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Extension properties and the Niemytzki plane

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Abstract. The first part of the paper is a brief survey on recent topics concerning the relationship between C^*-embedding and C-embedding for closed subsets. The second part studies extension properties of the Niemytzki plane NP. A zero-set, z-, C^*-, C-, and P-embedded subsets of NP are determined. Finally, we prove that every C^*-embedded subset of NP is a P-embedded zero-set, which answers a problem raised in the first part.

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1. Introduction

All spaces are assumed to be completely regular T_1-spaces. A subset Y of a space X is said to be *C-embedded* in X if every real-valued continuous function on Y can be continuously extended over X, and Y is said to be *C^*-embedded* in X if every bounded real-valued continuous function on Y can be continuously extended over X. Obviously, every C-embedded subset is C^*-embedded, but the converse is not true, in general. In the first part of the paper, formed by Sections 2, 3 and 4, we discuss several problems concerning the relationship between C^*-embedding and C-embedding for closed subsets. For example, the following problem is still open as far as the author knows:

Problem 1.1. *Does there exist a first countable space having a closed C^*-embedded subset which is not C-embedded?*

Since a space which answers the above problem positively cannot be normal, the following problem naturally arises:

Problem 1.2. *Let X be one of the following spaces: The Niemytzki plane (i.e., the space NP defined in Section 4 below); the Sorgenfrey plane (* [3, Example 2.3.12]*); Michael's product space (*[3, Example 5.1.32]*). Then, does the space X have a closed C^*-embedded subset which is not C-embedded?*

In the second part, formed by Sections 5, 6 and 7, we answer Problem 1.2 for the Niemytzki plane NP negatively by determining a zero-set, z-, C^*-, C- and P-embedded subsets of NP. The problem, however, remains open for the Sorgenfrey plane and Michael's product space.

Throughout the paper, let $\mathbb{R}$ denote the real line with the Euclidean topology, $\mathbb{Q}$ the subspace of rational numbers and $\mathbb{N}$ the subspace of positive integers. The cardinality of a set A is denoted by $|A|$. As usual, a cardinal is the initial ordinal and an ordinal is identified with the space of all smaller ordinals with the order topology. Let ω denote the first infinite ordinal and ω_1 the first uncountable ordinal. All undefined terms will be found in [3].

2. C^*-embedding versus C-embedding

It is an interesting problem to find a closed C^*-embedded subset which is not C-embedded. We begin by showing typical examples of such subsets. First, let us consider the subspace $\Lambda = \beta\mathbb{R} \setminus (\beta\mathbb{N} \setminus \mathbb{N})$ of $\beta\mathbb{N}$. The subset $\mathbb{N}$ is closed C^*-embedded but not C-embedded in Λ, because Λ is pseudocompact (cf. [4, 6P, p.97]). More generally, Noble proved in [16] that every space Y can be embedded in a pseudocompact space pY as a closed C^*-embedded subspace. Thus, every non-pseudocompact space Y embeds in pY as a closed C^*-embedded subset which is not C-embedded. Shakhmatov [20] constructed a pseudocompact space X with a much stronger property that every countable subset of X is closed and C^*-embedded.

Now, we give another examples which does not rely on pseudocompactness. For every space X there exist an extremally disconnected space $E(X)$, called the *absolute* of X, and a perfect onto map $e_X : E(X) \to X$ (cf. [3, 6.3.20 (b)]). We now call a space X *weakly normal* if every two disjoint closed sets in X, one of which is countable discrete, have disjoint neighborhoods.

Lemma 2.1. *Let X be a space which is not weakly normal. Then $E(X)$ contains a closed C^*-embedded subset which is not C-embedded.*

Proof. By the assumption, X has a closed set A and a countable discrete closed set $B = \{p_n : n \in \mathbb{N}\}$ such that $A \cap B = \varnothing$ but they have no disjoint neighborhoods. We show that the closed set $F = e_X^{-1}[B]$ in $E(X)$ is C^*-embedded but not C-embedded. Since B is countable discrete closed in X, we can find a disjoint family $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of open-closed sets in $E(X)$ such that $e_X^{-1}(p_n) \subseteq U_n \subseteq E(X) \setminus e_X^{-1}[A]$ for each $n \in \mathbb{N}$. Let $U = \bigcup\{U_n : n \in \mathbb{N}\}$; then U is a cozero-set in $E(X)$. Since F and $E(X) \setminus U$ cannot be separated by disjoint open sets, it follows from Theorem 3.1 below that F is not C-embedded in $E(X)$. On the other hand, F is C-embedded in U, because each $e_X^{-1}(p_n)$ is compact and $\mathcal{U}$ is disjoint, and further, U is C^*-embedded in $E(X)$ by [4, 1H6, p.23]. Consequently, A is C^*-embedded in $E(X)$. □

Corollary 2.2. *Let X be one of the following spaces: The Niemytzki plane NP; the Sorgenfrey plane S^2; Michael's product space $\mathbb{R}_{\mathbb{Q}} \times P$; the Tychonoff plank T (see Example 3.3 below). Then $E(X)$ contains a closed C^*-embedded subset which is not C-embedded.*

Proof. It is well known (and easily shown) that the spaces NP, S^2 and T are not weakly normal. Now, we show that Michael's product space $\mathbb{R}_{\mathbb{Q}} \times P$ is not weakly normal. The space $\mathbb{R}_{\mathbb{Q}}$ is obtained from $\mathbb{R}$ by making each point of $P = \mathbb{R} \setminus \mathbb{Q}$ isolated. Enumerate $\mathbb{Q}$ as $\{x_n : n \in \mathbb{N}\}$ and choose $y_n \in P$ with $|x_n - y_n| < 1/n$ for each $n \in \mathbb{N}$. Let $A = \{\langle x_n, y_n\rangle : n \in \mathbb{N}\}$ and $B = \{\langle x, x\rangle : x \in P\}$. Then A and B have no disjoint neighborhoods in $\mathbb{R}_{\mathbb{Q}} \times P$. Since A is discrete closed in $\mathbb{R}_{\mathbb{Q}} \times P$, $\mathbb{R}_{\mathbb{Q}} \times P$ is not weakly normal. Hence, the corollary follows from Lemma 2.1. □

As another application, we have the following example concerning Problem 1.1.

Example 2.3. There exists a space X in which every point is a G_δ and there exists a closed C^*-embedded subset which is not C-embedded. In fact, let $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of $\mathbb{N}$. Pick a point $p_A \in \mathrm{cl}_{\beta\mathbb{N}} A \setminus A$ for each $A \in \mathcal{R}$ and let $R = \{p_A : A \in \mathcal{R}\}$. Then the subspace $X = \mathbb{N} \cup R$ of $\beta\mathbb{N}$ is extremally disconnected (i.e., $E(X) = X$) and R is discrete closed in X. Let E be a countable infinite subset of R. Then E and $R \setminus E$ have no disjoint neighborhoods in X by the maximality of $\mathcal{R}$. Hence, by the proof of Lemma 2.1, E is closed C^*-embedded in X but not C-embedded. □

We change the topology of the space $X = \mathbb{N} \cup R$ in Example 2.3 by declaring the sets $\{p_A\} \cup (A \setminus \{1, 2, \cdots, n\})$, $n \in \mathbb{N}$, to be basic neighborhoods of p_A for each $A \in \mathcal{R}$. The resulting space is first countable and is usually called a Ψ-*space* (see [4, 5I, p.79]). A positive answer to the following problem answers Problem 1.1 positively.

