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Flows equivalences

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Abstract. Given a differential equation on an open set $\mathcal{O}$ of an n-manifold we can associate to it a pseudo-flow, that is, a flow whose trajectories may not be defined in the entire real line. In this paper we prove that this pseudo-flow is always equivalent to a flow with its trajectories defined in all $\mathbb{R}$. This result extends a similar result of Vinograd stated in the n-dimensional euclidean space.

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1. Introduction.

It is well known that given a C^r-flow ($r \geq 1$) on an n-manifold $\mathcal{M}$, $\Psi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$, we can associate to it a C^{r-1}-autonomous differential equation $y' = f(y)$. Where f maps $\mathcal{M}$ onto its tangent bundle, $T\mathcal{M}$, in the following way $f(y) = \frac{\partial \Psi}{dt}(0, y)$.

The converse does not work in general because the solutions of a differential equation could not be defined in the entire real line. For example, if we take the autonomous differential equation $(x', y') = (1, 1 + \tan^2(x))$, the solutions are defined for each initial condition (x_0, y_0) in an interval of length π. Then we can not associate to this autonomous differential equation a flow. However, if the manifold is compact, the converse does work [1, Theorem 4, §1.9] and [5, p.11].

Let us introduce some terminology. Given two flows Ψ and Φ on an n-manifold $\mathcal{M}$ we say that they are C^r-equivalent if there exists a C^r-diffeomorphism $h : \mathcal{M} \to \mathcal{M}$ such that h conserves the orbits of Φ. That is, the subsets $h(\Phi(\mathbb{R}, p))$ and $\Psi(\mathbb{R}, h(p))$ of $\mathcal{M}$ are equal for any $p \in \mathcal{M}$. Moreover the orientations of the curves $\Psi_{h(p)}(t) = \Psi(t, h(p))$ and $h \circ \Phi_p(t) = h \circ \Phi(t, p)$ coincide for any $p \in \mathcal{M}$, that is, there exists a continuous increasing map $i_p : \mathbb{R} \to \mathbb{R}$ for which $h \circ \Phi_p(t) = \Psi_{h(p)}(i_p(t))$. When we use the norm of a vector $x \in \mathbb{R}^n$ we are always using the norm $\|x\| = \|x\|_\infty = \max_{i \in \{1,2,\ldots,n\}} \{|x_1|, |x_2|, \ldots, |x_n|\}$.

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By $\mathbb{R}_+$ we denote the set of positive real numbers. As usual, given $\mathcal{O} \subset \mathcal{M}$, $\mathrm{Bd}(\mathcal{O})$ denote the topological boundary of the set $\mathcal{O}$.

From now on, when speaking of C^0-differential equations they are supposed to be continuous and locally Lipschitz. Let $r \in \mathbb{N} \cup \{0\}$ and take a C^r-differential equation $y' = f(y)$ on an open set $\mathcal{O} \subset \mathcal{M}$, $f : \mathcal{O} \to T\mathcal{O}$. The classical theory of differential systems assures that there exists a C^r-map $\psi : \mathcal{D} \to \mathcal{O}$ called pseudo-flow, where $\mathcal{D}$ is an open subset of $\mathbb{R} \times \mathcal{O}$ and for each $p \in \mathcal{O}$ the curve $\psi_p(t) = \psi(t, p)$ is the solution of the equation $y' = f(y)$ with initial condition $y(0) = p$. Analogously to the definition of equivalence between flows we say that two pseudo-flows $\phi : \mathcal{D} \subset \mathbb{R} \times \mathcal{O} \to \mathcal{O}$ and $\psi : \mathcal{E} \subset \mathbb{R} \times \mathcal{O} \to \mathcal{O}$ are *C^r-equivalent* if there exists a C^r-diffeomorphism $h : \mathcal{O} \to \mathcal{O}$ such that h conserves the orbits of ϕ and the orientations of the curves $\psi_{h(p)}(t)$ and $h \circ \phi_p(t)$ coincide for all $p \in \mathcal{O}$. With this terminology we will say that two autonomous differential equations are C^r-equivalent if their associated pseudo-flows are C^r-equivalent, moreover the diffeomorphism h will be called *equivalence diffeomorphism*.

The basic question in which we are interested is to prove that for any C^r-autonomous differential system in an open set $\mathcal{O}$, we can find a C^r-equivalent autonomous differential equation such that the associated pseudo-flow is in fact a flow, that is, defined in all $\mathbb{R} \times \mathcal{O}$. This question was solved by Vinograd [4, pp. 19-21] when the phase space is $\mathbb{R}^n$.

