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Weak completeness of the Bourbaki quasi-uniformity

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Abstract. The concept of semicompleteness (weaker than half-completeness) is defined for the Bourbaki quasi-uniformity of the hyperspace of a quasi-uniform space. It is proved that the Bourbaki quasi-uniformity is semicomplete in the space of nonempty sets of a quasi-uniform space $(X,\mathcal{U})$ if and only if each stable filter on $(X,\mathcal{U}^*)$ has a cluster point in $(X,\mathcal{U})$. As a consequence the space of nonempty sets of a quasi-pseudometric space is semicomplete if and only if the space itself is half-complete. It is also given a characterization of semicompleteness of the space of nonempty $\mathcal{U}^*$-compact sets of a quasi-uniform space $(X,\mathcal{U})$ which extends the well known Zenor-Morita theorem.

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1. Introduction

Our basic reference for quasi-uniform spaces is [8].

A (base $\mathcal{B}$ of a) quasi-uniformity $\mathcal{U}$ on a set X is a (base $\mathcal{B}$ of a) filter $\mathcal{U}$ of binary relations (called entourages) on X such that (a) each element of $\mathcal{U}$ contains the diagonal Δ_X of $X \times X$ and (b) for any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ satisfying $V \circ V \subseteq U$.

Let us recall that if $\mathcal{U}$ is a quasi-uniformity on a set X, then $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is also a quasi-uniformity on X called *the conjugate* of $\mathcal{U}$. The uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ will be denoted by $\mathcal{U}^*$. If $U \in \mathcal{U}$, the entourage $U \cap U^{-1}$ of $\mathcal{U}^*$ will be denoted by U^*.

Each quasi-uniformity $\mathcal{U}$ on X induces a topology $\mathcal{T}(\mathcal{U})$ on X, defined as follows:

$$\mathcal{T}(\mathcal{U}) = \{A \subseteq X : \text{for each } x \in A \text{ there is } U \in \mathcal{U} \text{ such that } U(x) \subseteq A\}.$$

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If $(X,\mathcal{T})$ is a topological space and $\mathcal{U}$ is a quasi-uniformity on X such that $\mathcal{T}=\mathcal{T}(\mathcal{U})$ we say that $\mathcal{U}$ is compatible with $\mathcal{T}$.

A quasi-uniform space $(X,\mathcal{U})$ is *precompact* if for each $U\in\mathcal{U}$ there exists a finite subset F of X such that $X=U(F)$. $(X,\mathcal{U})$ is $\mathcal{U}^{-1}$*-precompact* if $(X,\mathcal{U}^{-1})$ is precompact and $(X,\mathcal{U})$ is $\mathcal{U}^*$*-precompact* (totally bounded) if the uniform space $(X,\mathcal{U}^*)$ is precompact.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-pseudometric space (X,d) is called right K-Cauchy [12] if for each $\varepsilon>0$ there is $k\in\mathbb{N}$ such that $d(x_n,x_m)<\varepsilon$ for each $n\geq m\geq k$. (X,d) is said to be right K-sequentially complete if each right K-Cauchy sequence converges. A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called right K-Cauchy [13] if for each $U\in\mathcal{U}$ there is an $F\in\mathcal{F}$ such that $U^{-1}(x)\in\mathcal{F}$ for each $x\in F$. $(X,\mathcal{U})$ is said to be right K-complete if each right K-Cauchy filter converges.

Obviously a quasi-pseudometric space (X,d) is right K-sequentially complete if the quasi-uniformity $\mathcal{U}_d$ is right K-complete. It is known that the converse holds for regular spaces [2].

A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called left K-Cauchy [13] if for each $U\in\mathcal{U}$ there is an $F\in\mathcal{F}$ such that $U(x)\in\mathcal{F}$ for each $x\in F$. $(X,\mathcal{U})$ is said to be left K-complete if each left K-Cauchy filter converges.

A quasi-uniform space $(X,\mathcal{U})$ is half complete [7], if each Cauchy filter on $(X,\mathcal{U}^*)$ converges in $(X,\mathcal{U})$.

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two quasi-uniform spaces. A mapping $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ is said to be quasi-uniformly continuous if for each $V\in\mathcal{V}$ there is $U\in\mathcal{U}$ such that $(f(x),f(y))\in V$ whenever $(x,y)\in U$.

Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\mathcal{P}_0(X)$ be the collection of all nonempty subsets of X. The Bourbaki (Hausdorff) quasi-uniformity on $\mathcal{P}_0(X)$ is defined by $\mathcal{U}_H=\{U_H:U\in\mathcal{U}\}$, where U_H is defined by $U_H=\{(A,B)\in\mathcal{P}_0(X):B\subseteq U(A)\text{ and }A\subseteq U^{-1}(B)\}$ for each $U\in\mathcal{U}$ (see [3] and [11]).

Let $(X,\mathcal{U})$ be a quasi-uniform space. Let denote by $\mathcal{K}_0(X)$ (resp. $\mathcal{K}_0^{-1}(X)$, $\mathcal{K}_0^*(X)$) the family of nonempty compact (resp. $\mathcal{U}^{-1}$-compact, $\mathcal{U}^*$-compact) subsets of X, by $\mathcal{F}_0(X)$ the family of nonempty finite subsets of X, by $\mathcal{C}_0(X)$ (resp. $\mathcal{C}_0^{-1}(X)$, $\mathcal{C}_0^*(X)$) the family of nonempty closed (resp. $\mathcal{U}^{-1}$-closed, $\mathcal{U}^*$-closed) subsets of X and by $\mathcal{PC}_0(X)$ (resp. $\mathcal{PC}_0^{-1}(X)$, $\mathcal{PC}_0^*(X)$) the family of nonempty precompact (resp. $\mathcal{U}^{-1}$-precompact, $\mathcal{U}^*$-precompact) subsets of X. We will use the same symbol $\mathcal{U}_H$ to denote the restriction of $\mathcal{U}_H$ to any of the previous subspaces.

In this paper the concept of semicompleteness of the Bourbaki quasi uniformity is introduced and used to extend the main theorems concerning completeness in uniform (metric) spaces to the setting of quasi-uniform (quasi-pseudometric) spaces.

The well-known Zenor-Morita theorem states that a uniform space $(X,\mathcal{U})$ is complete if and only if $(\mathcal{K}_0(X),\mathcal{U}_H)$ is complete. In [5] it is proved that a compactly symmetric quasi-uniform space $(X,\mathcal{U})$ is complete if and only if $(\mathcal{K}_0(X),\mathcal{U}_H)$ is complete, providing a generalization of the Zenor-Morita theorem for compactly symmetric quasi-uniform spaces. Here completeness is meant in the sense used by Fletcher and Lindgren in their monograph [8]. In section 3 it is given a generalization of the Zenor-Morita theorem for quasi-uniform spaces in terms of semicompleteness.

