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Some properties of o-bounded and strictly o-bounded groups

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Abstract. We continue the study of (strictly) o-bounded topological groups initiated by the first listed author and solve two problems posed earlier. It is shown here that the product of a Comfort-like topological group by a (strictly) o-bounded group is (strictly) o-bounded. Some non-trivial examples of strictly o-bounded free topological groups are given. We also show that o-boundedness is not productive, and strict o-boundedness cannot be characterized by means of second countable continuous homomorphic images.

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1. Introduction

The class of σ-compact topological groups has many nice properties. For example, every σ-compact group is countably cellular [11] and perfectly κ-normal [13, 15]. The subgroups of σ-compact groups inherit these properties, but clearly need not be σ-compact. The notions of o-boundedness and strict o-boundedness introduced by O. Okunev and M. Tkachenko respectively, were considered in [9]. The idea was to find a wider class of topological groups as close to the class of σ-compact groups as possible which is additionally closed under taking subgroups. Let us recall the corresponding definitions.

A topological group G is called *o-bounded* if for every sequence $\{U_n : n \in \mathbb{N}\}$ of open neighborhoods of the neutral element in G, there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of G such that $G = \bigcup_{n\in\mathbb{N}} F_n \cdot U_n$. It is clear that all σ-compact groups as well as their subgroups are o-bounded. In a sense, o-bounded groups have to be small: the group $\mathbb{R}^\omega$ fails to be o-bounded [9,

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Example 2.6]. The class of o-bounded groups has good categorical properties: all subgroups and all continuous homomorphic images of an o-bounded group are o-bounded [9]. It was not known, however, whether this class was finitely productive [9, Problem 5.2]. We show in Example 2.12 that there exists a second countable o-bounded group G whose square is not o-bounded. Actually, the group G first appeared in [9, Example 6.1] in order to distinguish the classes of o-bounded and *strictly o-bounded* groups. However, the properties of this group were not completely exhausted there. As is shown in [4], the group G is additionally *analytic,* that is, G is a continuous image of a separable complete metric space.

To define strictly o-bounded groups, we need to describe the OF-game (see [9] or [14]). Suppose that G is a topological group and that two players, say I and II, play the following game. Player I chooses an open neighborhood U_1 of the identity in G, and player II responds choosing a finite subset F_1 of G. In the second turn, player I chooses another neighborhood U_2 of the identity in G and player II chooses a finite subset F_2 of G. The game continues this way until we have the sequences $\{U_n : n \in \mathbb{N}\}$ and $\{F_n : n \in \mathbb{N}\}$. Player II wins if $G = \bigcup_{n=1}^{\infty} F_n \cdot U_n$. Otherwise, player I wins. The group G is called *strictly o-bounded* if player II has a winning strategy in the OF-game on G. It is easy to see that σ-compact groups are strictly o-bounded and every strictly o-bounded group is o-bounded. As we mentioned above, o-bounded groups need not be strictly o-bounded. In addition, there are lots of strictly o-bounded groups that are neither σ-compact nor isomorphic to subgroups of σ-compact groups [9, Example 3.1]. However, an o-bounded continuous homomorphic image of a Weil-complete group is σ-bounded, hence strictly o-bounded [3]. All this makes the problem of studying the properties of these two classes of topological groups fairly interesting.

The class of o-bounded groups is not productive in view of Example 2.12. However, we have no examples of strictly o-bounded groups G and H such that the product $G \times H$ is not strictly o-bounded (see Problem 4.1). On the other hand, it was known that a product of an o-bounded group by a σ-compact group was o-bounded [9, Theorem 5.3], and a similar result for strictly o-bounded groups was recently proved by Jian He (see Theorem 2.7) who in fact has proved the result with 'σ-bounded' instead of 'σ-compact' and by a method that extends the o-bounded result as well. It turns out that there are many topological groups G (far from being σ-compact) with the property that the product $G \times H$ is (strictly) o-bounded for every (strictly) o-bounded group H. Let G be a σ-product of countable discrete groups endowed with the $\aleph_0$-box topology. We shall call any subgroup of such a group G a *Comfort-like* group. (It was W. Comfort who proved that every σ-product of countable discrete spaces with the $\aleph_0$-box topology inherited from the whole product is Lindelöf, see [5]). We prove in Section 2 that multiplication by a Comfort-like group G does not destroy (strict) o-boundedness: the product $G \times H$ is (strictly) o-bounded for every (strictly) o-bounded group H. It is also shown that the free topological group $F(X)$ is strictly o-bounded whenever X is the one-point

Lindelöfication of any uncountable discrete space (Theorem 2.8). In fact, the product $F(X) \times H$ is strictly o-bounded for every strictly o-bounded group H (see Theorem 2.11).

It is clear that every o-bounded group is $\aleph_0$-bounded in the sense of [6], that is, it can be covered by countably many translates of any neighborhood of the identity. By Theorem 4.1 of [9], if G is $\aleph_0$-bounded and all second countable continuous homomorphic images of G are o-bounded, then G itself is o-bounded. In Section 3 we use $\Diamond$ to construct an o-bounded group G whose second countable continuous homomorphic images are countable (hence strictly o-bounded), but G itself is not strictly o-bounded. Therefore, the class of strictly o-bounded groups is considerably more complicated than that of o-bounded groups. In other words, strict o-boundedness is not reflected in the class of second countable groups.

The group G in Theorem 3.1 has another interesting feature. Let us call a topological group H *OF-undetermined* if neither player I nor player II has a winning strategy in the OF-game in H. It was an open problem whether there exist OF-undetermined groups. It turns out that the group G in Example 3.1 is OF-undetermined. We do not know, however, if such a group can be constructed in ZFC. Another problem is considered by T. Banakh in [4]: Does there exist a metrizable OF-undetermined group? He shows that such groups exist under Martin's Axiom and have necessarily to be second countable.

1.1. **Notation and terminology.** We denote by $\mathbb{N}$ the positive integers, by $\mathbb{Z}$ the additive group of integers, and by $\mathbb{R}$ the group of reals. A topological group G is called *$\aleph_0$-bounded* [6] if countably many translates of every neighborhood of the identity in G cover the group G. By a result of [6], G is $\aleph_0$-bounded if and only if it is topologically isomorphic to a subgroup of a direct product of second countable topological groups. This class of groups is closed under taking direct products, subgroups and continuous homomorphic images.