Theorem 1.1 (Vinograd). *Let $\mathcal{O}$ be an open set of $\mathbb{R}^n$ and let $f : \mathcal{O} \to \mathbb{R}^n$ be a C^r-map ($r \geq 0$). Then there exists a C^r-map $g : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}^n$ such that the equations $y' = f(y)$ and $y' = g(y)$ are C^r-equivalent and the associated pseudo-flow to g is a flow. Moreover, the equivalence diffeomorphism is the identity map. (When $r = 0$ we consider f and g to be locally Lipschitz)*

The aim of this paper is to prove the following theorem that generalizes the previous one:

Theorem 1.2 (Main Result). *Let $\mathcal{M}$ be an n-manifold, $\mathcal{O}$ an open set of $\mathcal{M}$ and $f : \mathcal{O} \to T\mathcal{O}$ a C^r-map ($r \geq 0$) . Then there exists a C^r-map $g : \mathcal{O} \to T\mathcal{O}$ such that the equations $y' = f(y)$ and $y' = g(y)$ are C^r-equivalent and the associated pseudo-flow to g is a flow. (When $r = 0$ we consider f and g to be locally Lipschitz)*

Section 2 is devoted to state some classical results that we need in the proof of our result. We also construct a positive C^∞-function that vanish only in the boundary of $\mathcal{O}$. This function will be essential in the proof of the Main Theorem in Section 3.

2. Preliminary Results

In the sequel we are going to use the Whitney theorem that provides a C^∞ n-manifold $\mathcal{M}$ embedded in $\mathbb{R}^{2n+1}$ (see [2, §1.3]). Another Whitney theorem about function extensions is stated and used in the proof of Lemma 2.4 to construct a scalar C^∞-function $f : \mathbb{R}^n \to \mathbb{R}$ vanishing only in the boundary of an open set $\mathcal{O}$ and being strictly positive in $\mathcal{O}$.

Theorem 2.1 (Whitney). *Let $\mathcal{M}$ be an n-manifold of class C^r, $r \geq 1$. Then there exists a C^r-embedding $f : \mathcal{M}^n \to \mathbb{R}^{2n+1}$ such that $f(\mathcal{M})$ is a closed C^∞-submanifold of $\mathbb{R}^{2n+1}$.*

We will use another less known Whitney's Theorem. Its proof can be found combining [7, p. 177,Th. 4] and [8]. We introduce some necessary terminology for its statement: if $\beta \in (\{0\} \cup \mathbb{N})^n$, $y \in \mathbb{R}^n$ and f is a map defined on an open subset of $\mathbb{R}^n$, we denote $\beta! = \beta_1!\beta_2! \dots \beta_n!$, $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$, $y^\beta = y_1^{\beta_1} y_2^{\beta_2} \dots y_n^{\beta_n}$ and $D_\beta f(y) = \frac{\partial^{\beta_1+\beta_2+\cdots+\beta_n}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \dots \partial x_n^{\beta_n}} f(y)$. As usual we mean $D_0 f = f$.

Theorem 2.2 (Whitney). *Let $C \subset \mathbb{R}^n$ be a closed set (as a subset of $\mathbb{R}^n$). Then the following statements hold.*

(1) *Let $f^0 : C \to \mathbb{R}^m$ be a bounded Lipschitz map. Then there is a bounded Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(x) = f^0(x)$ for any $x \in C$.*
(2) *Let $1 \leq k \leq \infty$ and let $f^\beta : C \to \mathbb{R}^m$ be arbitrary maps for any $\beta \in (\{0\} \cup \mathbb{N})^n$ with $0 \leq |\beta| \leq k$. Let $F^{\gamma,r} : C \times C \to \mathbb{R}^m$ be defined by*

$$F^{\gamma,r}(x,y) = \frac{f^\gamma(y) - \sum_{0 \leq |\beta| \leq r} \frac{f^{\gamma+\beta}(x)(y-x)^\beta}{\beta!}}{\|x-y\|^r}$$

if $x \neq y$ and

$$F^{\gamma,r}(x,x) = 0$$

otherwise, for any $\gamma \in (\{0\} \cup \mathbb{N})^n$ and $0 \leq r < \infty$ with $|\gamma| + r \leq k$. Suppose that all maps $F^{\gamma,r}$ are continuous. Then there is a C^k map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $D_\beta f = f^\beta$ for any $\beta \in (\{0\} \cup \mathbb{N})^n$, $0 \leq |\beta| \leq k$.