Burdick [4, Corollary 2], based on former work of Isbell [9], answered a question of Császár [6] in the affirmative by proving the following characterization: The Hausdorff uniformity on $\mathcal{P}_0(X)$ of a uniform space $(X,\mathcal{U})$ is complete if and only if each stable filter on $(X,\mathcal{U})$ has a cluster point. In [11] it is proved a satisfactory generalization of this result to the setting of quasi-uniform spaces, since it was proved that $(\mathcal{P}_0(X),\mathcal{U}_H)$ is right K-complete if and only if each stable filter on the quasi-uniform space $(X,\mathcal{U})$ has a cluster point in $(X,\mathcal{U})$. In section 3 it is given another generalization of Isbell-Burdick theorem for quasi-uniform spaces. In particular it is proved that $(\mathcal{P}_0(X),\mathcal{U}_H)$ is semicomplete if and only if each stable filter on $(X,\mathcal{U}^*)$ has a cluster point in $(X,\mathcal{U})$. Moreover, a characterization of half completeness of $(\mathcal{P}_0(X),\mathcal{U}_H)$ is obtained in terms of doubly stable filters on $(X,\mathcal{U})$.

It is known (see e.g. [4, Corollary 6]) that the Hausdorff metric of a (bounded) metric space (X,d) is complete if and only if (X,d) is complete. In [11] it is proved a satisfactory generalization of this result to the setting of quasi-pseudometric spaces, since it was proved that $(\mathcal{P}_0(X),d_H)$ is right K-sequentially complete if and only if (X,d) is right K-sequentially complete. In section 3 a simpler proof of this result is given. It is also proved that $(\mathcal{P}_0(X),d_H)$ is semicomplete if and only if (X,d) is half complete.

2. Preliminary results

Let us denote $\mathcal{NPC}_0^{-1}(X) = \{A \in \mathcal{P}_0(X)$: for each $U \in \mathcal{U}$ there exists a finite subset F of X such that $A \subseteq U^{-1}(F)\}$.

$\mathcal{NPC}_0^{-1}(X)$ can be used to describe the closure of $\mathcal{F}_0(X)$ in $(\mathcal{P}_0(X),\mathcal{U}_H)$.

Proposition 2.1. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $Cl_{\mathcal{T}(\mathcal{U}_H)}(\mathcal{F}_0(X)) = \mathcal{NPC}_0^{-1}(X)$.*

Proof. Let $A \in Cl_{\mathcal{T}(\mathcal{U}_H)}(\mathcal{F}_0(X))$, and let $U \in \mathcal{U}$. Then there exists $F \in \mathcal{F}_0(X)$ such that $F \in U_H(A)$, and hence $A \subseteq U^{-1}(F)$. Therefore $A \in \mathcal{NPC}_0^{-1}(X)$.

Conversely, let $A \in \mathcal{NPC}_0^{-1}(X)$ and let $U \in \mathcal{U}$. Then there exists $F \in \mathcal{F}_0(X)$ such that $A \subseteq U^{-1}(F)$. Let $F' = F \cap U(A)$. It is easy to check that $F' \in U_H(A)$ and hence $A \in Cl_{\mathcal{T}(\mathcal{U}_H)}(\mathcal{F}_0(X))$. □

Corollary 2.2. *Let $(X,\mathcal{U})$ be a quasi-uniform space such that $(X,\mathcal{U}^{-1})$ is precompact. Then $\mathcal{K}_0(X)$ is dense in $(\mathcal{P}_0(X),\mathcal{U}_H)$.*

Proof. It is clear that $Cl_{\mathcal{T}(\mathcal{U}_H)}(\mathcal{F}_0(X)) \subseteq Cl_{\mathcal{T}(\mathcal{U}_H)}(\mathcal{K}_0(X))$. Since $(X,\mathcal{U}^{-1})$ is precompact then $X \in \mathcal{NPC}_0^{-1}(X)$, and hence $A \in \mathcal{NPC}_0^{-1}(X)$ for each $A \in \mathcal{P}_0(X)$. By the previous result we conclude that $Cl_{\mathcal{T}(\mathcal{U}_H)}(\mathcal{K}_0(X)) = \mathcal{P}_0(X)$. □

Proposition 2.3. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then it holds that $Cl_{\mathcal{T}((\mathcal{U}^*)_H)}(\mathcal{F}_0(X)) = \mathcal{PC}_0^*(X)$ and hence $Cl_{\mathcal{T}((\mathcal{U}^*)_H)}(\mathcal{K}_0^*(X)) = \mathcal{PC}_0^*(X)$.*

Proof. Let $A \in Cl_{\mathcal{T}((\mathcal{U}^*)_H)}(\mathcal{F}_0(X))$, and let $U \in \mathcal{U}$. Then there exists $F \in \mathcal{F}_0(X)$ such that $F \in (U^*)_H(A)$, and hence $A \subseteq U^*(F)$. Therefore $A \in \mathcal{NPC}_0^*(X) = \mathcal{PC}_0^*(X)$.

Conversely, let $A \in \mathcal{PC}_0^*(X)$ and let $U \in \mathcal{U}$. Then there exists $F \in \mathcal{F}_0(X)$ such that $F \subseteq A$ and $A \subseteq U^*(F)$. Then $F \in (U^*)_H(A)$ and hence $A \in Cl_{\mathcal{T}((\mathcal{U}^*)_H)}(\mathcal{F}_0(X))$. □

Corollary 2.4. *Let $(X,\mathcal{U})$ be a totally bounded quasi-uniform space. Then $\mathcal{K}_0^*(X)$ is dense in $(\mathcal{P}_0(X),(\mathcal{U}^*)_H)$ and hence in $(\mathcal{P}_0(X),(\mathcal{U}_H)^*)$.*

Let us denote $C^*(\mathcal{F}_0(X)) = \{A \in \mathcal{P}_0(X)$: there is a $(U^*)_H$-Cauchy net in $\mathcal{F}_0(X)$ which $\mathcal{T}(\mathcal{U}_H)$-converges to $A\}$, $C(\mathcal{F}_0(X)) = \{A \in \mathcal{P}_0(X)$: there is a left K-Cauchy net in $(\mathcal{F}_0(X),\mathcal{U}_H)$ which $\mathcal{T}(\mathcal{U}_H)$-converges to $A\}$ and $C^{-1}(\mathcal{F}_0(X)) = \{A \in \mathcal{P}_0(X)$: there is a right K-Cauchy net in $(\mathcal{F}_0(X),\mathcal{U}_H)$ which $\mathcal{T}(\mathcal{U}_H)$-converges to $A\}$.

The proof of the following result is a slight modification of [10, Lemma 1].