We say that H is a *P-group* if the intersection of any countable family of open sets in H is open. Every topological group H admits a finer group topology that makes it a P-group: a base of such a topology consists of all G_δ-subsets of H.

If X is a subset of a group G, we use $\langle X\rangle$ to denote the subgroup of G generated by X. Finally, the families of all non-empty finite and countable subsets of a set A will be denoted by $[A]^{<\omega}$ and $[A]^{\le\omega}$, respectively.

2. Productive properties of o-bounded groups

Here we introduce the class of Comfort-like topological groups and show that the product $G \times H$ is (strictly) o-bounded whenever G is a Comfort-like group and H is (stricly) o-bounded. We start with a simple but useful lemma.

Lemma 2.1. *Suppose that G, H and K are groups that $\varphi\colon G \to H$ and $\psi\colon G \to K$ are homomorphisms such that $\ker\psi \subseteq \ker\varphi$. Then there exists a homomorphism $f\colon K \to H$ such that $\varphi = f \circ \psi$. If in addition, G, H and K are topological groups, φ and ψ are continuous, and for each neighborhood U*

of the identity e_H in H there exists a neighborhood V of the identity e_K in K such that $\psi^{-1}(V) \subseteq \varphi^{-1}(U)$, then f is continuous.

Proof. The algebraic part of the lemma is well known. Let us verify the continuity of f in the second part of the lemma. Suppose that U is a neighborhood of e_H in H. By our assumption, there exists a neighborhood V of e_K in K such that $W = \psi^{-1}(V) \subseteq \varphi^{-1}(U)$. Then $f(V) = \varphi(W) \subseteq U$, that is, f is continuous at the identity of K. Therefore, f is continuous. □

Our second auxiliary result concerns continuous homomorphic images of $\aleph_0$-bounded P-groups.

Lemma 2.2. *Let $\varphi\colon G \to H$ be a continuous homomorphism of an $\aleph_0$-bounded P-group G to a topological group H of countable pseudocharacter. Then the image $\varphi(G)$ is countable.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable pseudobase at the identity e_H of H. Since G is a P-group, the kernel $N = \ker\varphi = \bigcap_{n\in\mathbb{N}} \varphi^{-1}(U_n)$ is a normal open subgroup of G. Let $\psi\colon G \to G/N$ be the quotient homomorphism. By Lemma 2.1, there exists a homomorphism $f\colon G/N \to H$ such that $\varphi = f \circ \psi$. Since G is $\aleph_0$-bounded, the quotient group G/N is countable, and hence $|\varphi(G)| = |f(\psi(G))| \le |G/N| \le \omega$. □

Let $\Pi = \prod_{i\in I} G_i$ be the direct product of topological groups G_i and let e be the identity of Π. For every $x \in \Pi$, put $\mathrm{supp}(x) = \{i \in I : x(i) \ne e_i\}$, where e_i is the identity of G_i, $i \in I$. Then we define

$$\sigma_\Pi = \{x \in \Pi : |\,\mathrm{supp}(x)| < \omega\}.$$

It is clear that σ_Π is a subgroup of Π. This subgroup is called the *σ-product* of the groups G_i, $i \in I$. Suppose that Π carries the *$\aleph_0$-box* topology $\mathcal{T}_\omega$ the standard base of which consists of the sets $\pi_J^{-1}(V)$, where J is a countable subset of I, $\pi_J\colon \Pi \to \Pi_J = \prod_{j\in J} G_j$ is the projection, and $V = \prod_{j\in J} V_j$ is a product of open subsets $V_j \subseteq G_j$, $j \in J$. Then Π with the topology $\mathcal{T}_\omega$ becomes a topological group. Note that if all groups G_i are discrete, then every G_δ-set in $(\Pi, \mathcal{T}_\omega)$ is open. In the special case when the groups G_i are countable and discrete, we shall call σ_Π (as well as every subgroup of σ_Π) a *Comfort-like group.* Therefore, every Comfort-like group is a P-group. In particular, such a group is zero-dimensional.

Let us show that Comfort-like groups form a subclass of $\aleph_0$-bounded P-groups. It is helpful to note that by Theorem 2.4 of [9], every $\aleph_0$-bounded P-group is o-bounded (in precise terms, the result in [9] was formulated for Lindelöf P-groups, but its proof remains valid for $\aleph_0$-bounded groups as well).

Corollary 2.3. *Every Comfort-like group G is $\aleph_0$-bounded. Therefore, G is an $\aleph_0$-bounded P-group, hence o-bounded.*

Proof. Every Comfort-like group G is a P-group. We show that the group G is $\aleph_0$-bounded. Since a subgroup of an $\aleph_0$-bounded group is also $\aleph_0$-bounded [6], we can assume that $G = \sigma_\Pi$, where $\Pi = \prod_{i\in I} G_i$ is the product of countable discrete groups G_i, and Π is endowed with the $\aleph_0$-box topology. Let $U =$

$G \cap \pi_J^{-1}(x)$ be a non-empty basic open set in G, where $J \in [I]^{\leq\omega}$, $\pi_J \colon \Pi \to \Pi_J$ is the projection and $x \in \Pi_J$. Since $\pi_J(G)$ is countable, there exists a countable subset K of G such that $\pi_J(K) = \pi_J(G)$. One easily verifies that $G = K \cdot U$. This proves that G is $\aleph_0$-bounded.

To finish the proof, note that the class of $\aleph_0$-bounded P-groups is closed with respect to taking arbitrary subgroups, and every group in this class is o-bounded by Theorem 2.4 of [9]. □

Now we present one of the main results of this section.