Problem 2.4. *Does there exist a Ψ-space having a closed C^*-embedded subset which is not C-embedded?*

For an infinite cardinal γ, a subset Y of a space X is said to be P^γ-*embedded* in X if for every Banach space B with the weight $w(Y) \leq \lambda$, every continuous map $f : Y \to B$ can be continuously extended over X. A subset Y of X is said to be P-*embedded* in X if Y is P^γ-embedded in X for every γ. It is known that Y is P^γ-embedded in X if and only if for every locally finite cozero-set cover $\mathcal{U}$ of Y with $|\mathcal{U}| \leq \gamma$, there exists a locally finite cozero-set cover $\mathcal{V}$ of X such that $\{V \cap Y : V \in \mathcal{V}\}$ refines $\mathcal{U}$. In particular, Y is C-embedded in X if and only if Y is P^ω-embedded in X. For further information about P^γ-embedding, the reader is referred to [1]. The following problem concerning the relationship between C-embedding and P-embedding is also open:

Problem 2.5. *Does there exist an example in ZFC of a space X, with $|X| = \omega_1$, having a closed C-embedded subset which is not P-embedded?*

Problem 2.6. *Does there exist an example in ZFC of a first countable space having a closed C-embedded subset which is not P-embedded?*

It is known that under certain set-theoretic assumption such as $MA + \neg CH$, there exists a first countable, normal space X which is not collectionwise normal (see [21]). Since a space is collectionwise normal if and only if every closed subset is P-embedded, such a space X has a closed C-embedded subset which is not P-embedded (cf. Remark 6.4 in Section 6 below).

3. Spaces in which every closed C^*-embedded set is C-embedded

We say that a space X has the *property* $(C^* = Q)$ if every closed C^*-embedded subset of X is Q-embedded in X, where $Q \in \{C, P^\gamma, P\}$. A subset Y of a space X is said to be *z-embedded* in X if every zero-set in Y is the restriction of a zero-set in X to Y (cf. [2]). Every C^*-embedded subset is z-embedded. Two subsets A and B are said to be *completely separated* in X if there exists a real-valued continuous function f on X such that $f[A] = \{0\}$ and $f[B] = \{1\}$. The following theorem was proved by Blair and Hager in [2, Corollary 3.6.B].

Theorem 3.1. [Blair-Hager] *A subset Y of a space X is C-embedded in X if and only if Y is z-embedded in X and Y is completely separated from every zero-set in X disjoint from Y.*

Recall from [11] that a space X is *δ-normally separated* if every two disjoint closed sets, one of which is a zero-set, are completely separated in X. All normal spaces and all countably compact spaces are δ-normally separated. By Theorem 3.1, we have the following corollary:

Corollary 3.2. *Every δ-normally separated space has the property $(C^* = C)$.*

The converse of Corollary 3.2 does not hold as the next example shows:

Example 3.3. The Tychonoff plank $T = ((\omega_1+1)\times(\omega+1))\setminus\{\langle\omega_1,\omega\rangle\}$ is not δ-normally separated but every closed C^*-embedded subset of T is P-embedded, i.e., T has the property $(C^* = P)$. To prove these facts, let $A = \{\omega_1\} \times \omega$ and $B = \omega_1 \times \{\omega\}$; then A is closed in T and B is a zero-set in T. Since A and B cannot be completely separated in T, T is not δ-normally separated. Next, let F be a closed C^*-embedded subset of T. We have to show that F is P-embedded in T. Since there is no uncountable discrete closed set in T, every locally finite cozero-set cover of F is countable. Hence, it suffices to show that F is C-embedded in T. Since F is closed in T, either F includes a closed unbounded subset of B or $F \cap \{\langle\beta, m\rangle : \alpha < \beta < \omega_1, n < m \le \omega\} = \varnothing$ for some $\alpha < \omega_1$ and some $n < \omega$. In the former case, every zero-set in T disjoint from F must be compact. In the latter case, $A \cap F$ is finite since F is C-embedded, which implies that F is compact. In both cases, F is completely separated form a zero-set disjoint from it. Hence, it follows from Theorem 2.1 that F is C-embedded.

The following example shows that the product of a space with the property $(C^* = P)$ and a compact space need not have the property $(C^* = C)$.

Example 3.4. Let T be the Tychonoff plank. As we showed in Example 3.3, T has the property $(C^* = P)$. We show that $T \times \beta E(T)$ fails to have the property $(C^* = C)$, where $E(T)$ is the absolute of T. Let $e_T : E(T) \to T$ be the perfect onto map. Then the subspace $G = \{\langle e_T(x), x\rangle : x \in E(T)\}$ is closed in $T \times \beta E(T)$, because e_T is perfect. Since T is not weakly normal, it follows from Lemma 2.1 that $E(T)$ does not have the property $(C^* = C)$, and hence, G also fails to have the property $(C^* = C)$, because G is homeomorphic to $E(T)$. Hence, if we prove that G is C^*-embedded in $T \times \beta E(T)$, then it would follow that $T \times \beta E(T)$ does not have the property $(C^* = C)$. For this end, let f be a bounded real-valued continuous function on G and define $g : E(T) \to \mathbb{R}$ by $g(x) = f(\langle e_T(x), x\rangle)$ for $x \in E(T)$. Since g is bounded continuous, g extends to a continuous function h on $\beta E(T)$. Then $h \circ \pi$ is a continuous extension of f over $T \times \beta E(T)$, where $\pi : T \times \beta E(T) \to \beta E(T)$ is the projection. Hence, G is C^*-embedded in $T \times \beta E(T)$. □

Problem 3.5. *Does there exist a space X with the property $(C^* = C)$ and a metric space M such that $X \times M$ fails to have the property $(C^* = C)$?*

The positive answer to Problem 1.2 for Michael's product space answers Problem 3.5 positively. We conclude this section by giving a class of spaces having the property $(C^* = P^\gamma)$. Recall from [10, 14] that a family $\mathcal{F}$ of subsets of a space X is *uniformly locally finite* in X if there exists a locally finite cozero-set cover $\mathcal{U}$ of X such that every $U \in \mathcal{U}$ intersects only finitely many members of $\mathcal{F}$. Let γ be an infinite cardinal. A subset Y of a space X is said to be *U^γ-embedded* in X if every uniformly locally finite family $\mathcal{F}$ of subsets in Y with $|\mathcal{F}| \leq \gamma$ is uniformly locally finite in X (cf. [7]). The following theorem was proved in [15] (see also [7, Proposition 1.6]).