The following result is an easy consequence of the previous Theorem:

Corollary 2.3. *Let $C \subset \mathbb{R}^n$ be a closed set decomposed into disjoint sets A and B, $C = A \cup B$. Given two real numbers a and b define $f_* : C \to \mathbb{R}^m$ as follows: $f_*(x) = a$ for any $x \in A$ and $f_*(x) = b$ for any $x \in B$. Then there is a C^∞-map $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = f_*(x)$ for any $x \in C$ and any partial derivate of f is equal to 0 in C.*

Proof. Take for each $\beta \in (\{0\} \cup \mathbb{N})^n$, $f^\beta : C \to \mathbb{R}$ with $f^\beta \equiv 0$ for any $0 < |\beta| < \infty$ and $f^0 = f_*$. It is clear that the functions f^β satisfy the conditions of part 2 of Theorem 2.2. Then there exists a C^∞-function $f : \mathbb{R}^n \to \mathbb{R}$ that extends f^0 and whose derivates are 0 in C. □

We also need some previous lemmas:

Lemma 2.4. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a nonclosed set. There exists a C^∞-map $f : \mathbb{R}^n \to [0,1[$ such that $f(x) = 0$ for any $x \in \mathrm{Bd}(\mathcal{O})$ and $f(x) \in]0,1[$ for any $x \in \mathcal{O}$.*

Proof. We are going to construct the C^∞-map as the sum of a function series. Thus we are going to construct C^∞-functions $f_i : \mathbb{R}^n \to [0,1[$ for every $i \in \mathbb{N}$. Define $C_j = \varnothing$ for $j \in \mathbb{Z} \backslash \mathbb{N}$, $C_1 = \{x \in \mathcal{O} : 1 < d(x, \mathrm{Bd}(\mathcal{O})\}$ and for $j \in \mathbb{N} \backslash \{1\}$ consider $C_j = \{x \in \mathcal{O} : \frac{1}{j} < d(x, \mathrm{Bd}(\mathcal{O})) < \frac{1}{j-1}\}$ (eventually $C_j = \varnothing$ for j

small). Let $A = \overline{C_i}$ and $B = \cup_{j>i+1}\overline{C_j} \bigcup \cup_{j<i-1}\overline{C_j} \bigcup \mathrm{Bd}(\mathcal{O})$. It is clear that $A \cap B = \varnothing$ and A and B are closed sets. We define

$$g_{i*}(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in B. \end{cases}$$

Using that $\mathbb{R}$ and $]-2,2[$ are C^∞-diffeomorphic (being 0 a fixed point of the diffeomorphism) and the precedent Corollary, we extend g_{i*} to $\mathbb{R}^n$ obtaining the C^∞-function $g_i : \mathbb{R}^n \to]-2,2[$ that vanishes in $\mathrm{Bd}(\mathcal{O})$ and verifies $g_i(x) = 1$ if $x \in \overline{C_i}$. Define $f_i = \frac{g_i^2}{4}$ and obtain the C^∞-function $f_i : \mathbb{R}^n \to [0,1[$ with $f_i(x) = 0$ for every $x \in \mathrm{Bd}(\mathcal{O})$.

Let us define $f(x) = \sum_{i=0}^{\infty} \frac{1}{i^2} f_i(x)$ which is clearly uniformly convergent because it is bounded by the series of real number $\sum_{i=0}^{\infty} \frac{1}{i^2}$. Moreover, the function f is C^∞ in each point $x \in \mathcal{O}$ because, in a neighborhood of x the function f is at most the sum of three C^∞-functions f_j different of 0. Then f is C^∞. Obviously the function f can not vanish in the set $\mathcal{O}$. That concludes the proof. □

Finally we state a Theorem about maximal solutions of differential equations. Find the proof, e.g., in [3, p.195, Th. 25.9].

Theorem 2.5. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous bounded map. Then every maximal solution of $y' = f(y)$ is defined in the entire real line.*

3. Proof of The Main Result

Theorem (Main Result). *Let $\mathcal{M}$ be an n-manifold, $\mathcal{O}$ an open set of $\mathcal{M}$ and $f : \mathcal{O} \to T\mathcal{O}$ a C^r-map $(r \geq 0)$. Then there exists a C^r-map $g : \mathcal{O} \to T\mathcal{O}$ such that the equations $y' = f(y)$ and $y' = g(y)$ are C^r-equivalent and the associated pseudo-flow to g is a flow. (When $r = 0$ we consider f and g to be locally Lipschitz)*

Proof. As $\mathcal{M}$ is embedded in $\mathbb{R}^{2n+1}$ thanks to Whitney theorem, $i : \mathcal{M} \to \mathbb{R}^{2n+1}$, we can see the manifold $\mathcal{M}$ and the vector field f in $\mathbb{R}^{2n+1}$. Thus we have $f : \mathcal{O} \subset \mathcal{M} \to \mathbb{R}^{2n+1}$. Denote by f_j the j-th component of f, $f_j : \mathcal{M} \to \mathbb{R}$.