Theorem 2.4. *Let G be an $\aleph_0$-bounded P-group and H be an o-bounded topological group. Then $G \times H$ is o-bounded.*

Proof. Let us show that if $\varphi \colon G \times H \to K$ is a continuous epimorphism, where K is a second countable group, then K is o-bounded. Suppose that $W_1, W_2, \ldots$ is a neighborhood basis at the identity e_K of K. For each $i \in \mathbb{N}$, we take a neighborhood $U_i \times V_i$ of (e_G, e_H), where U_i and V_i are neighborhoods of the identities e_G and e_H of G and H respectively, such that $U_i \times V_i \subseteq \varphi^{-1}(W_i)$. Since G and H are $\aleph_0$-bounded, there exist continuous homomorphisms $f_i \colon G \to G_i$ and $h_i \colon H \to H_i$, where G_i and H_i are second countable groups, and neighborhoods U_i' and V_i' of the identities e_{G_i} and e_{H_i} of G_i and H_i respectively, such that $f_i^{-1}(U_i') \subseteq U_i$ and $h_i^{-1}(V_i') \subseteq V_i$ (see [14, Lemma 3.7]). Let $f = \Delta_{i\in\mathbb{N}} f_i \colon G \to \prod_{i\in\mathbb{N}} G_i$ and $h = \Delta_{i\in\mathbb{N}} h_i \colon H \to \prod_{i\in\mathbb{N}} H_i$ be the diagonal products of the families $\{f_i : i \in \mathbb{N}\}$ and $\{h_i : i \in \mathbb{N}\}$, respectively. The groups $G' = f(G)$ and $H' = h(H)$ are second countable and by Theorem 2.3 of [9], are o-bounded as continuous homomorphic images of o-bounded groups G and H, respectively. In addition, $G' = f(G)$ is countable by Lemma 2.2, and so, σ-compact. It then follows from [9, Theorem 5.3] that $G' \times H'$ is o-bounded. Observe that $\ker(f \times h) \subseteq \ker \varphi$ and then, for each neighborhood W of the identity in K, there exists a neighborhood V of the identity $(e_{G'}, e_{H'})$ in $G' \times H'$ such that $(f \times h)^{-1}(V) \subseteq \varphi^{-1}(W)$. So, by Lemma 2.1, there exists a continuous homomorphism $\psi \colon G' \times H' \to K$ such that $\varphi = (f \times h) \circ \psi$. Applying again Theorem 2.3 of [9], we infer that K is o-bounded. Therefore, all continuous homomorphic images of $G \times H$ are o-bounded, so [9, Theorem 4.1] implies that the group $G \times H$ is o-bounded. □

The above theorem and Corollary 2.3 together imply the following.

Corollary 2.5. *If G is a Comfort-like group, then the product $G \times H$ is o-bounded for every o-bounded group H.*

It is shown in [9, Example 3.1] that every σ-product of countable discrete groups that carries the $\aleph_0$-box topology is strictly o-bounded. Since subgroups of a strictly o-bounded group inherit this property [9, Theorem 2.1], every Comfort-like group is strictly o-bounded. Here we strengthen this result by considering the product of a Comfort-like group by a strictly o-bounded group.

Theorem 2.6. *If G is a Comfort-like group and H is a strictly o-bounded group, then $G \times H$ is strictly o-bounded.*

Proof. By Theorem 2.1 of [9], a subgroup of a strictly o-bounded group is strictly o-bounded. Therefore, it suffices to consider the case $G = \sigma_\Pi \subseteq \Pi$, where the product $\Pi = \prod_{\alpha<\tau} G_\alpha$ of countable discrete groups G_α is equipped with the $\aleph_0$-box topology. For every $A \in [\tau]^{\leq\omega}$, let $\pi_A \colon \Pi \to \Pi_A = \prod_{\alpha\in A} G_\alpha$ be the projection. Denote by e_A the identity of Π_A. Then the family $\mathcal{U} = \{U_A : A \in [\tau]^{\leq\omega}\}$ is a base at the identity e of G, where $U_A = G \cap \pi_A^{-1}(e_A)$. One easily verifies the following:

(1) every U_A is a normal subgroup of G;
(2) the sets U_A are clopen in G;
(3) $|G/U_A| \leq \aleph_0$ for each $A \in [\tau]^{\leq\omega}$.

It was proved in [9, Example 3.1] that G is strictly o-bounded. We need to refer to aspects of that proof, so the needed parts are reproduced here with appropriate adaptation for our present proof. Suppose that player I chooses in the turn i the neighborhood W_i of the identity in $G\times H$. We can assume without loss of generality that each W_i has the form $W_i = U_i \times V_i$ where U_i and V_i are neighborhoods of the identity in G and H respectively. In addition, we can assume that $U_i = U_{A_i}$, where $A_i \in [\tau]^{\leq\omega}$, and that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq \cdots$.

First consider the group G. If $x \in G$, put $\mathrm{supp}(x) = \{\alpha < \tau : x_\alpha \neq e_\alpha\}$, where e_α is the identity of G_α. Clearly, $\mathrm{supp}(x)$ is a finite subset of τ for each $x \in G$.

Since each U_A has a countable number of cosets in G we may do as follows:

For every $A \in [\tau]^{\leq\omega}$, we define a countable set $B_A = \{x_1^A, x_2^A, \dots\}$ choosing elements x_i^A in every coset of U_A in G in such a way that $\mathrm{supp}(x_i^A) \subseteq A$ for each $i \in \mathbb{N}$.

Choose $x \in G$. Then of course $x \in x_i^A U_A$ for some $x_i^A \in B_A$. We note that as any element $u \in U_A$ has $u_\alpha = e_\alpha$, for $\alpha \in A$, therefore $x_\alpha = (x_i^A)_\alpha$ for $\alpha \in A$, and moreover, $\mathrm{supp}(x_i^A) \subseteq \mathrm{supp}(x)$. We further note that if $A, B \in [\tau]^{\leq\omega}$ and $A \subseteq B$ then $U_B \subseteq U_A$, and if $x \in x_j^B U_B$ then $x_j^B U_B \subseteq x_i^A U_A$, so $\mathrm{supp}(x_i^A) \subseteq \mathrm{supp}(x_j^B) \subseteq \mathrm{supp}(x)$.

With reference to the $U_i = U_{A_i}$ above, and writing x_j^i for $x_j^{A_i}$ we see that for each n, we have that $x \in x_{j_n}^n U_n \subseteq \cdots \subseteq x_{j_2}^2 U_2 \subseteq x_{j_1}^1 U_1$ and $\mathrm{supp}(x_{j_1}^1) \subseteq \mathrm{supp}(x_{j_2}^2) \subseteq \cdots \subseteq \mathrm{supp}(x_{j_n}^n) \subseteq \cdots \subseteq \mathrm{supp}(x)$.

As all of these sets are finite we must have some $n = n(x)$ such that $\mathrm{supp}(x_{j_{n+k}}^{n+k}) = \mathrm{supp}(x_{j_n}^n)$, for each $k = 1, 2, \dots$. Since also by our remarks above x must agree with each one of the $x_{j_n}^n$ at each one of their coordinates of support, therefore $x_{j_{n+k}}^{n+k} = x_{j_n}^n$ for all $k \geq 1$.