Theorem 3.6. [Morita-Hoshina] *For every infinite cardinal γ, a subset Y of a space X is P^γ-embedded in X if and only if Y is both z-embedded and U^γ-embedded in X.*

Recall from [7] that a space X has the *property* (U^γ) (resp. *property* $(U^\gamma)^*$) if every locally finite (resp. discrete) family $\mathcal{F}$ of subsets of X with $|\mathcal{F}| \leq \gamma$ is uniformly locally finite in X. All γ-collectionwise normal and countably paracompact spaces have the property (U^γ), and all γ-collectionwise normal spaces have the property $(U^\gamma)^*$. Hoshina [7] proved that a space X has the property $(U^\gamma)^*$ if and only if every closed subset of X is U^γ-embedded. Combining this with Theorem 3.6, we have the following corollary:

Corollary 3.7. *For every infinite cardinal γ, every space having the property $(U^\gamma)^*$ has the property $(C^* = P^\gamma)$.*

It will be worth noting that every γ-collectionwise normal Dowker space (see [17]) has the property $(U^\gamma)^*$ for every γ but does not have the property (U^ω).

4. Products

It is quite interesting to consider the relationship between C^*- and C-embeddings in the realm of product spaces. In spite of extensive studies, the following problem is still unanswered.

Problem 4.1. *Let A be a closed C-embedded subset of a space X, Y a space, and assume that $A \times Y$ is C^*-embedded in $X \times Y$. Then, is $A \times Y$ C-embedded in $X \times Y$?*

In this section, we summarize partial answers to Problem 4.1 and also discuss the following problem:

Problem 4.2. *Let X and Y be spaces with the property $(C^* = C)$. Under what conditions on X and Y does $X \times Y$ have the property $(C^* = C)$?*

First, we consider product spaces with a compact factor. Morita-Hoshina [15] proved the following theorem which answers Problem 4.1 positively when Y is a compact space.

Theorem 4.3. [Morita-Hoshina] *Let A be a subset of a space X, Y an infinite compact space, and assume that $A \times Y$ is C^*-embedded in $X \times Y$. Then $A \times Y$ is $P^{w(Y)}$-embedded in $X \times Y$, where $w(Y)$ is the weight of Y.*

From now on, let γ denote an infinite cardinal. The next theorem is an answer to Problem 4.2.

Theorem 4.4. *If a space X has the property (U^γ), then $X \times Y$ has the property $(C^* = P^\gamma)$ for every compact space Y.*

Proof. If X has the property (U^γ) and Y is a compact space, then it is easily proved that $X \times Y$ has the property (U^γ). Hence, $X \times Y$ has the property $(C^* = P^\gamma)$ by Corollary 3.7. □

Example 3.4 shows that 'property (U^γ)' in Theorem 4.4 cannot be weakened to 'property $(C^* = P^\gamma)$'. The following problem remains open:

Problem 4.5. *If $X \times Y$ has the property $(C^* = P^\gamma)$ for every compact space Y, then does X have the property (U^γ)? More specially, does Theorem 4.4 remain true if 'property (U^γ)' is weakened to 'property $(U^\gamma)^*$'?*

A space is called *σ-locally compact* if it is the union of countably many closed locally compact subspaces. Concerning products with a σ-locally compact, paracompact factor, the following theorem was proved by Yamazaki in [23] and [25]:

Theorem 4.6. [Yamazaki] *Let A be a C-embedded subset of a space X, Y a σ-locally compact, paracompact space, and assume that $A \times Y$ is C^*-embedded in $X \times Y$. Then $A \times Y$ is C-embedded in $X \times Y$. Moreover, if A is P^γ-embedded in X in addition, then $A \times Y$ is also P^γ-embedded in $X \times Y$.*

Problem 4.7. *Does Theorem 4.4 remain true if 'compact' is weakened to 'σ-locally compact, paracompact'?*

Next, we consider products with a metric factor. The difficulty of this case is in the fact that $A \times Y$ need not be U^ω-embedded in $X \times Y$ even if A is P-embedded in X (consider Michael's product space). Nevertheless, the following Theorems 4.8 and 4.9 were proved by Gutev-Ohta [6]:

Theorem 4.8. [Gutev-Ohta] *Let A be a subset of a space X, Y a non-discrete metric space, and assume that $A \times Y$ is C^*-embedded in $X \times Y$. Then $A \times Y$ is C-embedded in $X \times Y$.*

Theorem 4.9. [Gutev-Ohta] *Let A be a P^γ-embedded subset of a space X and Y a metric space. Then the following conditions are equivalent*:

(1) $A \times Y$ *is* P^γ*-embedded in* $X \times Y$;
(2) $A \times Y$ *is* C^**-embedded in* $X \times Y$;
(3) $A \times Y$ *is* U^ω*-embedded in* $X \times Y$.

Corollary 4.10. *Let A be a P^γ-embedded subset of a space X, Y the product of a σ-locally compact, paracompact space K with a metric space M, and assume that $A \times Y$ is C^*-embedded in $X \times Y$. Then $A \times Y$ is P^γ-embedded in $X \times Y$.*

Proof. Since $(A \times K) \times M$ is C^*-embedded in $(X \times K) \times M$, $A \times K$ is C^*-embedded in $X \times K$. Hence, $A \times K$ is P^γ-embedded in $X \times K$ by Theorem 4.6. Finally, it follows from Theorem 4.9 that $(A \times K) \times M$ is P^γ-embedded in $(X \times K) \times M$. □

Problem 4.11. *Does Theorem 4.8 remain true if 'metric space' is weakened to 'paracompact M-space' or 'Lašnev space'?*

Problem 4.12. *Let A be a P^γ-embedded subset of a space X and Y a paracompact M-space. Then, does the condition (2) in Theorem 4.9 imply the condition (1)?*

Problem 4.13. *Let A be a P^γ-embedded subset of a space X and let Y be one of the following spaces (i)–(iii): (i) a Lašnev space; (ii) a stratifiable space; (iii) a paracompact σ-space. Then, are the conditions (1), (2), (3) in Theorem 4.9 equivalent?*

For the definitions of the spaces (i), (ii) and (iii) in Problem 4.13, we refer the reader to [5]. Problems 4.12 and 4.13 were raised in [6].

Now, we try to extend Theorems 4.3 and 4.8 to products with a factor space in wider class of spaces. For this end, we write $Y \in \Pi(Q)$ if for every space X and every closed subset A of X, if $A \times Y$ is C^*-embedded in $X \times Y$, then $A \times Y$ is Q-embedded in $X \times Y$, where $Q \in \{C, P^\gamma\}$. By Theorem 4.3, $Y \in \Pi(P^{w(Y)})$ for every infinite compact space Y, and by Theorem 4.8, $Y \in \Pi(C)$ for every non-discrete metric space Y. The following results show that the classes $\Pi(P^\gamma)$ and $\Pi(C)$ are much wider than we expected.