Proposition 2.5. *Let $(X,\mathcal{U})$ be a quasi-uniform space.*

(1) $\mathcal{PC}_0(X) \subseteq C(\mathcal{F}_0(X))$.
(2) $\mathcal{PC}_0^{-1}(X) \subseteq C^{-1}(\mathcal{F}_0(X))$.
(3) $\mathcal{PC}_0^*(X) = C^*(\mathcal{F}_0(X))$.

Proof. Let us prove that $\mathcal{PC}_0(X) \subseteq C(\mathcal{F}_0(X))$. The proofs of $\mathcal{PC}_0^{-1}(X) \subseteq C^{-1}(\mathcal{F}_0(X))$ and $\mathcal{PC}_0^*(X) \subseteq C^*(\mathcal{F}_0(X))$ are analogous to this one.

Let $A \in \mathcal{PC}_0(X)$. Let $[A]^{<\omega}$ be the set of nonempty finite subsets of A directed by set-theoretic inclusion. Then $[A]^{<\omega}$ can be considered a left K-Cauchy net in $(\mathcal{F}_0(X),\mathcal{U}_H)$. Indeed, since $A \in \mathcal{PC}_0(X)$ for each $U \in \mathcal{U}$ there exists $A_U \in [A]^{<\omega}$ such that $A \subseteq U(A_U)$. Then for each $B, C \in [A]^{<\omega}$ with $A_U \subseteq B \subseteq C$ we have that $C \subseteq A \subseteq U(A_U) \subseteq U(B)$ and $B \subseteq C \subseteq U^{-1}(C)$, and hence $C \in U_H(B)$. On the other hand, it is clear that $[A]^{<\omega}$ $\mathcal{T}(\mathcal{U}_H)$-converges to A and hence $A \in C(\mathcal{F}_0(X))$.

Now, let us prove that $C^*(\mathcal{F}_0(X)) \subseteq \mathcal{PC}_0^*(X)$. Let $A \in C^*(\mathcal{F}_0(X))$, then there exists a $(\mathcal{U}^*)_H$-Cauchy net $(F_i)_{i\in I}$ in $\mathcal{F}_0(X)$ which $\mathcal{T}(\mathcal{U}_H)$-converges to A. Given $U \in \mathcal{U}$, let $V \in \mathcal{U}$ with $V^2 \subseteq U$, then there exists $i \in I$ such that $F_i \in (V^*)_H(A)$. Then $A \subseteq V^*(F_i)$ and $F_i \subseteq V^*(A)$. Since F_i is finite there exists $B \subseteq A$ finite and such that $F_i \subseteq V^*(B)$ and hence $A \subseteq V^* \circ V^*(B) \subseteq U^*(B)$. Therefore A is totally bounded, and hence $A \in \mathcal{PC}_0^*(X)$. □

3. Semicompleteness of the Bourbaki quasi-uniformity

The following concept is the main key of this paper.

Definition 3.1. *Let $(X,\mathcal{U})$ be a quasi-uniform space. $(\mathcal{P}_0(X),\mathcal{U}_H)$ is said to be semi-complete if each $(\mathcal{U}^*)_H$-Cauchy net is $\mathcal{T}(\mathcal{U}_H)$-convergent.*

Note that if $(X,\mathcal{U})$ is a uniform space then $(\mathcal{P}_0(X),\mathcal{U}_H)$ is semi-complete if and only if it is complete.

Since $(\mathcal{U}_H)^* \subseteq (\mathcal{U}^*)_H$ ([11]), it follows that if $(\mathcal{P}_0(X),\mathcal{U})$ is half complete then it is semi-complete.

The following result shows that semicompleteness of $(\mathcal{K}_0(X), \mathcal{U}_H)$ is a strong condition.

Proposition 3.2. *Let $(X, \mathcal{U})$ be a Hausdorff quasi-uniform space.*

(1) *If $(\mathcal{K}_0(X), \mathcal{U}_H)$ is semicomplete then $\mathcal{C}_0(X) \cap C^*(\mathcal{F}_0(X)) \subseteq \mathcal{K}_0(X)$.*
(2) *If $(\mathcal{K}_0^*(X), \mathcal{U}_H)$ is semicomplete then $\mathcal{C}_0(X) \cap C^*(\mathcal{F}_0(X)) \subseteq \mathcal{K}_0^*(X)$.*
(3) *If $(\mathcal{K}_0(X), \mathcal{U}_H)$ is left K-complete then $\mathcal{C}_0(X) \cap C(\mathcal{F}_0(X)) \subseteq \mathcal{K}_0(X)$.*
(4) *If $(\mathcal{K}_0^*(X), \mathcal{U}_H)$ is left K-complete then $\mathcal{C}_0(X) \cap C(\mathcal{F}_0(X)) \subseteq \mathcal{K}_0^*(X)$.*
(5) *If $(\mathcal{K}_0(X), \mathcal{U}_H)$ is right K-complete then $\mathcal{C}_0(X) \cap C^{-1}(\mathcal{F}_0(X)) \subseteq \mathcal{K}_0(X)$.*
(6) *If $(\mathcal{K}_0^*(X), \mathcal{U}_H)$ is right K-complete then $\mathcal{C}_0(X) \cap C^{-1}(\mathcal{F}_0(X)) \subseteq \mathcal{K}_0^*(X)$.*

Proof. Let us prove the first item. Suppose that $(\mathcal{K}_0(X), \mathcal{U}_H)$ is semicomplete and let $A \in \mathcal{C}_0(X) \cap C^*(\mathcal{F}_0(X))$. Then there exists a $(U^*)_H$-Cauchy net $(F_d)_{d\in D}$ in $\mathcal{F}_0(X)$ which $\mathcal{T}(\mathcal{U}_H)$-converges to A. Since $(\mathcal{K}_0(X), \mathcal{U}_H)$ is semicomplete, there exists $K \in \mathcal{K}_0(X)$ such that the net $\mathcal{T}(\mathcal{U}_H)$-converges also to K. Then it is easy to check that $(F_d)_{d\in D}$ $\mathcal{T}(\mathcal{U}_H)$-converges to $A \cup K$.

If $A \subseteq K$, since K is compact and A is closed then A is compact and hence $A \in \mathcal{K}_0(X)$. Suppose that $A \not\subseteq K$. Then there exists $x \in A \setminus K$. Since X is Hausdorff and K is compact, there exists $U \in \mathcal{U}$ such that $U(x) \cap U(K) = \varnothing$. Then $U_H(K \cup A) \cap U_H(K) = \varnothing$. Indeed, if there is $B \in U_H(K \cup A) \cap U_H(K)$, then $A \subseteq A \cup K \subseteq U^{-1}(B) \subseteq U^{-1} \circ U(K)$, and hence $x \in U^{-1} \circ U(K)$ which contradicts that $U(x) \cap U(K) = \varnothing$. Therefore $U_H(K \cup A) \cap U_H(K) = \varnothing$, but $(F_d)_{d\in D}$ converges to $K \cup A$ and converges to K, so there exists $d_0 \in D$ such that $F_{d_0} \in U_H(K \cup A) \cap U_H(K)$. The contradiction shows that $A \subseteq K$, and $A \in \mathcal{K}_0(X)$ (note that this implies that $A = K$) .