Thus we have our main point as follows: $x \in x_{j_{n(x)}}^{n(x)} U_n$ for each $n \geq n(x)$. For each n, let $E_n = \{x_j^i : i, j \leq n\}$. Then certainly $x \in E_n \cdot U_n$ for each $n \geq n' = \max\{n(x), j_{n(x)}\}$. To prove that G is strictly o-bounded we only need one $n \geq n'$, but to prove our present theorem we need the full set of $n \geq n'$ as we will now see.

Since H is strictly o-bounded, player II has a winning strategy in the OF-game on H. So, we are able to construct finite non-empty subsets $F_{i,j}$ of H as set out in the following scheme:

$$\begin{array}{ccccccccc} V_1 & V_2 & V_3 & \dots & V_p & \dots & V_j & \dots \\ F_{1,1} & F_{2,1} & F_{3,1} & \dots & F_{p,1} & \dots & F_{j,1} & \dots \\ & F_{2,2} & F_{3,2} & \dots & F_{p,2} & \dots & F_{j,2} & \dots \\ & & & & \vdots & & \vdots & \\ & & & & F_{p,p} & \dots & F_{j,p} & \dots \\ & & & & & & \vdots & \\ & & & & & & F_{j,j} & \dots \end{array}$$

such that for each $p \geq 1$,

$$H = \bigcup_{q=p}^{\infty} F_{q,p} \cdot V_q.$$

If we put $F_i = \bigcup_{j=1}^{i} F_{i,j}$, clearly then $H = \bigcup_{i=p}^{\infty} F_i \cdot V_i$ for each $p \geq 1$. Now we shall prove that $G \times H = \bigcup_{i=1}^{\infty} (E_i \times F_i) \cdot (U_i \times V_i)$. Let $(x, y) \in G \times H$. Then $x \in E_{n'} \cdot U_{n'}$ where $n' = \max\{n(x), j_{n(x)}\}$. Now, $H = \bigcup_{i=n'}^{\infty} F_i \cdot V_i$. So $y \in F_j \cdot V_j$, for some $j \geq n'$. But then $x \in E_j \cdot U_j$. Hence $(x, y) \in (E_j \cdot U_j) \times (F_j \cdot V_j) = (E_j \times F_j) \cdot (U_j \times V_j)$, as required. □

It is clear that every σ-compact group is strictly o-bounded. By [9, Theorem 2.1], subgroups of σ-compact groups inherit this property. The following theorem proved by Jian He strengthens this result and complements Theorem 2.6. We present its proof with his kind permission.

Theorem 2.7. *If G is a strictly o-bounded group and H is a subgroup of a σ-compact group, then $G \times H$ is strictly o-bounded.*

Proof. Since H is a subgroup of a σ-compact group it is *σ-bounded,* that is, $H = \bigcup_{i=1}^{\infty} X_i$, where the sets X_i are precompact in H and may be taken such that X_i is included in X_j whenever $i \leq j$. Therefore, for each sequence $\{V_i : i \in \mathbb{N}\}$ of neighborhoods of the identity, e_H, of H, there exists a sequence $\{Q_i : i \in \mathbb{N}\}$ of finite non-empty subsets of H such that for each i, $X_i \subseteq Q_i \cdot V_i$. Now we suppose that at turn i, player I chooses a neighborhood $U_i \times V_i$ of the identity in $G \times H$. Since player II has a winning strategy in the OF-game on G, we are able to construct finite non-empty subsets $F_{i,j}$ of G as set out in the following scheme:

$$\begin{array}{ccccccccc} U_1 & U_2 & U_3 & \dots & U_p & \dots & U_q & \dots \\ F_{1,1} & F_{1,2} & F_{1,3} & \dots & F_{1,p} & \dots & F_{1,q} & \dots \\ & F_{2,2} & F_{2,3} & \dots & F_{2,p} & \dots & F_{2,q} & \dots \\ & & & & \vdots & & \vdots & \\ & & & & F_{p,p} & \dots & F_{p,q} & \dots \\ & & & & & & \vdots & \\ & & & & & & F_{q,q} & \dots \end{array}$$

such that for each $p \geq 1$,

$$G = \bigcup_{q=p}^{\infty} F_{p,q} \cdot U_q.$$

For every $i \geq 1$, let

$$F_i = F_{1,i} \cup F_{2,i} \cup \cdots \cup F_{i,i} \text{ and } K_i = F_i \times Q_i.$$

We claim that $G \times H = \bigcup_{i=1}^{\infty} K_i \cdot (U_i \times V_i)$. That is because if $(x, y) \in G \times H$, then there exists $p \geq 1$ such that $y \in X_p$. But then there exists $q \geq p$ such that $x \in F_{p,q} \cdot U_q$. Now as $Q_q \cdot V_q \supseteq X_q \supseteq X_p$, then $y \in Q_q \cdot V_q$. Since $F_{p,q} \subseteq F_q$, we have

$$(x, y) \in K_q \cdot (U_q \times V_q),$$

and so $G \times H$ is strictly o-bounded. □

Another interesting problem is to characterize the spaces X such that the free topological group $F(X)$ is strictly o-bounded. Here we find a special class of spaces X with this property.

Let D be a discrete space of uncountable cardinality. Denote by $D^* = D \cup \{x^*\}$ the space obtained by adjoining to D the point x^* not in D, whose topology consists of all subsets of D and all subsets of D^* with countable complements. Such a space D^* is known as the *one-point Lindelöfication* of D.

Theorem 2.8. *The free topological group $F(D^*)$ is strictly o-bounded.*

To prove the above theorem we need two auxiliary results.