Theorem 4.14. *Let Y be a space with $Y \in \Pi(P^\gamma)$. Then $Y \times Z \in \Pi(P^\gamma)$ for every space Z.*

Proof. Let X be a space with a closed subset A such that $A \times (Y \times Z)$ is C^*-embedded in $X \times (Y \times Z)$. Then, it is obvious that $(A \times Z) \times Y$ is C^*-embedded in $(X \times Z) \times Y$. Since $Y \in \Pi(P^\gamma)$, $(A \times Z) \times Y$ is P^γ-embedded in $(X \times Z) \times Y$, which means that $A \times (Y \times Z)$ is P^γ-embedded in $X \times (Y \times Z)$. Hence, $Y \times Z \in \Pi(P^\gamma)$. □

Corollary 4.15. *For every space Y, $Y \times (\omega + 1) \in \Pi(C)$.*

Proof. Since $\omega + 1 \in \Pi(C)$ by Theorem 4.3 (or Theorem 4.8), this follows immediately from Theorem 4.14. □

The next theorem and its corollary were proved by Hoshina and Yamazaki in [9].

Theorem 4.16. [Hoshina-Yamazaki] *Let Y be a space which is homeomorphic to $Y \times Y$ and contains an infinite compact subset K. Then $Y \in \Pi(P^{w(K)})$.*

Corollary 4.17. [Hoshina-Yamazaki] *For every space Y with $|Y| \geq 2$, $Y^\gamma \in \Pi(P^\gamma)$.*

Finally, we consider some miscellaneous products. The following theorem was proved by Yamazaki in [24] and [25]. By a P-space, we mean a P-space in the sense of Morita [13]. For the definition of a Σ-space, see [5].

Theorem 4.18. [Yamazaki] *Let A be a closed subset of a normal P-space X, Y a paracompact Σ-space, and assume that $A \times Y$ is C^*-embedded in $X \times Y$. Then $A \times Y$ is C-embedded in $X \times Y$. Moreover, if A is P^γ-embedded in X in addition, then $A \times Y$ is P^γ-embedded in $X \times Y$.*

Since a P-space is countably paracompact, all normal P-spaces have the property (U^ω) and all γ-collectionwise normal P-spaces have the property (U^γ). Hence, the following problem naturally arises after Theorem 4.18.

Problem 4.19. *Let X be a normal P-space and Y a paracompact Σ-space. Then, does $X \times Y$ have the property $(C^* = C)$? Moreover, if X is γ-collectionwise normal in addition, then does $X \times Y$ have the property $(C^* = P^\gamma)$?*

Recently, a partial answer to Problem 4.19 was given by Yajima [22].

Theorem 4.20. [Yajima] *Let X be a collectionwise normal P-space and Y a paracompact Σ-space. Then every closed C-embedded subset of $X \times Y$ is P-embedded in $X \times Y$.*

5. Zero-sets in the Niemytzki plane

In the remainder of this paper, we consider extension properties of the Niemytzki plane NP, and in the final section, we answer Problem 1.2 for NP negatively. The Niemytzki plane NP is the closed upper half-plane $\mathbb{R} \times [0, +\infty)$ with the topology defined as follows: For each $p = \langle x, y \rangle \in NP$ and $\varepsilon > 0$, let

$$S_\varepsilon(p) = \begin{cases} \{q \in NP : d(\langle x, \varepsilon \rangle, q) < \varepsilon\} \cup \{p\} & \text{for } y = 0, \\ \{q \in NP : d(p, q) < \varepsilon\} & \text{for } y > 0, \end{cases}$$

where d is the Euclidean metric on the plane. The topology of NP is generated by the family $\{S_\varepsilon(p) : p \in NP, \varepsilon > 0\}$. Let $L = \{\langle x, 0 \rangle : x \in \mathbb{R}\} \subseteq NP$.

From now on, we always consider a subset of $\mathbb{R}$ to be a subspace of $\mathbb{R}$, and consider a subset of NP to be a subspace of NP unless otherwise stated. For example, an interval I is a subspace of $\mathbb{R}$ but $I \times \{0\}$ is a subspace of NP. When $A \subseteq X \subseteq NP$, we say that A is *ε-open* in X if A is open with respect to the relative topology on X induced from the Euclidean topology. The words *ε-closed* and *ε-continuous* are used similarly.

In this section, we determine a zero-set in NP. We first state the main results in this section, then proceed to the proofs.

Theorem 5.1. *Let F be a closed subset of NP. Then F is a zero-set in NP if and only if the set $\{x \in \mathbb{R} : \langle x, 0\rangle \in F\}$ is a G_δ-set in $\mathbb{R}$.*

Corollary 5.2. *If S is a subset of NP with $S \cap L = \varnothing$, then $\mathrm{cl}_{NP} S$ is a zero-set in NP. In particular, every closed subset S of NP with $S \cap L = \varnothing$ is a zero-set in NP.*

Proof. This follows from Theorem 5.1 above and Lemma 5.11 below. □

The next corollary follows from Corollary 5.2, since $F = \mathrm{cl}_{NP}(F \setminus L)$ for every regular-closed set F in NP.

Corollary 5.3. *Every regular-closed set in NP is a zero-set.*

Theorem 5.1 also shows that every zero-set in NP is a G_δ-set with respect to the Euclidean topology. On the other hand, every ε-closed set in the closed upper half-plane is a zero-set in NP. Hence, we have the following corollary.

Corollary 5.4. *For a subset S of NP, S is a Baire set in NP if and only if S is a Borel set with respect to the Euclidean topology.*

The final theorem of this section describes a zero-set in a subspace of NP.

Theorem 5.5. *Let Y be a subspace of NP and $Y_0 = \mathrm{cl}_Y(Y \setminus L)$. Let F be a closed subset of Y. Then F is a zero-set in Y if and only if A is a G_δ-set in B, where $A = \{x \in \mathbb{R} : \langle x, 0\rangle \in F \cap Y_0\}$ and $B = \{x \in \mathbb{R} : \langle x, 0\rangle \in Y_0\}$.*

Before proving Theorems 5.1 and 5.5, let us observe some examples of non-trivial zero-sets in NP.

Example 5.6. (1) The first one is a zero-set E in NP such that $E \cap L = \varnothing$ but the set $\{x \in \mathbb{R} : \langle x, 0\rangle \in \mathrm{cl}_\varepsilon E\}$ is the Cantor set $\mathcal{K}$, where $\mathrm{cl}_\varepsilon E$ is the closure of E with respect to the Euclidean topology. Let $\mathcal{I}$ be the set of all components of $[0,1] \setminus \mathcal{K}$. For each open interval $I = (a, b) \in \mathcal{I}$, define

$$E_I = \{\langle x, y\rangle : a < x < b,\ y = \min\{1 - \sqrt{1-(x-a)^2}, 1 - \sqrt{1-(x-b)^2}\}\}.$$

Then E_I is a closed set in NP such that $\mathrm{cl}_\varepsilon E_I \setminus E_I = \{\langle a, 0\rangle, \langle b, 0\rangle\}$. Define $E = \bigcup\{E_I : I \in \mathcal{I}\}$. Then E is a closed set in NP such that $E \cap L = \varnothing$ and $\mathcal{K} = \{x \in \mathbb{R} : \langle x, 0\rangle \in \mathrm{cl}_\varepsilon E\}$, as required. By Corollary 5.2, E is a zero-set in NP.