The rest of the items have an analogous proof. We only note that $\mathcal{K}_0^*(X) \subseteq \mathcal{K}_0(X)$ and $\mathcal{C}_0(X) \subseteq \mathcal{C}_0^*(X)$. $\square$

The next result provides a generalization of the Zenor-Morita theorem to the setting of Hausdorff quasi-uniform spaces.

Theorem 3.3. *Let $(X, \mathcal{U})$ be a Hausdorff quasi-uniform space. It follows that $(K_0^*(X), \mathcal{U}_H)$ is semicomplete if and only if $(X, \mathcal{U})$ is half-complete and $\mathcal{C}_0(X) \cap C^*(\mathcal{F}_0(X)) \subseteq \mathcal{K}_0^*(X)$.*

Proof. Suppose that $(K_0^*(X), \mathcal{U}_H)$ is semicomplete. It is easy to prove that $(X, \mathcal{U})$ is half complete and $\mathcal{C}_0(X) \cap C^*(\mathcal{F}_0(X)) \subseteq \mathcal{K}_0^*(X)$ by Proposition 3.2.

Conversely, suppose that $(X, \mathcal{U})$ is half-complete. Let $\{C_\alpha : \alpha \in D\}$ be a $(U^*)_H$-Cauchy net in $\mathcal{K}_0^*(X)$. Let us show that $\{C_\alpha : \alpha \in D\}$ is convergent in $(\mathcal{K}_0^*(X), \mathcal{U}_H)$.

For each $\alpha \in D$, let $F_\alpha = \bigcup_{\beta \geq \alpha} C_\beta$. Let $\mathcal{F} = fil\{F_\alpha : \alpha \in D\}$.

(1) Let $\mathcal{F}'$ be an ultrafilter containing $\mathcal{F}$. Let us prove that $\mathcal{F}'$ is $(\mathcal{U}^*)_H$-Cauchy. Let $U \in \mathcal{U}$ and $V \in U$ with $V^2 \subseteq U$. Since $\{C_\alpha : \alpha \in D\}$ is $(\mathcal{U}^*)_H$-Cauchy, there exists $\alpha_0 \in D$ such that $C_\alpha \in (V^*)_H(C_{\alpha_0})$ for each $\alpha \geq \alpha_0$ and hence $C_\alpha \subseteq V^*(C_{\alpha_0})$ for each $\alpha \geq \alpha_0$. Since $C_{\alpha_0} \in \mathcal{K}_0^*(X)$ there exists a finite subset B of C_{α_0} such that $C_{\alpha_0} \subseteq V^*(B)$ and hence $F_{\alpha_0} \subseteq V^*(C_{\alpha_0}) \subseteq V^* \circ V^*(B) \subseteq U^*(B)$. Since $F_{\alpha_0} \in \mathcal{F}'$, B is finite and

$\mathcal{F}'$ is an ultrafilter there exists $b \in B$ such that $U^*(b) \in \mathcal{F}'$. Therefore $\mathcal{F}'$ is $\mathcal{U}^*$-Cauchy.

Set $C = \bigcap_{\alpha \in D} \overline{F_\alpha}$. First, we note that $C \neq \varnothing$ by (1).

(2) Let us prove that $\{C_\alpha : \alpha \in D\}$ $\mathcal{T}(\mathcal{U}_H)$-converges to C. Let $U \in \mathcal{U}$ (we can suppose that $U(x)$ is open for each $x \in X$), then there exist $V \in \mathcal{U}$ with $V^2 \subseteq U$ and $\alpha_0 \in D$ such that $C \subseteq \overline{F_\alpha} \subseteq V^{-1} \circ V^*(C_\alpha) \subseteq U^{-1}(C_\alpha)$ for each $\alpha \geq \alpha_0$. Suppose that $F_\alpha \not\subseteq U(C)$ for each $\alpha \geq \alpha_0$. Let $\mathcal{G} = fil\{(X \setminus U(C)) \cap F : F \in \mathcal{F}\}$. It is clear that $\mathcal{F} \subseteq \mathcal{G}$. Let $\mathcal{G}'$ be an ultrafilter containing $\mathcal{G}$. Analogous to (1), it can be proved that $\mathcal{G}'$ is $\mathcal{U}^*$-Cauchy and hence it $\mathcal{T}(\mathcal{U})$-converges to $y_0 \in X$. Note that $y_0 \in C$ since $\mathcal{F} \subseteq \mathcal{G}'$ and $y_0 \in X \setminus U(C)$ since $X \setminus U(C)$ is closed. The contradiction shows that there exists $\alpha_1 \geq \alpha_0$ such that $C_\alpha \subseteq F_{\alpha_1} \subseteq U(C)$ for each $\alpha \geq \alpha_1$. Therefore $C_\alpha \in U_H(C)$ for each $\alpha \geq \alpha_1$ and so $\{C_\alpha : \alpha \in D\}$ $\mathcal{T}(\mathcal{U}_H)$-converges to C.

(3) Let us prove that $C \in \mathcal{K}_0^*(X)$. For each U and $\alpha \in D$ there exists $F_{\alpha,U} \subseteq C_\alpha$ finite such that $C_\alpha \in (U^*)_H(F_{\alpha,U})$. Then the net $\{F_{\alpha,U} : (\alpha, U) \in D \times \mathcal{U}\}$ (where $(\alpha, U) \leq (\alpha', U')$ if and only if $\alpha \leq \alpha'$ and $U' \subseteq U$) is clearly $(U^*)_H$-Cauchy and $\mathcal{T}(\mathcal{U}_H)$-convergent to C. Then $C \in \mathcal{C}_0(X) \cap C^*(\mathcal{F}_0(X)) \subseteq \mathcal{K}_0^*(X)$.

Combining the previous arguments, $(\mathcal{K}_0^*(X), \mathcal{U}_H)$ is semicomplete. $\square$

Remark 3.4. In the proof of the previous theorem the hypothesis that X is Hausdorff is only used in the only if part. Also note that $\mathcal{K}_0^*(X) \subseteq \mathcal{C}_0(X) \cap C^*(\mathcal{F}_0(X))$ if X is Hausdorff.

Remark 3.5. Note that if $(X,\mathcal{U})$ is a uniform space and $(X,\mathcal{U})$ is complete then $\mathcal{C}_0(X) \cap C(\mathcal{F}_0(X)) = \mathcal{C}_0(X) \cap \mathcal{PC}_0(X)$ which is clearly a subset of $\mathcal{K}_0(X)$, since a closed totally bounded subspace in a complete uniform space is compact (also note that $C(\mathcal{F}_0(X)) = C^{-1}(\mathcal{F}_0(X)) = C^*(\mathcal{F}_0(X))$ and $\mathcal{K}_0(X) = \mathcal{K}_0^{-1}(X) = \mathcal{K}_0^*(X)$). Therefore Theorem 3.3 is a generalization of Zenor-Morita theorem.