Lemma 2.9. *The family $\gamma = \{U_K : K \in [D]^{\leq\omega}\}$, where U_K is the normal subgroup of $F(D^*)$ generated by $D^* \setminus K$, is a base at the identity of $F(D^*)$.*

Proof. Let $K \in [D]^{\leq\omega}$ and $K^* = K \cup \{x^*\}$. Define the natural retraction $r \colon D^* \to K^*$ by the formula $r(x) = x$ if $x \in K^*$ and $r(x) = x^*$ if $x \notin K^*$. Now, consider $\hat{r} \colon F(D^*) \to F(K^*)$, the extension of r to a continuous homomorphism. It is easy to see that $\ker \hat{r} = U_K$. Since K^* is a discrete space, the group $F(K^*)$ is discrete, so U_K is a normal open subgroup of $F(D^*)$. Let us prove that $\bigcap \gamma = \{e\}$, where e is the identity of $F(D^*)$. Note that an element $x = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \ldots a_n^{\varepsilon_n} \in F(D^*)$, with $a_i \in D^*$ and $\varepsilon_i = \pm 1$, belongs to U_K if and only if the word representing x becomes equal to e when all elements $a_i \in D^* \setminus K^*$ are replaced by x^*. Hence if we choose $K = \{a_1, \ldots, a_n\}$, then $x \neq e$ implies $x \notin U_K$.

Clearly, D^* is a Lindelöf P-space, and hence all finite powers of D^* are also Lindelöf [10]. Therefore, the group $F(D^*)$ is Lindelöf (see also [11]). Suppose that U is an open neighborhood of the neutral element e. Since the group $F(D^*)$ is Lindelöf and $\{e\} = \bigcap \gamma \subseteq U$, we can find a countable subfamily μ of γ such that $\bigcap \mu \subseteq U$. Finally, observe that the intersection of any countable subfamily of γ is in γ. So, γ is a base at e for $F(D^*)$. □

Lemma 2.10. *Let U_K be the normal subgroup of $F(D^*)$ generated by $D^* \setminus K$, where $K \in [D]^{\leq\omega}$. Then $F(D^*) = \langle K \rangle \cdot U_K$.*

Proof. Let g be an element of $F(D^*)$. Then $g = x_1 \cdots x_n$, where each x_i is in $D^* \cup (D^*)^{-1}$. We use mathematical induction on n to prove that g is in $\langle K \rangle \cdot U_K$. If $n = 1$, then $g = x_1$ is either in $K \cup K^{-1} \subseteq \langle K \rangle$ or in $(D^* \setminus K) \cup (D^* \setminus K)^{-1} \subseteq U_K$. In either case, $g \in \langle K \rangle \cdot U_K$. We suppose that $n > 1$ and $x_1 \cdots x_{n-1} = f \cdot u$, where $f \in \langle K \rangle$ and $u \in U_K$.

If $x_n \in D^* \setminus K$, then it is clear that $ux_n \in U_K$, hence $g = f \cdot ux_n \in \langle K \rangle \cdot U_n$. On the other hand, if x_n is in K, then for U_K to be a normal subgroup, $u' = x_n^{-1} u x_n$ is in U_K. Therefore, $g = f x_n \cdot u' \in \langle K \rangle \cdot U_K$. This finishes the proof. □

Proof of Theorem 2.8. For every $K \in [D]^{\leq \omega}$, let $\{g_n^K : n \in \mathbb{N}\}$ be an enumeration of the group $\langle K \rangle$. Without loss of generality, we may suppose that player I chooses open sets of the form U_K (Lemma 2.9 applies here). If player I chooses U_{K_1}, player II chooses $F_1 = \{g_1^{K_1}\}$. In general, if player I chooses U_{K_n}, then player II chooses $F_n = \{g_j^{K_i} : 1 \leq i, j \leq n\}$. We can also assume that $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$. Let $K = \bigcup_{i=1}^{\infty} K_i$. Observe that $\langle K \rangle = \bigcup_{n=1}^{\infty} \langle K_n \rangle = \bigcup_{n=1}^{\infty} F_n$. Finally, since $U_K \subseteq U_{K_n}$ for each n, Lemma 2.10 implies that

$$F(D^*) = \langle K \rangle \cdot U_K = \Big(\bigcup_{n=1}^{\infty} F_n\Big) \cdot U_K = \bigcup_{n=1}^{\infty} (F_n \cdot U_K) \subseteq \bigcup_{n=1}^{\infty} F_n \cdot U_{K_n}.$$

Then, $F(D^*)$ is strictly o-bounded. □

In fact, the above theorem admits a stronger form.

Theorem 2.11. *The product $F(D^*) \times H$ is strictly o-bounded for each strictly o-bounded group H.*

Proof. We can modify slightly the proof of Theorem 2.6 and obtain the proof of our theorem. Indeed, the sets U_K are now the normal subgroups of $F(D^*)$ generated by $D^* \setminus K$, where $K \in [D]^{\leq \omega}$. These sets were used in the proof of Theorem 2.8 and, as before, form a base for the identity that has the following properties:

(1) each U_K is a normal subgroup of $F(D^*)$;
(2) the subsets U_K are clopen in $F(D^*)$;
(3) $|F(D^*)/U_K| \leq \aleph_0$ for each $K \in [D]^{\leq \omega}$.

We may suppose that player I chooses neighborhoods of the form $U_i \times V_i$, where U_i and V_i are neighborhoods of the identity e of $F(D^*)$ and e_H of H respectively, and $U_i = U_{K_i}$, where K_i is in $[D]^{\leq \omega}$, $i \in \mathbb{N}$.

As in the proof of Theorem 2.8, we choose an enumeration $\{g_n^K : n \in \mathbb{N}\}$ of $\langle K \rangle$, and put $E_n = \{x_j^{K_i} : i, j \leq n\}$. At this point, the proof of the theorem continues in the same way as the proof of Theorem 2.6. □

The following example shows that the class of o-bounded groups is not finitely multiplicative. This answers the corresponding problem posed in [9] in the negative. It turns out that the o-bounded group G from [9, Example 8] suits.

Example 2.12. There exists a second countable o-bounded topological group G such that $G \times G$ is not o-bounded.

For every $x \in \mathbb{R}^\omega$, define $\operatorname{supp} x = \{n \in \mathbb{N} : x(n) \neq 0\}$. Let $\{n_k(x) : k \in \omega\}$ be the enumeration of $\operatorname{supp} x$ in the increasing order. Denote by X the set of all $x \in \mathbb{R}^\omega$ such that

$$\lim_{k\to\infty} \frac{x(n_k)}{n_{k+1}(x)} = 0.$$

Consider the subgroup G of $\mathbb{R}^\omega$ generated by X, i.e., $G = \langle X \rangle$. In what follows we use the additive notation for the group operation in $\mathbb{R}^\omega$.