(2) The second one is a zero-set F of NP such that $F = \mathrm{cl}_{NP}(F \setminus L)$ and $\{x \in \mathbb{R} : \langle x, 0\rangle \in F\} = \mathbb{R} \setminus \mathbb{Q}$. Since $\mathbb{Q} \times \{0\}$ is countable and discrete closed in NP, we can find a disjoint family $\mathcal{S} = \{S_{\varepsilon(x)}(\langle x, 0\rangle) : x \in \mathbb{Q}\}$ of basic open sets in NP. Define $F = NP \setminus \bigcup\{S : S \in \mathcal{S}\}$. Then, $\{x \in \mathbb{R} : \langle x, 0\rangle \in F\} = \mathbb{R} \setminus \mathbb{Q}$ clearly. To show that $F = \mathrm{cl}_{NP}(F \setminus L)$, consider a point $q = \langle x, 0\rangle \in (\mathbb{R} \setminus \mathbb{Q}) \times \{0\}$. Then, $S_\varepsilon(q) \cap (F \setminus L) \neq \varnothing$ for each $\varepsilon > 0$, because $\mathcal{S}$ is disjoint and the open interval $\{y \in \mathbb{R} : \langle x, y\rangle \in S_\varepsilon(q) \setminus \{q\}\}$ cannot be covered by disjoint open intervals J with $\inf J > 0$. Hence, $q \in \mathrm{cl}_{NP}(F \setminus L)$, which implies that $F = \mathrm{cl}_{NP}(F \setminus L)$. Finally, F is a zero-set in NP by Corollary 5.2. □

To prove Theorems 5.1 and 5.5, we need some definitions and lemmas. Let $\mathbb{R}^\sharp = \mathbb{R} \cup \{-\infty, +\infty\}$ and consider $-\infty < x < +\infty$ for each $x \in \mathbb{R}$. For each $a \in \mathbb{R}^\sharp$, we define a function $h_a : \mathbb{R} \to [0,1]$ as follows: For $a \in \mathbb{R}$, define

$$h_a(x) = \begin{cases} 1 - \sqrt{1-(x-a)^2} & \text{if } |x-a| \leq 1, \\ 1 & \text{otherwise,} \end{cases}$$

and define $h_{+\infty}(x) = h_{-\infty}(x) = 1$ for $x \in \mathbb{R}$. By an *open interval* in $\mathbb{R}$, we mean a set of the form $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ for $a, b \in \mathbb{R}^\sharp$ with $a < b$. For an open interval $J = (a,b)$ in $\mathbb{R}$, we define

$$U_J = \{\langle x, y\rangle : a < x < b, 0 \leq y < \min\{h_a(x), h_b(x)\}\}.$$

Lemma 5.7. *For every open interval $J = (a,b)$ in $\mathbb{R}$, the following are valid:*

(1) $J \times \{0\} \subseteq U_J$,

(2) $J \times \{0\}$ *is a zero-set in NP and U_J is a cozero-set in NP.*

Proof. (1) is obvious. To prove (2), let $H = J \times [0, +\infty)$. Since U_J is ε-open in H, there is an ε-continuous function $f : H \to [0,1]$ such that $f^{-1}(0) = J \times \{0\}$ and $f^{-1}(1) = H \setminus U_J$. We extend f to the function $f_* : NP \to [0,1]$ by letting $f_*|_H = f$ and $f_*(p) = 1$ for each $p \in NP \setminus H$. Then f_* is continuous on NP by the definition of U_J. Since $J \times \{0\} = f_*^{-1}(0)$ and $U_J = f_*^{-1}[[0,1)]$, we have (2). □

Lemma 5.8. *If $\mathcal{J}$ is a family of disjoint open intervals in $\mathbb{R}$, then the family $\mathcal{U} = \{U_J : J \in \mathcal{J}\}$ is discrete in NP.*

Proof. Let $p = \langle x, y\rangle \in NP$. If $y = 0$, then $S_1(p)$ meets at most one member of $\mathcal{U}$. If $y > 0$, then $S_{y/2}(p)$ meets at most one member of $\mathcal{U}$. □

Let $\mathcal{F}$ be a family of subsets of a space X. It is known [14, 18] that $\mathcal{F}$ is uniformly locally finite in X if and only if there exist a locally finite family $\{G(F) : F \in \mathcal{F}\}$ of cozero-sets in X and a family $\{Z(F) : F \in \mathcal{F}\}$ of zero-sets in X such that $F \subseteq Z(F) \subseteq U(F)$ for each $F \in \mathcal{F}$. Now, we say that $\mathcal{F}$ is *uniformly discrete* in X if there exist a discrete family $\{U(F) : F \in \mathcal{F}\}$ of cozero-sets in X and a family $\{Z(F) : F \in \mathcal{F}\}$ of zero-sets in X such that $F \subseteq Z(F) \subseteq U(F)$ for each $F \in \mathcal{F}$.

Lemma 5.9. [15, Lemma 2.3] *The union of a uniformly locally finite family of zero-sets in a space X is a zero-set in X.*

Lemma 5.10. *If A is a G_δ-set in $\mathbb{R}$, then $A \times \{0\}$ is a zero-set in NP.*

Proof. There exist open sets G_n, $n \in \mathbb{N}$, in $\mathbb{R}$ such that $A = \bigcap_{n\in\mathbb{N}} G_n$. For each $n \in \mathbb{N}$, G_n is the union of a family $\{J_i : i \in M\}$ of disjoint open intervals in $\mathbb{R}$. By Lemmas 5.7 and 5.8, $\{J_i \times \{0\} : i \in M\}$ is a uniformly discrete family of zero-sets in NP. Hence, $G_n \times \{0\}$ is a zero-set in NP by Lemma 5.9. Since the intersection of countably many zero-sets is a zero-set, $A \times \{0\}$ is a zero-set in NP. □

Lemma 5.11. *If S is a subset of NP with $S \cap L = \varnothing$, then the set $A = \{x \in \mathbb{R} : \langle x, 0\rangle \in \mathrm{cl}_{NP}\, S\}$ is a G_δ-set in $\mathbb{R}$.*

Proof. For each $x \in \mathbb{R} \setminus A$, there exists $n(x) \in \mathbb{N}$ such that $S_{1/n(x)}(\langle x, 0 \rangle) \cap S = \varnothing$. For each $n \in \mathbb{N}$, let $B_n = \{x \in \mathbb{R} : n(x) = n\}$. Then it is easily proved that $A \cap \mathrm{cl}_{\mathbb{R}} B_n = \varnothing$. Since $\mathbb{R} \setminus A = \bigcup_{n \in \mathbb{N}} \mathrm{cl}_{\mathbb{R}} B_n$, A is a G_δ-set in $\mathbb{R}$. □