In order to generalize the Burdick-Isbell theorem, we will need the following result. Its proof is based on [11, Lemma 6].

Lemma 3.6. *Let $(X,\mathcal{U})$ be a quasi-uniform space such that each stable filter on $(X,\mathcal{U}^*)$ has a cluster point in $(X,\mathcal{U})$. Let $\mathcal{F}$ be a stable filter on $(X,\mathcal{U}^*)$ and let $C = \bigcap_{F \in \mathcal{F}} \overline{F}$. Then $U(C) \in \mathcal{F}$ for each $U \in \mathcal{U}$.*

Proof. Suppose that there exists $U_0 \in \mathcal{U}$ such that $E \setminus U_0^2(C) \neq \varnothing$ for each $E \in \mathcal{F}$. Let $H_{U,E} = \{a \in X :$ There is $V \in \mathcal{U}$ such that $V^2 \subseteq U$; $V^*(V^*(a)) \cap U_0(C) = \varnothing$ and $a \in \bigcap_{F \in \mathcal{F}} V^*(F) \cap E\}$. Given $U \in \mathcal{U}$ and $E \in \mathcal{F}$, let $V \in \mathcal{U}$ with $V^2 \subseteq U_0 \cap U$. Then it is easy to check that $\varnothing \neq (\bigcap_{F \in \mathcal{F}} V^*(F) \cap E) \setminus U_0^2(C) \subseteq H_{U,E}$, and hence $H_{U,E} \neq \varnothing$ for each $U \in \mathcal{U}$ and $E \in \mathcal{F}$.

Note also that for any $U_1, U_2 \in \mathcal{U}$ such that $U_1 \subseteq U_2$ and any $E_1, E_2 \in \mathcal{F}$ such that $E_1 \subseteq E_2$ we have that $H_{U_1,E_1} \subseteq H_{U_2,E_2}$.

Thus $\{H_{U,E} : U \in \mathcal{U}; E \in \mathcal{F}\}$ is a base for a filter $\mathcal{H}$ on X. In order to show that $\mathcal{H}$ is stable on $(X,\mathcal{U}^*)$, let $U, V \in \mathcal{U}$ and $E \in \mathcal{F}$ and let us prove that $H_{U,X} \subseteq U^*(H_{V,E})$. Let $a \in H_{U,X}$, then there exists $W \in \mathcal{U}$ such that

$W^2 \subseteq U$, $W^*(W^*(a)) \cap U_0(C) = \varnothing$ and $a \in \bigcap_{F\in\mathcal{F}} W^*(F)$. Choose $Z \in \mathcal{U}$ such that $Z^2 \subseteq V \cap W$. Since $a \in \bigcap_{F\in\mathcal{F}} W^*(F)$ and $E \cap \bigcap_{F\in\mathcal{F}} Z^*(F) \in \mathcal{F}$ it follows that there exists $y \in [E \cap \bigcap_{F\in\mathcal{F}} Z^*(F)] \cap W^*(a)$. On the other hand, since $Z^*(Z^*(y)) \subseteq W^*(y) \subseteq W^*(W^*(a))$ it follows that $Z^*(Z^*(y)) \cap U_0(C) = \varnothing$. We conclude that $y \in H_{V,E}$ and since $a \in W^*(y) \subseteq U^*(y)$ we have that $H_{U,X} \subseteq U^*(H_{V,E})$ and hence $\mathcal{H}$ is stable in $(X,\mathcal{U}^*)$. Hence it has a cluster point x in $(X,\mathcal{U})$ and since $H_{X\times X,F} \subseteq F$ whenever $F \in \mathcal{F}$, it follows that $x \in C$, but this is a contradiction, since $H_{V,E} \cap U_0(C) = \varnothing$ and hence $\overline{H_{V,E}} \cap C = \varnothing$ for each $V \in \mathcal{U}$ and $E \in \mathcal{F}$. The contradiction shows that $U_0^2(C) \in \mathcal{F}$ for each $U_0 \in \mathcal{U}$ and hence $U(C) \in \mathcal{F}$ for each $U \in \mathcal{U}$. $\square$

Now, we generalize the Burdick-Isbell theorem to the setting of quasi-uniform spaces.

Theorem 3.7. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{P}_0(X),\mathcal{U}_H)$ is semicomplete if and only if each stable filter on $(X,\mathcal{U}^*)$ has a cluster point in $(X,\mathcal{U})$.*

Proof. Suppose that $(\mathcal{P}_0(X),\mathcal{U}_H)$ is semicomplete, and let $\mathcal{F}$ be a stable filter on $(X,\mathcal{U}^*)$. Consider the net $(F)_{F\in(\mathcal{F},\supseteq)}$ on $\mathcal{P}_0(X)$. Let $U \in \mathcal{U}$. Since $\mathcal{F}$ is stable on $(X,\mathcal{U}^*)$ there exists $F_U \in \mathcal{F}$ such that $F_U \subseteq U^*(F)$ for each $F \in \mathcal{F}$. Thus, for each $F \in \mathcal{F}$ with $F \subseteq F_U$, we have that $F_U \subseteq U^*(F)$ and $F \subseteq F_U \subseteq U^*(F_U)$, so $F \in (U^*)_H(F_U)$ for each $F \subseteq F_U$ and hence $(F)_{F\in(\mathcal{F},\supseteq)}$ is a $(\mathcal{U}^*)_H$-Cauchy net on $\mathcal{P}_0(X)$. Since $(\mathcal{P}_0(X),\mathcal{U}_H)$ is semicomplete, the net $\mathcal{T}(\mathcal{U}_H)$-converges to some $C \in \mathcal{P}_0(X)$. It is easy to see that x is a cluster point of $\mathcal{F}$ for each $x \in C$.

Conversely, suppose that each stable filter on $(X,\mathcal{U}^*)$ has a cluster point in $(X,\mathcal{U})$. Let $(C_d)_{d\in D}$ be a $(\mathcal{U}^*)_H$-Cauchy net on $\mathcal{P}_0(X)$. For each $d \in D$, let $F_d = \bigcup_{e\geq d} C_e$ and set $\mathcal{F} = fil\{F_d : d \in D\}$ on X. Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ with $V^2 \subseteq U$, then there exists $d_V \in D$ such that $C_d \in (V^*)_H(C_{d_V})$ for each $d \geq d_V$. Then $C_d \subseteq V^*(C_{d_V})$ and $C_{d_V} \subseteq V^*(C_d)$ for each $d \geq d_V$. It follows that $F_d \subseteq V^*(C_{d_V})$ for each $d \geq d_V$. In order to prove that $F_{d_V} \subseteq \bigcap_{d\in D} U^*(F_d)$, let $x \in F_{d_V}$ and $d \in D$. Let $h \geq d, d_V$, then $x \in F_{d_V} \subseteq V^*(C_{d_V}) \subseteq V^*(V^*(C_h)) \subseteq U^*(F_d)$ and hence $F_{d_V} \subseteq \bigcap_{d\in D} U^*(F_d)$. Therefore $\mathcal{F}$ is a stable filter on $(X,\mathcal{U}^*)$.