Let $\lambda : \mathbb{R}^{2n+1} \to \mathbb{R}$ be a C^∞-function equal to 0 in $\mathrm{Bd}(\mathcal{O})$ and strictly positive outside (see precedent section, Lemma 2.3). Define $\gamma : \mathcal{O} \to \mathbb{R}$ as $\gamma(x) = \frac{1}{\exp\left(\sum_{j=0}^{2n+1} f_j(x)^2\right)}$ and $G : \mathcal{M} \to \mathbb{R}^{2n+1}$ as

$$G(x) = \begin{cases} \gamma(x)\lambda(x)f(x) & \text{si } x \in \mathcal{O}, \\ 0 & \text{si } x \notin \mathcal{M}\backslash\mathcal{O}. \end{cases}$$

G is bounded by $\frac{1}{\sqrt{2e}}$

$$\begin{aligned} \|\gamma(x)\lambda(x)f(x)\|_\infty &\leq \|\gamma(x)f(x)\|_\infty \\ &\leq \sup_{a\in\mathbb{R}}\left(\frac{a}{e^{a^2}}\right) \\ &\leq \frac{1}{\sqrt{2e}} \end{aligned}$$

and it is locally lipstchitz in $\mathcal{M}$: it is clear that G is locally Lipschitz inside and outside of $\mathcal{O}$, let us now show that G is locally Lipschitz in $\mathrm{Bd}(\mathcal{O})$.

Take $y \in \mathrm{Bd}(\mathcal{O})$ and x in a neighborhood U_y of y. Then

$$\begin{aligned} \|G(x) - G(y)\| &= \|\gamma(x)\lambda(x)f(x)\| \\ &= \|\gamma(x)\lambda(x)f(x)\| \\ &\leq \|\gamma(x)f(x)\|\|\lambda(x)\| \\ &\leq \tfrac{1}{\sqrt{2e}}\|\lambda(x)\|. \end{aligned}$$

Since λ is C^∞ it will be locally Lipschitz and taking U_x small enough we will have

$$\|G(x) - G(y)\| \leq \frac{1}{\sqrt{2e}}\|\lambda(x)\| \leq \frac{1}{\sqrt{2e}}M\|x-y\|.$$

Therefore G is locally Lipschitz in $\mathcal{M}$.

By Theorem 2.1 $\mathcal{M}$ is closed in $\mathbb{R}^{2n+1}$. Now we can use Theorem 2.2 for extending G to a locally Lipschitz function $G^1 : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ with $G^1|_{\mathcal{M}} = G$. Hence the autonomous differential equation $y' = G^1(y)$ has uniqueness of solutions and using Theorem 2.5, the solutions are defined in the entire real line.

Denote by ψ the pseudo-flow associated to f and by ϕ the restriction of the flow associated to G^1 to $\mathbb{R} \times \mathcal{O}$. Notice that ϕ is the pseudo-flow associated to $y' = g(y)$ where $g : \mathcal{O} \to T\mathcal{O}$ is defined by $g(y) = G^1(y)$. We must see that ϕ and ψ are C^r-equivalent with equivalence diffeomorphism $\mathrm{Id} : \mathcal{O} \to \mathcal{O}$, that is, we must prove that the orbits of ψ are orbits of ϕ with the same orientation. Let $y : I = (a,b) \to \mathcal{O}$ be an orbit of ψ, that is $y'(t) = f(y(t))$. Consider the real function $s : I = (a,b) \to \mathbb{R}$ defined by $s(t) = c + \int_c^t \frac{1}{\gamma(y(u))\lambda(y(u))}\mathrm{d}u$ with $c \in I$. As $s'(t) = \frac{1}{\gamma(y(t))\lambda(y(t))} > 0$, s is strictly increasing and there exists its inverse $t : s(I) \to (a,b)$. Define $z : s(I) \to \mathcal{O}$ as $z(s) = y(t(s))$ and notice that

$$z'(s) = y'(t(s))\frac{1}{s'(t(s))} = f(y(t(s)))\gamma(y(t(s)))\lambda(y(t(s))) = G(z(s)) = g(z(s))$$

and $z(c) = y(t(c)) = y(c)$. Thus the orbits of ψ and ϕ coincide and also their orientations because s is strictly increasing. □

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