Lemma 5.12. *Let Y be a subspace of NP such that $Y = \mathrm{cl}_Y(Y \setminus L)$ and let F be a zero-set in Y. Then A is a G_δ-set in B, where $A = \{x \in \mathbb{R} : \langle x, 0 \rangle \in F\}$ and $B = \{x \in \mathbb{R} : \langle x, 0 \rangle \in Y\}$.*

Proof. Since F is a zero-set in Y, there exist open sets G_n, $n \in \mathbb{N}$, in Y such that $F = \bigcap_{n \in \mathbb{N}} \mathrm{cl}_Y G_n$. Let $H = \bigcap_{n \in \mathbb{N}} \mathrm{cl}_{NP}(G_n \setminus L)$; then $F = H \cap Y$ by the condition of Y. Moreover, the set $C = \{x \in \mathbb{R} : \langle x, 0 \rangle \in H\}$ is a G_δ-set in $\mathbb{R}$ by Lemma 5.11. Since $A = B \cap C$, A is a G_δ-set in B. □

Lemma 5.13. *Let E and F be closed sets in NP such that $L \subseteq E$ and $E \cap F = \varnothing$. Then there exists an open set U in NP such that $E \subseteq U \subseteq \mathrm{cl}_{NP} U \subseteq NP \setminus F$.*

Proof. For each $p \in E$, there is $n(p) \in \mathbb{N}$ such that $S_{1/n(p)}(p) \cap F = \varnothing$. For each $n \in \mathbb{N}$, let $E_n = \{p \in E : n(p) = n\}$ and $U_n = \bigcup\{S_{1/2n}(p) : p \in E_n\}$. Then U_n is an open set in NP with $E_n \subseteq U_n$. We show that $\mathrm{cl}_{NP} U_n \cap F = \varnothing$ for each $n \in \mathbb{N}$. Suppose on the contrary that there is a point $q = \langle x, y \rangle \in \mathrm{cl}_{NP} U_n \cap F$ for some $n \in \mathbb{N}$. Then $y > 0$, because $F \cap L = \varnothing$. Thus, we can find $\delta > 0$ such that for every $x \in \mathbb{R}$, if $q \notin S_{1/n}(\langle x, 0 \rangle)$, then $S_\delta(q) \cap S_{1/2n}(\langle x, 0 \rangle) = \varnothing$. If we put $\varepsilon = \min\{\delta, 1/2n\}$, then

$$\forall p \in NP \ (q \notin S_{1/n}(p) \Rightarrow S_\varepsilon(q) \cap S_{1/2n}(p) = \varnothing).$$

Now, since $q \in \mathrm{cl}_{NP} U_n$, $S_\varepsilon(q) \cap S_{1/2n}(p) \neq \varnothing$ for some $p \in E_n$. By (1), this implies that $q \in S_{1/n}(p)$, which contradicts the fact that $S_{1/n}(p) \cap F = \varnothing$. Hence, $\mathrm{cl}_{NP} U_n \cap F = \varnothing$ for every $n \in \mathbb{N}$, and obviously, $E \subseteq \bigcup_{n \in \mathbb{N}} U_n$. On the other hand, since F is Lindelöf, there exists a countable family $\{V_n : n \in \mathbb{N}\}$ of open sets in NP such that $F \subseteq \bigcup_{n \in \mathbb{N}} V_n$ and $\mathrm{cl}_{NP} V_n \cap E = \varnothing$ for each $n \in \mathbb{N}$. Finally, the set $U = \bigcup_{n \in \mathbb{N}}(U_n \setminus \bigcap_{i \leq n} \mathrm{cl}_{NP} V_i)$ is a required open set in NP. □

We are now ready to prove Theorems 5.1 and 5.5.

Proof. (of Theorem 5.1) Let $A = \{x \in \mathbb{R} : \langle x, 0 \rangle \in F\}$. If F is a zero-set in NP, then A is a G_δ-set in $\mathbb{R}$ by Lemma 5.12. Conversely, assume that A is a G_δ-set in $\mathbb{R}$, i.e., there exist open sets G_n, $n \in \mathbb{N}$, in $\mathbb{R}$ with $A = \bigcap_{n \in \mathbb{N}} G_n$. For each $n \in \mathbb{N}$, let $K_n = (\mathbb{R} \setminus G_n) \times \{0\}$. Then both $A \times \{0\}$ and K_n are zero-sets in NP by Lemma 5.10. Hence, there exists a continuous function $f_n : NP \to [0, 1]$ such that $f_n[A \times \{0\}] = \{0\}$ and $f_n[K_n] = \{1\}$. Let $H_n = F \cap f^{-1}[[1/2, 1]]$. Then H_n is a closed set in NP with $H_n \cap L = \varnothing$. By using Lemma 5.13 and the technique used in the proof of Urysohn's lemma, we can define another continuous function $g_n : NP \to [0, 1]$ such that $g_n[L] = \{0\}$ and $g_n[H_n] = \{1\}$. Define $Z_n = f_n^{-1}[[0, 1/2]] \cup g_n^{-1}[\{1\}]$. Then Z_n is a zero-set in NP such that $F \subseteq Z_n$ and $Z_n \cap K_n = \varnothing$. On the other hand, $F \cup L$ is a zero-set in NP, because it is an ε-closed set. Since $F = (F \cup L) \cap \bigcap_{n \in \mathbb{N}} Z_n$, F is a zero-set in NP. □

Proof. (of Theorem 5.5) If F is a zero-set in Y, then $F \cap Y_0$ is a zero-set in Y_0. Since $Y_0 = \mathrm{cl}_Y(Y_0 \setminus L)$, it follows from Lemma 5.12 that A is a G_δ-set in B. To prove the converse, assume that A is a G_δ-set in B. Since $Y \setminus Y_0$ is a discrete, open and closed subset of Y, $F \setminus Y_0$ is a zero-set in Y. Hence, it suffices to show that $F \cap Y_0$ is a zero-set in Y. To show this, let $Z_1 = \mathrm{cl}_{NP}(F \setminus L) \cap Y_0$ and $Z_2 = (F \cap Y_0) \cap L$. Then $F \cap Y_0 = Z_1 \cup Z_2$. By Corollary 5.2, Z_1 is a zero-set in Y_0. On the other hand, by the assumption, there exists a G_δ-set C in $\mathbb{R}$ such that $A = B \cap C$. Since $Z_2 = (C \times \{0\}) \cap Y_0$, Z_2 is a zero-set in Y_0 by Lemma 5.10. Consequently, $F \cap Y_0$ is a zero-set in Y_0, and hence, in Y, because Y_0 is open and closed in Y. □

6. z-EMBEDDED SUBSETS IN NP

A subset A of $\mathbb{R}$ is called a *Q-set* if every subset of A is a G_δ-set in A. All countable sets are Q-sets and the existence of an uncountable Q-set is known to be independent of the usual axioms of set theory (cf. [12]). It is quite easy to determine a z-embedded set Y in NP such that $Y \subseteq L$. Indeed, the first theorem immediately follows from Theorem 5.1:

Theorem 6.1. *For a subset A of $\mathbb{R}$, $A \times \{0\}$ is z-embedded in NP if and only if A is a Q-set in $\mathbb{R}$.*

Next, we consider a z-embedded subset in NP which is not necessarily a subset of L.