Let $C = \bigcap_{F\in\mathcal{F}} \overline{F}$. Since $\mathcal{F}$ is stable on $(X,\mathcal{U}^*)$ it follows from the hypothesis that $C \neq \varnothing$. Let us prove that $(C_d)_{d\in D}$ converges to C in $(\mathcal{P}_0(X),\mathcal{U}_H)$. Let $U \in \mathcal{U}$ and let $V \in \mathcal{U}$ such that $V^3 \subseteq U$. It is clear that $C \subseteq \overline{F_d} \subseteq V^{-1}(F_d) \subseteq V^{-1}(V^*(C_{d_V})) \subseteq V^{-1}(V^*(V^*(C_d))) \subseteq U^{-1}(C_d)$ for each $d \geq d_V$.

On the other hand, by Lemma 3.6 we have that $U(C) \in \mathcal{F}$, and hence there exists $d_0 \geq d_V$ such that $F_{d_0} \subseteq U(C)$, and hence $C_d \subseteq U(C)$ for each $d \geq d_0$. Therefore $(C_d)_{d\in D}$ converges to C in $(\mathcal{P}_0(X),\mathcal{U}_H)$. $\square$

The next result is a generalization of the following result: a metric space (X,d) is complete if and only if $(\mathcal{P}_0(X), d_H)$ is complete.

Corollary 3.8. *Let (X,d) be a quasi-pseudometric space. Then (X,d) is half complete if and only if $(\mathcal{P}_0(X),(\mathcal{U}(d))_H)$ is semicomplete (where $\mathcal{U}(d)$ denotes the quasi-uniformity induced by d).*

Proof. Let $U_n = \{(x,y) \in X \times X : d(x,y) < \frac{1}{2^n}\}$.

By Theorem 3.7 we only have to prove that any stable filter on (X,d^*) has a cluster point in (X,d) if X is half complete. Let $\mathcal{F}$ be a stable filter on (X,d^*), then for each $n \in \mathbb{N}$ there exists $F_n \in \mathcal{F}$ such that $F_n \subseteq \bigcap_{F\in\mathcal{F}} U_n^*(F)$. Let $x_1 \in F_1$, suppose that we have defined x_n, and define x_{n+1} as follows: Since $F_n \subseteq U_n^*(F_{n+1})$, let $x_{n+1} \in F_{n+1} \cap U_n^*(x_n)$. Then it is easy to check that (x_n) is a Cauchy sequence in (X,d^*) and since X is half complete it converges to some point $x \in X$. It is clear that x is a cluster point of $\mathcal{F}$ in (X,d). $\square$

Remark 3.9. Let $(X,\mathcal{U})$ be a quasi-uniform space. From the Burdick-Isbell theorem it follows that the uniform space $(\mathcal{P}_0(X),(\mathcal{U}^*)_H)$ is complete if and only if each stable filter on $(X,\mathcal{U}^*)$ has a cluster point in $(X,\mathcal{U}^*)$. Of course, $(\mathcal{P}_0(X),\mathcal{U}_H)$ is semicomplete if $(\mathcal{P}_0(X),(\mathcal{U}^*)_H)$ is complete.

On the other hand, for any half complete non-bicomplete quasi-metric space (X,d) it follows that $(\mathcal{P}_0(X),(\mathcal{U}(d)^*)_H)$ is not complete, but $(\mathcal{P}_0(X),\mathcal{U}_H)$ is semicomplete by Corollary 3.8.

The proof of the following proposition is analogous to the proof of [11, Proposition 7].

Proposition 3.10. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces and $f : (X,\mathcal{U}^*) \to (Y,\mathcal{V}^*)$ be a uniformly continuous surjection that is $\mathcal{T}(\mathcal{U})$-$\mathcal{T}(\mathcal{V})$-perfect. If $(\mathcal{P}_0(Y),\mathcal{V}_H)$ is semicomplete then $(\mathcal{P}_0(X),\mathcal{U}_H)$ is semicomplete.*

Proof. Let $\mathcal{F}$ be a stable filter on $(X,\mathcal{U}^*)$ and let $V \in \mathcal{V}$. Since f is $\mathcal{U}^*$-$\mathcal{V}^*$-uniformly continuous there is $U \in \mathcal{U}$ such that $(f \times f)(U^*) \subseteq V^*$. Since $\mathcal{F}$ is stable on $(X,\mathcal{U}^*)$, there is $F_0 \in \mathcal{F}$ such that $F_0 \subseteq U^*(F)$ for each $F \in \mathcal{F}$, and hence $f(F_0) \subseteq V^*(f(F))$ for each $F \in \mathcal{F}$. Therefore the filter $f(\mathcal{F}) := \{f(F) : F \in \mathcal{F}\}$ is stable on $(Y,\mathcal{V}^*)$. Since $(\mathcal{P}_0(Y),\mathcal{V}_H)$ is semicomplete, it follows from Theorem 3.7 that $f(\mathcal{F})$ has a cluster point y_0 in $(Y,\mathcal{V})$. Since f is $\mathcal{T}(\mathcal{U})$-$\mathcal{T}(\mathcal{V})$-perfect the filter $\mathcal{F}$ has a cluster point $x_0 \in f^{-1}(y_0)$ in $(X,\mathcal{U})$. Therefore $(\mathcal{P}_0(X),\mathcal{U}_H)$ is semicomplete by Theorem 3.7. $\square$

Corollary 3.11. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces and let $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a quasi-uniformly continuous surjection that is perfect. If $(\mathcal{P}_0(Y),\mathcal{V}_H)$ is semicomplete then $(\mathcal{P}_0(X),\mathcal{U}_H)$ is semicomplete.*

Corollary 3.12. *Let $(X,\mathcal{V})$ be a quasi-uniform space such that $(\mathcal{P}_0(X),\mathcal{V}_H)$ is semicomplete. Then for any compatible quasi-uniformity $\mathcal{U}$ finer than $\mathcal{V}$ on X, it follows that $(\mathcal{P}_0(X),\mathcal{U}_H)$ is semicomplete.*

A condition for bicompleteness of $(\mathcal{P}_0(X),\mathcal{U}_H)$ in terms of doubly stable filters was given in [11].