We already know that G is o-bounded. We shall prove that G^2 is not o-bounded describing a sequence $\{U_n : n \in \mathbb{N}\}$ of open neighborhoods of the identity $e \in G$ for which no sequence of finite subsets $\{E_n : n \in \mathbb{N}\}$ in G will make $G^2 = \bigcup_{n=1}^\infty [(E_n \times E_n) + (U_n \times U_n)]$. For every $n \in \mathbb{N}$, let $U_n = G \cap \prod_{j=1}^\infty V_{n,j}$, where $V_{n,j} = (-1,1)$ for $0 \le j \le n$ and $V_{n,j} = \mathbb{R}$ if $j > n$. Now, when considering $E_n + U_n$ the only coordinates of the elements E_n that matter are $0, 1, \dots, n$ since U_n is unrestricted on $\omega \setminus n$ coordinates. So, we may as well only consider E_n where the elements have 0 at each of the $\omega \setminus n$ places. Moreover, we can assume that $E_n \subseteq E_{n+1}$. Let $A_n = \max\{|z(i)| : z \in E_n,\ 0 \le i \le n\}$. Observe that $A_0 < A_1 < \cdots$. We shall prove that $G^2 \neq \bigcup_{n=1}^\infty [(E_n \times E_n) + (U_n \times U_n)]$ for any finite subsets $E_n \subseteq G$. That is, there exists at least one pair of elements $x,\, y \in G$ such that $(x, y) \notin \bigcup_{n=1}^\infty [(E_n \times E_n) + (U_n \times U_n)]$. We construct x and y as follows. Choose $n_0 = 0$, $n_1 = 1$ and set $x(0) = x_0 > A_{n_1}$. We now choose any n_2 such that $x_0/n_2 < 1/2$. Now, for all i, $0 < i < n_2$, we put $x(i) = 0$. Let $y(n_1) = y_{n_1} > A_{n_2}$. Then we choose $n_3 \in \omega$ so that $y_{n_1}/n_3 < 1/3$. We set $y(j) = 0$ if $0 \le j < n_1$ or $n_1 < j < n_3$. We continue in this way to define numbers $\{n_k : k \in \omega\}$. We put $x(n_k) = x_{n_k} > A_{n_{k+1}}$ for k even and such that $x(n_k)/n_{k+2} < 1/(k+2)$. The other values for $x(j)$ so far undefined for $j < n_{k+2}$ are set as 0. Similarly, if k is odd, then define $y(n_k) = y_{n_k} > A_{n_{k+1}}$ and n_{k+2} is defined so that $y(n_k)/n_{k+2} < 1/(k+2)$. It is clear that x,$y \in G$. We claim that $(x, y) \notin \bigcup_{n=1}^\infty [(E_n \times E_n) + (U_n \times U_n)]$. Indeed, suppose that $n \in \mathbb{N}$ and that $n_k \le n < n_{k+1}$. If k is even, then $x_{n_k} > A_{n_{k+1}} \ge A_n$, so $x \notin E_n + U_n$. If k is odd, then $y_{n_k} > A_{n_{k+1}} \ge A_n$, so $y \notin E_n + U_n$. Hence $(x, y) \notin \bigcup_{n=0}^\infty [(E_n \times E_n) + (U_n \times U_n)]$. This shows that G^2 is not o-bounded. □

3. An example of an OF-undetermined group

By Theorem 4.1 of [9], an $\aleph_0$-bounded group G is o-bounded if and only if all second countable continuous homomorphic images of G are o-bounded. Here we show that strictly o-bounded groups cannot be characterized this way, thus answering [9, Problem 4.2] in the negative. In addition, the group G we construct below will be OF-undetermined, that is, neither player I nor player II has a winning strategy in the OF-game on G.

Theorem 3.1. *Under $\diamondsuit$, there exists a topological group G with the following properties:*

(a) *every countable intersection of open sets in G is open;*
(b) *the image $f(G)$ is countable for every continuous homomorphism $f : G \to H$ to a second countable topological group H; in particular, G is o-bounded;*

(c) *G is OF-undetermined, hence not strictly o-bounded.*

Proof. We shall construct G as a subgroup of the group $\mathbb{Z}^{\omega_1}$ endowed with the $\aleph_0$-box topology, where the group $\mathbb{Z}$ has the discrete topology. This will guarantee (a). For every $\alpha < \omega_1$, let $\pi_\alpha \colon \mathbb{Z}^{\omega_1} \to \mathbb{Z}^\alpha$ be the projection and K_α be the kernel of π_α. Then K_α is an open subgroup of $\mathbb{Z}^{\omega_1}$, and we put $N_\alpha = G \cap K_\alpha$. Clearly, the family $\{N_\alpha : \alpha < \omega_1\}$ forms a decreasing base at the neutral element of G. The subgroup G of $\mathbb{Z}^{\omega_1}$ will also satisfy the following strong condition:

(B) $|G| = \aleph_1$, but $\pi_\alpha(G)$ is countable for each $\alpha < \omega_1$.

Let us show that (B) implies (b). Suppose that $f \colon G \to H$ is a continuous homomorphism to a second countable topological group H. Choose a countable base $\{U_n : n \in \mathbb{N}\}$ at the neutral element of H. For every $n \in \mathbb{N}$, there exists an ordinal $\alpha_n < \omega_1$ such that $N_{\alpha_n} \subseteq f^{-1}(U_n)$. Let α be a countable ordinal satisfying $\alpha_n < \alpha$ for each $n \in \mathbb{N}$. Then $N_\alpha \subseteq \ker f$, so by Lemma 2.1 there exists a homomorphism $g \colon \pi_\alpha(G) \to H$ such that $f = g \circ \pi_\alpha$. Since the group $\pi_\alpha(G)$ is countable by (B), we have $|f(G)| \le |\pi_\alpha(G)| \le \omega$. Clearly, every countable group is o-bounded, so Theorem 4.1 of [9] implies that G is o-bounded.

The difficult part of our construction is to guarantee (c). This requires some preliminary work. For a point $x \in \mathbb{Z}^{\omega_1}$, put $\operatorname{supp}(x) = \{\alpha < \omega_1 : x(\alpha) \neq 0\}$ and consider the subgroup Σ of $\mathbb{Z}^{\omega_1}$ defined by

$$\Sigma = \{x \in \mathbb{Z}^{\omega_1} : |\operatorname{supp}(x)| \le \omega\}.$$

It is clear that $|\Sigma| = \mathfrak{c} = \aleph_1$. Actually, our group G will be constructed as a subgroup of Σ. Since $\{N_\alpha : \alpha < \omega_1\}$ is a base at the neutral element of G, we can assume without loss of generality that player I always makes his choice from this family, and this choice, say N_α, is defined by the corresponding ordinal α. Therefore, every possible winning strategy for player II is a function $\psi \colon \mathit{Seq} \to [G]^{<\omega}$, where Seq is the family of all finite sequences $(\alpha_0, \alpha_1, \dots, \alpha_n)$ with $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \omega_1$ and $[G]^{<\omega}$ is the family of all non-empty finite subsets of G.