Lemma 6.2. *Let Y be a subset of NP such that $Y = \mathrm{cl}_Y(Y \setminus L)$. Then Y is z-embedded in NP.*

Proof. Let F be a zero-set in Y. Let $A = \{x \in \mathbb{R} : \langle x, 0\rangle \in F\}$ and $B = \{x \in \mathbb{R} : \langle x, 0\rangle \in Y\}$. Then by Lemma 5.12, A is a G_δ-set in B, i.e., there is a G_δ-set C in $\mathbb{R}$ with $A = B \cap C$. Let $Z = (C \times \{0\}) \cup \mathrm{cl}_{NP}(F \setminus L)$. Then Z is a zero-set in NP, because both $C \times \{0\}$ and $\mathrm{cl}_{NP}(F \setminus L)$ are zero-sets in NP by Lemma 5.10 and Corollary 5.2, respectively. Since $F = Z \cap Y$, Y is z-embedded in NP. □

Theorem 6.3. *Let Y be a subspace of NP and $Y_0 = \mathrm{cl}_Y(Y \setminus L)$. Then Y is z-embedded in NP if and only if A is a Q-set in $\mathbb{R}$ and is a G_δ-set in B, where $A = \{x \in \mathbb{R} : \langle x, 0\rangle \in Y \setminus Y_0\}$ and $B = \{x \in \mathbb{R} : \langle x, 0\rangle \in Y\}$.*

Proof. First, assume that Y is z-embedded in NP. Then $Y \setminus Y_0$ is z-embedded in NP, because Y_0 is open and closed in Y. Hence, it follows from Theorem 6.1 that A is a Q-set. Moreover, since Y is z-embedded in NP, there is a zero-set F in NP such that $Y \setminus Y_0 = F \cap Y$. By Theorem 5.1, the set $C = \{x \in \mathbb{R} : \langle x, 0\rangle \in F\}$ is a G_δ-set in $\mathbb{R}$. Since $A = B \cap C$, A is a G_δ-set in B. Next, we prove the converse. By the assumption, there is a G_δ-set D in $\mathbb{R}$ such that $A = B \cap D$. Let $Z_1 = D \times \{0\}$ and $Z_2 = \mathrm{cl}_{NP}(Y \setminus L)$. Then both Z_1 and Z_2 are zero-sets in NP by Lemma 5.10 and Corollary 5.2, respectively, and they satisfy that $Y \setminus Y_0 \subseteq Z_1$, $Y_0 \subseteq Z_2$, $Z_1 \cap Y_0 = \varnothing$ and $Z_2 \cap (Y \setminus Y_0) = \varnothing$. Hence, it suffices to show that both $Y \setminus Y_0$ and Y_0 are z-embedded in NP. Since A is a Q-set, $Y \setminus Y_0$ is z-embedded in NP by Theorem 6.1, and Y_0 is z-embedded in NP by Lemma 6.2. □

Remark 6.4. It is known that if $A \subseteq \mathbb{R}$ is a Q-set, then the subspace $Y = (A \times \{0\}) \cup (NP \setminus L)$ of NP is normal (cf. [21, Example F]). Hence, the closed set $A \times \{0\}$ is then C-embedded in Y. However, this does not mean that $A \times \{0\}$ is C-embedded in NP even if A is countable. In fact, it is known ([8, Example 3.14]) that $\mathbb{Q} \times \{0\}$ is not C^*-embedded in NP; this also follows from Theorem 7.1 below.

7. P-, C- and C^*-embedded subsets in NP

Recall from [6] that a subset Y of a space X is *CU-embedded* in X if for every pair of a zero-set E in Y and a zero-set F in X with $E \cap F = \varnothing$, E and $F \cap Y$ are completely separated in X. The extension properties we have considered are related as the following diagram, where the arrow '$A \to B$' means that every A-embedded subset is B-embedded:

$$\begin{array}{ccccccc} P & \longrightarrow & C & \longrightarrow & C^\star & \longrightarrow & z \\ & & \downarrow & & \downarrow & & \\ & & U^\omega & \longrightarrow & CU & & \end{array}$$

Moreover, we say that a subset $Y \subseteq X$ is *uniformly discrete* in X if the family $\{\{x\} : x \in Y\}$ is uniformly discrete in X, in other words, there exists a discrete family $\{U(x) : x \in Y\}$ of cozero-sets in X such that $x \in U(x)$ for each $x \in Y$. As is easily shown, every uniformly discrete set in X is P-embedded in X. Finally, we briefly review scattered sets in $\mathbb{R}$. Let $A \subseteq \mathbb{R}$. For every ordinal α, we define the set $A^{(\alpha)}$ inductively as follows: $A^{(0)} = A$; if $\alpha = \beta + 1$, then $A^{(\alpha)}$ is the derived set of $A^{(\beta)}$; and if α is a limit, then $A^{(\alpha)} = \bigcap\{A^{(\beta)} : \beta < \alpha\}$. A subset A of $\mathbb{R}$ is called *scattered* if $A^{(\alpha)} = \varnothing$ for some α, and then we write $\kappa(A) = \min\{\alpha : A^{(\alpha)} = \varnothing\}$. It is known that $\kappa(A) < \omega_1$ for every scattered set A in $\mathbb{R}$.

Theorem 7.1. *For a subset A of $\mathbb{R}$, the following conditions are equivalent*:

(1) *A is a scattered set in $\mathbb{R}$;*
(2) *$A \times \{0\}$ is uniformly discrete in NP;*
(3) *$A \times \{0\}$ is P-embedded in NP;*
(4) *$A \times \{0\}$ is CU-embedded in NP.*