Recall that a filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is said to be doubly stable ([11]) provided that $\bigcap_{F\in\mathcal{F}}(U(F) \cap U^{-1}(F))$ belongs to $\mathcal{F}$ for each $U \in \mathcal{U}$.

The next result characterizes half completeness of $(\mathcal{P}_0(X),\mathcal{U}_H)$ in terms of doubly stable filters.

Proposition 3.13. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{P}_0(X),\mathcal{U}_H)$ is half complete if and only if each doubly stable filter on $(X,\mathcal{U})$ verifies that*

$U(C) \in \mathcal{F}$ for each $U \in \mathcal{U}$ (where C denotes the set of $\mathcal{T}(\mathcal{U})$-cluster points of $\mathcal{F}$).

Proof. Suppose that $(\mathcal{P}_0(X), \mathcal{U}_H)$ is half complete, and let $\mathcal{F}$ be a doubly stable filter on $(X, \mathcal{U})$. Consider the net $(F)_{F\in(\mathcal{F},\supseteq)}$ on $\mathcal{P}_0(X)$. Let $U \in \mathcal{U}$. Since $\mathcal{F}$ is doubly stable on $(X, \mathcal{U})$ there exists $F_U \in \mathcal{F}$ such that $F_U \subseteq U(F) \cap U^{-1}(F)$ for each $F \in \mathcal{F}$. Thus, for each $F \in \mathcal{F}$ with $F \subseteq F_U$, we have that $F_U \subseteq U(F) \cap U^{-1}(F)$ and $F \subseteq F_U \subseteq U(F_U) \cap U^{-1}(F_U)$, so $F \in (U_H)^*(F_U)$ for each $F \subseteq F_U$ and hence $(F)_{F\in(\mathcal{F},\supseteq)}$ is a $(\mathcal{U}_H)^*$-Cauchy net on $\mathcal{P}_0(X)$. Since $(\mathcal{P}_0(X), \mathcal{U}_H)$ is half complete, the net $\mathcal{T}(\mathcal{U}_H)$-converges to some $D \in \mathcal{P}_0(X)$. It is easy to see that x is a cluster point of $\mathcal{F}$ in $(X, \mathcal{U})$ whenever $x \in D$, and hence, if C is the set of $\mathcal{T}(\mathcal{U})$-cluster points of $\mathcal{F}$ we have that $D \subseteq C$. On the other hand, given $U \in \mathcal{U}$ there exists $F_0 \in \mathcal{F}$ such that $F_0 \in U_H(D)$, and hence $F_0 \subseteq U(D) \subseteq U(C)$. Therefore $U(C) \in \mathcal{F}$ for each $U \in \mathcal{U}$.

Conversely, suppose that each doubly stable filter on $(X, \mathcal{U})$ verifies $U(C) \in \mathcal{F}$ (where C denotes the set of $\mathcal{T}(\mathcal{U})$-cluster points of $\mathcal{F}$). Let $(C_d)_{d\in D}$ be a $(\mathcal{U}_H)^*$-Cauchy net on $\mathcal{P}_0(X)$. For each $d \in D$, let $F_d = \bigcup_{e\geq d} C_e$ and set $\mathcal{F} = fil\{F_d : d \in D\}$ on X. Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ with $V^2 \subseteq U$, then there exists $d_V \in D$ such that $C_d \in (V_H)^*(C_{d_V})$ for each $d \geq d_V$. Then $C_d \subseteq V(C_{d_V}) \cap V^{-1}(C_{d_V})$ and $C_{d_V} \subseteq V(C_d) \cap V^{-1}(C_d)$ for each $d \geq d_V$. It follows that $F_d \subseteq V(C_{d_V}) \cap V^{-1}(C_{d_V})$ for each $d \geq d_V$. In order to prove that $F_{d_V} \subseteq \bigcap_{d\in D} U(F_d) \cap U^{-1}(F_d)$, let $d \in D$, and $h \geq d, d_V$, then $F_{d_V} \subseteq V(C_{d_V}) \cap V^{-1}(C_{d_V}) \subseteq V \circ V(C_h) \cap V^{-1} \circ V^{-1}(C_h) \subseteq U(F_d) \cap U^{-1}(F_d)$ and hence $F_{d_V} \subseteq \bigcap_{d\in D} U(F_d) \cap U^{-1}(F_d)$. Therefore $\mathcal{F}$ is a doubly stable filter on $(X, \mathcal{U})$.

Let $C = \bigcap_{F\in\mathcal{F}} \overline{F}$ (that is, C is the set of $\mathcal{T}(\mathcal{U})$-cluster points of $\mathcal{F}$). Since $\mathcal{F}$ is doubly stable on $(X, \mathcal{U})$ it follows from the hypothesis that $C \neq \varnothing$ and that $U(C) \in \mathcal{F}$ for each $U \in \mathcal{U}$. Let us prove that $(C_d)_{d\in D}$ converges to C in $(\mathcal{P}_0(X), \mathcal{U}_H)$. Let $U \in \mathcal{U}$ and let $V \in \mathcal{U}$ such that $V^3 \subseteq U$. It is clear that $C \subseteq \overline{F_d} \subseteq V^{-1}(F_d) \subseteq V^{-1}(V(C_{d_V}) \cap V^{-1}(C_{d_V})) \subseteq V^{-1}(V^{-1}(V^{-1}(C_d))) \subseteq U^{-1}(C_d)$ for each $d \geq d_V$.

On the other hand, since $U(C) \in \mathcal{F}$, there exists $d_0 \in D$ such that $F_{d_0} \subseteq U(C)$ and hence $C_d \subseteq U(C)$ for each $d \geq d_0$. Therefore $(C_d)_{d\in D}$ converges to C in $(\mathcal{P}_0(X), \mathcal{U}_H)$. □

Example 3.14. Let $\mathbb{Q}$ be the rationals with the Sorgenfrey quasi-metric d_S. Then $(\mathbb{Q}, d_S)$ is bicomplete and hence $(\mathcal{P}_0(\mathbb{Q}), \mathcal{U}_H)$ is semicomplete by Corollary 3.8 (in fact, $(\mathcal{P}_0(\mathbb{Q}), (\mathcal{U}^*)_H)$ is complete), but $(\mathcal{P}_0(X), \mathcal{U}_H)$ is not half complete. Indeed, by [11, Example 7] there is a doubly stable filter $\mathcal{F}$ on $(\mathbb{Q}, d_S)$ without cluster point in $(\mathbb{Q}, d_S)$. Then $(\mathcal{P}_0(X), \mathcal{U}_H)$ is not half complete by Proposition 3.13.

The next result is a simpler proof of [11, Proposition 5]. First, we prove a lemma.

