitzpatrick, *On separation axioms for certain types of topological spaces*, Mathematical Proceedings of the Cambridge Philosophical Society **82** (1977), 59-65.

[5] J.C. Candeal and E. Induráin and G.B. Mehta, *Some utility theorems on inductive limits of preordered topological spaces*, Bulletin of the Australian Mathematical Society **52** (1995), 235-246.

[6] J.C. Candeal, C. Hervés and E. Induráin, *Useful topologies on preordered metric spaces*, Preprint, Universidad de Zaragoza, 1995.

[7] J.C. Candeal, C. Hervés and E. Induráin, *Some results on representation and extension of preferences*, Journal of Mathematical Economics **29** (1998), 75-81.

[8] J. Cigler and H.C. Reichel, Topologie, BI Hochschultaschenbücher (BI, Mannheim, 1986, second edition).

[9] G. Debreu, *Representation of a preference ordering by a numerical function*, in: R. Thrall, C. Coombs and R. Davies (eds.), Decision Processes. New York: John Wiley, 1954.

[10] G. Debreu, *Continuity properties of Paretian utility*, International Economic Review **5** (1964) 285-293.

[11] S. Eilenberg, *Ordered topological spaces*, American Journal of Mathematics 63, 39-45 (1941).

[12] R. Engelking, General Topology, (Warzawa, 1977, second edition).

[13] M. Estévez and C. Hervés, *On the existence of continuous preference orderings without utility representations*, Journal of Mathematical Economics **24** (1995), 305-309.

[14] G. Herden, *On the existence of utility functions II*, Mathematical Social Sciences **18** (1989), 107-117.

[15] G. Herden, *Topological spaces for which every continuous total preorder can be represented by a continuous utility function*, Mathematical Social Sciences **22** (1991), 123-136.

[16] G. Herden, *On some equivalent approaches to mathematical utility theory*, Mathematical Social Sciences **29** (1995), 19-31.

[17] G. Herden and G.B. Mehta, *Order, Topology and Utility*, Preprint, Universität–GH Essen, 2000.

[18] T. Husain, Introduction to Topological Groups, W.B. Saunders Company (Saunders, Philadephia, 1966.

[19] L. Jones, *A competitive Model of product differentiation*, Econometrica **52** (1984), 507-530.

[20] A. Mas-Colell, *A model of equilibrium with differentiation*, Journal of Mathematical Economics **2** (1975), 263-295.

[21] A. Mas-Colell, *The price equilibrium existence problem in topological vector lattices*, Econometrica **54** (1986), 1039-2053.

[22] G.B. Mehta, *Existence of an order-preserving function on a normally preordered space*, Bulletin of the Australian Mathemataical Society **34** (1986), 141-147.

[23] G.B. Mehta, *Infinite dimensional Arrow-Hahn theorem*, Preprint, University of Brisbane, 1989.

[24] G.B. Mehta, *Birkhoff- and Debreu separability*, Oral communication, Essen-Brisbane (November 1999).

[25] G.B. Mehta and P.K. Monteiro, *Infinite dimensional utility representation theorems*, Economic Theory **53** (1996), 169-173.

[26] A. Milgram, *Partially ordered sets, separating systems and inductiveness*, in: Reports of a mathematical colloquium (second series, No. 1),K. Menger ed. (University of Notre Dame, 1939), 18-30.

[27] A. Milgram, *Partially ordered sets and topology*, Reports of a Mathematical Colloquium Second Series (University of Notre Dame, 1940), 3-9.

[28] P.K. Monteiro, *Some results on the existence of utility functions on path connected spaces*, Journal of Mathematical Economics **16** (1987), 147-156.

[29] L. Nachbin, Topology and Order, Van Nostrand Reinhold (Van Nostrand, New York, 1965).

[30] B. Peleg, *Utility functions for partially ordered topological spaces*, Econometrica **38** (1970), 93-96.

[31] W. Shafer, *Representations of preorders on normed spaces*, Preprint, University of Southern California, 1984.

[32] L. Steen and J. Seebach, Counterexamples in Topology, (Rinehart and Winston, New York, 1978, second edition).

[33] E. Szpilrajn, *Sur l'extension de l'ordre partial*, Fundamenta Mathematica **16** (1930), 386-389.

[34] P. Urysohn, *Über die Mächtigkeit der zusammenhängenden Mengen*, Mathem. Annalen **94** (1925), 262-295.

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