Proof. (1) $\Rightarrow$ (2): We prove this implication by induction on $\kappa(A)$. If $\kappa(A) = 0$, it is obviously true since $A = \varnothing$. Now, let $\alpha > 0$ and assume that the implication holds for every subset $A' \subseteq \mathbb{R}$ with $\kappa(A') < \alpha$. Let $A \subseteq \mathbb{R}$ be a scattered set with $\kappa(A) = \alpha$. In case $\alpha = \beta + 1$, $(A \setminus A^{(\beta)}) \times \{0\}$ is uniformly discrete in NP by inductive hypothesis, because $\kappa(A \setminus A^{(\beta)}) < \alpha$. Since $A^{(\beta)}$ is discrete, there is a family $\{I_x : x \in A^{(\beta)}\}$ of disjoint open intervals in $\mathbb{R}$ such that $x \in I_x$ for each $x \in A^{(\beta)}$. Hence, it follows from Lemmas 5.7 and 5.8 that $A^{(\beta)} \times \{0\}$ is also uniformly discrete in NP. Since the union of finitely many uniformly discrete subsets is uniformly discrete, $A \times \{0\}$ is uniformly discrete in NP. In case α is a limit, then $\mathcal{U} = \{A \setminus A^{(\beta)} : \beta < \alpha\}$ is an open cover of A. Since every scattered set in $\mathbb{R}$ is zero-dimensional, there exists a disjoint open refinement $\mathcal{V}$ of $\mathcal{U}$. By considering order components of each member of $\mathcal{V}$, we can find a family $\mathcal{J} = \{J_n : n \in M\}$ of disjoint open intervals in $\mathbb{R}$ such that $\mathcal{J}$ covers A and $\{J_n \cap A : n \in M\}$ refines $\mathcal{V}$. By Lemmas 5.7 and 5.8 again, the family $\{J_n \times \{0\} : n \in M\}$ is uniformly discrete in NP, and hence, so is $\{(J_n \cap A) \times \{0\} : n \in M\}$. Moreover, each $(J_n \cap A) \times \{0\}$ is uniformly discrete in NP by inductive hypothesis. Since the union of a uniformly discrete family of uniformly discrete subsets is also uniformly discrete, $A \times \{0\}$ is uniformly discrete in NP.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (1): Suppose that A is not scattered; then A includes a perfect subset B which is closed in A. Let $K = \mathrm{cl}_{\mathbb{R}} B$ and take a countable dense subset B_0 of B such that the set $B_1 = B \setminus B_0$ is also dense in B, i.e., $K = \mathrm{cl}_{\mathbb{R}} B_0 = \mathrm{cl}_{\mathbb{R}} B_1$. Let $E = (K \setminus B_0) \times \{0\}$; then E is a zero-set in NP by Lemma 5.10. Now, $B_0 \times \{0\}$ is a zero-set in $A \times \{0\}$, because $A \times \{0\}$ is discrete. On the other hand, $B_1 \times \{0\} = E \cap (A \times \{0\})$. Since $A \times \{0\}$ is CU-embedded in NP, there exists a continuous function $f : NP \to [0,1]$ such that $f[B_i \times \{0\}] = i$ for $i = 0, 1$. Let $C_i = \{x \in \mathbb{R} : f(\langle x, 0\rangle) = i\}$ for each $i = 0, 1$. Then C_0 and C_1 are disjoint G_δ-sets in $\mathbb{R}$ by Theorem 5.1. Hence, we can write $K \setminus C_i = \bigcup_{j\in\mathbb{N}} D_{i,j}$, where each $D_{i,j}$ is ε-closed in K, for each $i = 0, 1$. Since $B \subseteq C_0 \cup C_1$ and both B_0 and B_1 are dense in K, $D_{i,j}$ is nowhere dense in K for all i and j. This contradicts the completeness of K. □ □

Lemma 7.2. *Every CU-embedded subset Y in a first countable space X is closed.*

Proof. If Y is not closed in X, then there exists a sequence $\{p_n : n \in \mathbb{N}\} \subseteq Y$ which converges to a point $p \in X \setminus Y$. We may assume that $p_m \neq p_n$ if $m \neq n$. Let $E = \{p_{2n} : n \in \mathbb{N}\}$ and $F = \{p_{2n-1} : n \in \mathbb{N}\} \cup \{p\}$. It is easily proved that F is a compact G_δ-set in X, and hence, a zero-set in X, because X is completely regular. On the other hand, since $E \cup \{p\}$ is also a zero-set in X, E is a zero-set in Y. Since Y is CU-embedded in X, E and $F \setminus \{p\}$ must be completely separated in X, which is impossible. □

Lemma 7.3. *Every scattered subset A of $\mathbb{R}$ is a G_δ-set in $\mathbb{R}$.*

Proof. This is well-known and also follows from our results. In fact, by Theorem 7.1, $A \times \{0\}$ is uniformly discrete in NP, which implies that $A \times \{0\}$ is a zero-set in NP by Lemma 5.9. Hence, A is a G_δ-set in $\mathbb{R}$ by Theorem 5.1. □

By Lemma 7.2, we can restrict our attention to closed subsets of NP. The following theorem shows that every CU-embedded subset of NP is P-embedded, which answers Problem 1.2 for the Niemytzki plane negatively.

Theorem 7.4. *Let Y be a closed subspace of NP and let $Y_0 = \mathrm{cl}_Y(Y \setminus L)$. Then the following conditions are equivalent*:

(1) *The set $A = \{x \in \mathbb{R} : \langle x, 0\rangle \in Y \setminus Y_0\}$ is a scattered set in $\mathbb{R}$;*
(2) *$Y \setminus Y_0$ is uniformly discrete in NP;*
(3) *Y is P-embedded in NP;*
(4) *Y is CU-embedded in NP.*

Proof. (1) $\Leftrightarrow$ (2): This equivalence follows from Theorem 7.1.

(1) $\Rightarrow$ (3): Suppose that (1) is true. Then A is a G_δ-set in $\mathbb{R}$ by Lemma 7.3. Hence, the set $A \times \{0\}\,(= Y \setminus Y_0)$ is a zero-set in NP by Lemma 5.10. On the other hand, by the definition of Y_0, it follows from Corollary 5.2 that Y_0 is a zero-set. Consequently, Y_0 and $Y \setminus Y_0$ are completely separated in NP. Hence, it suffices to show that both Y_0 and $Y \setminus Y_0$ are P-embedded in NP. By Theorem 6.3 and Corollary 5.2, Y_0 is a z-embedded zero-set in NP, which implies that Y_0 is C-embedded in NP by Theorem 3.1. Since Y_0 is separable, Y_0 has no uncountable locally finite cozero-set cover. Hence, Y_0 is P-embedded in NP. On the other hand, $Y \setminus Y_0$ is P-embedded in NP by Theorem 7.1.

(3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (1): If Y is CU-embedded in NP, then the set $A \times \{0\}\,(= Y \setminus Y_0)$ is also CU-embedded in NP, because $Y \setminus Y_0$ is open and closed in Y. Hence, this implication follows from Theorem 7.1. □

By Theorem 7.4, both of the zero-sets E and F defined in Example 5.6 are P-embedded in NP.

Corollary 7.5. *Every CU-embedded subset in NP is a P-embedded zero-set.*

Proof. Let Y be a CU-embedded set in NP and let $Y_0 = \mathrm{cl}_Y(Y \setminus L)$. By Theorem 7.4, Y is P-embedded in NP. Moreover, as I showed in the proof of Theorem 7.4 (1) $\Rightarrow$ (3), both Y_0 and $Y \setminus Y_0$ are zero-sets in NP. Hence, Y is a zero-set in NP. □

Recall from [19] that a subset A of a space X is *π-embedded* in X if $A \times Y$ is C^*-embedded in $X \times Y$ for every space Y. The following problem is open:

Problem 7.6. *Is every P-embedded subset in NP π-embedded in NP?*

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