Lemma 3.15. *Let (X, d) be a quasi-pseudometric space. Then d is right K-sequentially complete if and only if whenever $(U_n^{-1}(x_n))$ is a decreasing sequence* *(that is, $x_{n+1} \in U_n^{-1}(x_n)$ for each $n \in \mathbb{N}$) it follows that $\bigcap_{n\in\mathbb{N}} U_{n-2}^{-1}(x_n) \neq \varnothing$ (where $U_n = \{(x,y) \in X \times X : d(x,y) < \frac{1}{2^n}\}$).*

Proof. Suppose that whenever $(U_n^{-1}(x_n))$ is a decreasing sequence it follows that $\bigcap_{n\in\mathbb{N}} U_{n-2}^{-1}(x_n) \neq \varnothing$, and let (x_n) be a right K-Cauchy sequence. Then there exists $(x_{m(n)})$ a subsequence of (x_n) such that $x_{m(n+1)} \in U_n^{-1}(x_{m(n)})$ for each $n \in \mathbb{N}$. By hypothesis there exists $x \in \bigcap_{n\in\mathbb{N}} U_{n-2}^{-1}(x_{m(n)})$, and hence $(x_{m(n)})$ converges to x. Since each right K-Cauchy sequence converges to its cluster points it follows that (x_n) is convergent. Therefore d is right K-sequentially complete.

Suppose that d is right K-sequentially complete and let (x_n) be a sequence with $x_{n+1} \in U_n^{-1}(x_n)$. Then $d(x_{n+h+l}, x_{n+h}) \leq \sum_{i=0}^{l-1} d(x_{n+h+i+1}, x_{x+h+i}) \leq \frac{1}{2^{n-1}}$, so $x_k \in U_{n-1}^{-1}(x_m)$ for each $k \geq m \geq n$. Therefore (x_n) is a right K-Cauchy sequence and hence it converges to some $x \in X$. Now, $d(x, x_n) \leq d(x, x_k) + d(x_k, x_n) < \frac{1}{2^n} + \frac{1}{2^{n-1}} < \frac{1}{2^{n-2}}$ for a suitable $k \in \mathbb{N}$ and hence $x \in U_{n-2}^{-1}(x_n)$ for each $n \in \mathbb{N}$. □

Theorem 3.16. [11, Proposition 5] *Let (X,d) be a quasi-pseudometric space. The following statements are equivalent:*

(1) *(X,d) is right K-sequentially complete.*
(2) *$(\mathcal{P}_0(X), (\mathcal{U}(d))_H)$ is right K-sequentially complete.*
(3) *$(Cl_{\mathcal{T}(U_H)}(\mathcal{F}_0(X)), (\mathcal{U}(d))_H)$ is right K-sequentially complete.*

Proof. Let $U_n = \{(x,y) \in X \times X : d(x,y) < \frac{1}{2^n}\}$.

2) implies 3) is clear.

3) implies 1). Let $(x_n)_{n\in\mathbb{N}}$ be a right K-Cauchy sequence in (X,d). Then it is clear that $(\{x_n\})_{n\in\mathbb{N}}$ is a right K-Cauchy sequence in $Cl_{\mathcal{T}(U_H)}(\mathcal{F}_0(X))$, and since it is right K-sequentially complete there exists $C \in Cl_{\mathcal{T}(U_H)}(\mathcal{F}_0(X))$ such that $(\{x_n\})_{n\in\mathbb{N}}$ $\mathcal{T}(\mathcal{U}_H)$-converges to C. Let $x_0 \in C$, then it is easy to check that x_0 is a cluster point of $(x_n)_{n\in\mathbb{N}}$, and since a right K-Cauchy sequence converges to its cluster points, it follows that $(x_n)_{n\in\mathbb{N}}$ is convergent and then (X,d) is right K-sequentially complete.

1) implies 2). Suppose that X is right K-sequentially complete, and let (F_n) be a sequence in $\mathcal{P}_0(X)$ such that $F_{n+1} \in (U_H)_n^{-1}(F_n)$. Let $F = \bigcap_{n\in\mathbb{N}} U_{n-2}^{-1}(F_n)$, then it is clear that $F \subseteq U_{n-2}^{-1}(F_n)$ for each $n \in \mathbb{N}$. On the other hand, given $x_n \in F_n$, since $F_{n+1} \in (U_H)_n^{-1}(F_n)$ and $F_n \in (U_H)_{n-1}^{-1}(F_{n-1})$, it follows that $F_{n+1} \subseteq U_n^{-1}(F_n)$ and $F_{n-1} \subseteq U_n(F_n)$, and then it is clear that we can construct a sequence (y_k), with $y_n = x_n$, $y_{k+1} \in U_k^{-1}(y_k)$ and $y_k \in F_k$ for each $k \in \mathbb{N}$, and hence $(U_k^{-1}(y_k))$ is a decreasing sequence, and since X is right K-sequentially complete, by Lemma 3.15 there exists $x \in \bigcap_{k\in\mathbb{N}} U_{k-2}^{-1}(y_k)$. It is clear that $x \in F$ and $x_n = y_n \in U_{n-2}(x)$. Therefore $F_n \subseteq U_{n-2}(F)$, and hence $F \in (U_H)_{n-2}^{-1}(F_n)$ for each $n \in \mathbb{N}$, that is, $F \in \bigcap_{n\in\mathbb{N}} (U_H)_{n-2}^{-1}(F_n)$, and by Lemma 3.15 $(\mathcal{P}_0(X), \mathcal{U}_H)$ is right K-sequentially complete. □

Remark 3.17. Let $(X, \mathcal{U})$ be a quasi-uniform space. We have noted that if $(\mathcal{P}_0(X), \mathcal{U}_H)$ is half complete or $(\mathcal{P}_0(X), (\mathcal{U}^*)_H)$ is complete then $(\mathcal{P}_0(X), \mathcal{U}_H)$ is semicomplete. In Example 3.14 we proved that completeness of $(\mathcal{P}_0(X), (\mathcal{U}^*)_H)$ does not imply half completeness of $(\mathcal{P}_0(X), \mathcal{U}_H)$.

On the other hand, by Theorem 3.16, if $(X, \mathcal{U})$ is any right K-sequentially complete non-bicomplete quasi-metrizable quasi-uniform space it follows that $(\mathcal{P}_0(X), \mathcal{U}_H)$ is half complete, but $(\mathcal{P}_0(X), (\mathcal{U}^*)_H)$ is not complete. Note ([14], [1]) that sequentially half-completeness and half-completeness are equivalent for quasi-pseudometrizable quasi-uniform spaces.

Finally, note that bicompleteness of $(\mathcal{P}_0(X), \mathcal{U}_H)$ implies that $(\mathcal{P}_0(X), \mathcal{U}_H)$ is half complete, and in the light of Example 3.14 completeness of $(\mathcal{P}_0(X), (\mathcal{U}^*)_H)$ does not imply bicompleteness of $(\mathcal{P}_0(X), \mathcal{U}_H)$.

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