Denote by Lim the set of all infinite limit ordinals in ω_1. For every $\alpha \in \mathit{Lim}$, denote by $\mathit{Seq}(\alpha)$ the family of all finite sequences $(\beta_0, \beta_1, \dots, \beta_n)$, where $\beta_0 < \beta_1 < \cdots < \beta_n < \alpha$. Using $\Diamond$ and Lemma 2 of [Fed], we can find a family $\{\psi_\alpha : \alpha \in \mathit{Lim}\}$ satisfying the following conditions:

(i) $\psi_\alpha \colon \mathit{Seq}(\alpha) \to [\mathbb{Z}^\alpha]^{<\omega}$ is a function for each $\alpha \in \mathit{Lim}$;

(ii) for every function $\psi \colon \mathit{Seq} \to [\Sigma]^{<\omega}$ satisfying $|\pi_\gamma(\psi(\mathit{Seq}))| \le \omega$ for each $\gamma < \omega_1$, there exists $\alpha \in \mathit{Lim}$ such that $\psi_\alpha = \pi_\alpha \circ \psi|_{\mathit{Seq}(\alpha)}$, i.e., $\psi_\alpha(\beta_0, \dots, \beta_n) = \pi_\alpha(\psi(\beta_0, \dots, \beta_n))$ for any sequence $\beta_0 < \cdots < \beta_n < \alpha$.

If $\beta < \alpha < \omega_1$, denote by π^α_β the projection of $\mathbb{Z}^\alpha$ to $\mathbb{Z}^\beta$. Now, we will construct a sequence $\{G_\alpha : \alpha < \omega_1\}$ satisfying the following conditions for each $\alpha < \omega_1$:

(1) G_α is a countable subgroup of $\mathbb{Z}^\alpha$;

(2) $\pi^\alpha_\beta(G_\alpha) = G_\beta$ if $\beta < \alpha$;

(3) $G_{\alpha+1} = G_\alpha \times \mathbb{Z}$;
(4) if $\alpha \in Lim$, then $\bigcup_{\beta<\alpha}(G_\beta \times \{0_\beta^\alpha\}) \subseteq G_\alpha$, where 0_β^α is the neutral element of $\mathbb{Z}^{\alpha\setminus\beta}$;
(5) if $\alpha \in Lim$, then ψ_α is not a winning strategy for player II in G_α.

Put $G_0 = \mathbb{Z}$. Suppose that for some $\alpha < \omega_1$, we have defined a sequence $\{G_\beta : \beta < \alpha\}$ satisfying (1)–(5). If α is non-limit, say $\alpha = \beta + 1$, then we put $G_\alpha = G_\beta \times \mathbb{Z} \le \mathbb{Z}^\beta \times \mathbb{Z} = \mathbb{Z}^\alpha$. Let us consider, therefore, the case $\alpha \in Lim$. Set $H = \bigcup_{\beta<\alpha}(G_\beta \times \{0_\beta^\alpha\})$. Clearly, the subgroup H of $\mathbb{Z}^\alpha$ is countable. Fix a sequence $\{\beta_n : n \in \mathbb{N}\} \subseteq \alpha$ such that $\lim_{n\in\mathbb{N}} \beta_n = \alpha$. For every $n \in \mathbb{N}$, put $F_n = \psi_\alpha(\beta_0, \ldots, \beta_n)$. We claim that there exists a point $x \in \mathbb{Z}^\alpha \setminus \bigcup_{n\in\mathbb{N}}(U_n + F_n)$ such that $\pi_{\beta_n}^\alpha(x) \in G_{\beta_n}$ for each $n \in \mathbb{N}$, where $U_n = \pi_\alpha(K_{\beta_n}) \subseteq \mathbb{Z}^\alpha$. Indeed, choose a point $x_0 \in G_{\beta_0} \setminus \pi_{\beta_0}^\alpha(F_0)$. By induction, with the help of (2) and (3), define a sequence $\{x_n : n \in \mathbb{N}\}$ such that $x_n \in G_{\beta_n} \setminus \pi_{\beta_n}^\alpha(F_n)$ and $\pi_{\beta_n}^{\beta_{n+1}}(x_{n+1}) = x_n$ for each $n \in \mathbb{N}$. This is possible because (2) and (3) together imply that for every β, γ with $\gamma < \beta < \alpha$ and every $z \in G_\gamma$, there exist infinitely many $y \in G_\beta$ with $\pi_\gamma^\beta(y) = z$. Let $x \in \mathbb{Z}^\alpha$ be a point satisfying $\pi_{\beta_n}^\alpha(x) = x_n$ for each $n \in \mathbb{N}$. It is easy to see then that $x \notin \bigcup_{n\in\mathbb{N}}(U_n + F_n)$. Put $G_\alpha = H \oplus \langle x \rangle$. Then $G_\alpha \setminus \bigcup_{n\in\mathbb{N}}(U_n + F_n) \neq \varnothing$, i.e., ψ_α is not a winning strategy for player II in G_α. Since $\pi_\beta^\alpha(H) = G_\beta$ and $\pi_\beta^\alpha(x) \in G_\beta$ for each $\beta < \alpha$, we conclude that $\pi_\beta^\alpha(G_\alpha) = G_\beta$ for all $\beta < \alpha$. Clearly, the group G_α is countable, so that the sequence $\{G_\gamma : \gamma \le \alpha\}$ satisfies (1)–(5). This finishes our construction.

Consider the subgroup $G = \bigcup_{\alpha<\omega_1}(G_\alpha \times \{0(\alpha)\})$ of $\mathbb{Z}^{\omega_1}$, where $0(\alpha)$ is the neutral element of $\mathbb{Z}^{\omega_1\setminus\alpha}$ for every $\alpha < \omega_1$. Then $G \subseteq \Sigma$, $|G| = \aleph_1$ and $\pi_\alpha(G) = G_\alpha$ for each $\alpha < \omega_1$, i.e., G satisfies (B). Let us verify that G is OF-undetermined.

First, we show that player II has no winning strategy. Let $\psi\colon Seq \to [G]^{<\omega}$ be a function. Then $|\pi_\alpha(\psi(Seq))| \le |[G_\alpha]^{<\omega}| \le \omega$ for each $\alpha < \omega_1$, so (ii) implies that there is $\alpha \in Lim$ such that $\psi_\alpha = \pi_\alpha \circ \psi|_{Seq(\alpha)}$. However, ψ_α fails to be a winning strategy for player II in G_α, and hence one can find an increasing sequence $\beta_0 < \cdots < \beta_n < \cdots < \alpha$ and a point $x \in G_\alpha \setminus \bigcup_{n\in\mathbb{N}}(U_n + F_n)$, where $F_n = \psi_\alpha(\beta_0, \ldots, \beta_n)$ and $U_n = \pi_\alpha(K_{\beta_n})$ for each $n \in \mathbb{N}$. Choose an element $y \in G$ with $\pi_\alpha(y) = x$. Since $F_n = \pi_\alpha(\psi(\beta_0, \ldots, \beta_n))$ for every integer n, we conclude that $y \notin \bigcup_{n\in\mathbb{N}}(K_{\beta_n} + \psi(\beta_0, \ldots, \beta_n))$, i.e., ψ is not a winning strategy for player II in G.

Let us show that player I has no winning strategy. Since player I always chooses elements of the base $\{N_\alpha : \alpha < \omega_1\}$, every possible winning strategy for him is a function $\phi\colon \mathcal{F} \to \omega_1$, where $\mathcal{F}$ is the family of all finite sequences $(F_0, \ldots, F_n)$ with $F_0, \ldots, F_n \in [G]^{<\omega}$. The equality $\alpha = \phi(F_0, \ldots, F_n)$ means that player I chooses the neighborhood N_α at the step $n + 1$. Suppose that $\phi\colon \mathcal{F} \to \omega_1$ is a function, and β_0 is the player I's choice at the first step. Since $G \subseteq \Sigma$, for every $x \in G$ there exists $\alpha < \omega_1$ such that $\mathrm{supp}(x) \subseteq \alpha$. Therefore, for every finite subset F of G, we can define

$$s(F) = \min\{\alpha < \omega_1 : \mathrm{supp}(x) \subseteq \alpha \text{ for each } x \in F\}.$$

Given an ordinal $\beta < \omega_1$, put

$$\lambda(\beta)=\sup\{\phi(F_0,\dots,F_n) : (n \in \mathbb{N}) \wedge (\forall i \le n)[F_i \in [G]^{<\omega} \wedge s(F_i) \le \beta]\}.$$

Define a strictly increasing sequence $\{\beta_n : n \in \mathbb{N}\} \subseteq \omega_1$ such that $\lambda(\beta_n) \le \beta_{n+1}$ for each $n \in \mathbb{N}$ and put $\alpha = \sup_{n\in\mathbb{N}} \beta_n$. Let $G_\alpha = \{x_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, $H_n = G_{\beta_n} \times \{0(n)\}$ is a subgroup of G, where $0(n)$ is the neutral element of $\mathbb{Z}^{\omega_1\setminus\beta_n}$ (see (4)). It is easy to define a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of G satisfying the following conditions for all $n \in \mathbb{N}$:

(6) $F_n \subseteq H_n$;

(7) $\pi_{\beta_n}(x_i) \in \pi_{\beta_n}(F_n)$ for $i = 0,\dots,n$.

We claim that if $x \in G$ and $\pi_\alpha(x) = x_i$, then $x \in N_{\beta_n} + F_n$ for each $n \ge i$. Indeed, by (7), we have $\pi_{\beta_n}(x_i) \in \pi_{\beta_n}(F_n)$, so $x|_{\beta_n} = x_i|_{\beta_n} = y|_{\beta_n}$ for some $y \in F_n$. This implies that $x \in K_{\beta_n} + y \subseteq K_{\beta_n} + F_n$, and hence $x \in N_{\beta_n} + F_n$. This proves the claim. From (6) it follows that $\gamma_{n+1} = \phi(F_0,\dots,F_n) \le \beta_{n+1}$ for each $n \in \mathbb{N}$, so we conclude that

$$G \subseteq \bigcup_{n=1}^{\infty}(N_{\beta_n} + F_n) \subseteq \bigcup_{n=1}^{\infty}(N_{\gamma_n} + F_n).$$

Therefore, ϕ cannot be a winning strategy for player I, and hence G is OF-undetermined. It remains to note that an OF-undetermined group is not strictly o-bounded. This completes the proof. □

4. Open problems

The class of o-bounded groups is not productive by Example 2.12. This motivates the following

Problem 4.1. *Does there exist strictly o-bounded groups G and H such that the product $G \times H$ is not strictly o-bounded?*

It is known that the product $G \times H$ is o-bounded whenever G is o-bounded and H is either a σ-compact or Comfort-like group (see [9, Theorem 5.3] and our Corollary 2.5). We do not know whether the multiplication by a strictly o-bounded group can destroy o-boundedness:

Problem 4.2. *Is it true that a product of an o-bounded group by a strictly o-bounded group is o-bounded?*

Theorem 2.8 suggests the following problem:

Problem 4.3. *Characterize the spaces X such that the free (Abelian) topological group $F(X)$ $(A(X))$ is o-bounded or strictly o-bounded.*

In fact, the above problem splits up into four distinct subproblems.

By [9, Theorem 4.1], an $\aleph_0$-bounded group G is o-bounded iff all second countable continuous homomorphic images of G are o-bounded. We do not know whether "$\aleph_0$-bounded" can be omitted here:

Problem 4.4. *Suppose that all second countable continuous homomorphic images of an (Abelian) topological group G are o-bounded. Is then G o-bounded?*

In Theorem 3.1 we constructed an example of an OF-undetermined topological group G. Our group G is very far from being metrizable: it is a non-discrete P-group. The following question is considered by T. Banakh in [4]:

Problem 4.5. *Does there exist an OF-undetermined metrizable group?*

Since every OF-undetermined group is o-bounded and o-bounded groups are $\aleph_0$-bounded, such a group has to be second countable. It is also shown in [4] that the answer to the above question is "yes" under Martin's Axiom.

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