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Continuous representability of complete preorders on the space of upper-continuous capacities

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Abstract. Given a compact metric space (X, d), and its Borel σ-algebra Σ, we discuss the existence of a (semi)continuous utility function U for a complete preorder $\preceq$ on a subset $M'(X)$ of the space $M(X)$ of all upper-continuous capacities on Σ, endowed with the weak topology.

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1. Introduction

In decision theory under uncertainty, it is usual to consider a separable metric space (X, d) of possible consequences of a game, endowed with its Borel σ-algebra Σ (i.e., the σ-algebra generated by the open subsets of X). A player is required to choose a probability measure from a set P of σ-additive probability measures on the measurable space (X, Σ), endowed with the induced weak topology. The preferences of the player among probability measures in P are expressed by a *complete preorder* (i.e., a reflexive, transitive and complete binary relation) $\preceq$ on P. This is the usual model for *expected utility* (see Grandmont [5]). In a more general setting, it may be assumed that player's uncertainty is reflected by *capacities* better than probability measures (see e.g. Epstein and Wang [4]). In particular, the notion of an *upper-continuous capacity* generalizes the notion of a σ-additive probability measure in the case of additive set functions. Topological properties of the space $M(X)$ of all upper-continuous capacities on the measurable space (X, Σ), endowed with the weak topology, have been studied by Zhou [7] in case that (X, d) is a compact (metric) space.

In this paper, we use the results proved by Zhou [7] in order to discuss the existence of a continuous or at least upper semicontinuous utility function U

for a complete preorder $\preceq$ on a subset $M'(X)$ of the space $M(X)$ of all upper-continuous capacities.

2. Notation and preliminaries

Throughout this paper, we shall always consider a compact metric space (X, d), endowed with its Borel σ-algebra, denoted by Σ. The space of all upper-continuous capacities on Σ will be denoted by $M(X)$ (see Zhou [7]). We recall that a *capacity* μ on Σ (i.e., a function from Σ into $[0,1]$ such that $\mu(\varnothing) = 0$, $\mu(X) = 1$, and $\mu(A) \leq \mu(B)$ for all $A \subseteq B$, $A, B \in \Sigma$) is said to be *upper-continuous* if $\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$ for any weakly decreasing sequence of sets $\{A_n\}$ with $A_n \in \Sigma$ for all n. A sequence $\{\mu_n\} \subseteq M(X)$ is said to *converge weakly* to $\mu \in M(X)$ if

$$\int_X f d\mu_n \to \int_X f d\mu \text{ for all } f \in C(X),$$

with $C(X)$ the space of all continuous real-valued functions on (X, d), and $\int_X f d\mu$ the *Choquet integral* of f with respect to μ, namely

$$\int_X f d\mu = \int_0^{\infty} \mu(f \geq t)dt + \int_{-\infty}^{0} (\mu(f \geq t) - 1)dt.$$

The corresponding topology (i.e., the *weak topology* on $M(X)$) will be denoted by τ^w. The reader could refer to the comprehensive book by Denneberg [2] for details concerning the basic properties of the Choquet integral. More recent results on the Choquet integral with respect to upper-continuous capacities are found in Zhou [7].

Given a *complete preorder* (i.e., a reflexive, transitive and complete binary relation) $\preceq$ on a subset $M'(X)$ of $M(X)$, we are interested in the existence of a utility function U for $\preceq$ (i.e., a real-valued function on $M'(X)$ such that, for every $\mu, \nu \in M'(X)$, $\mu \preceq \nu$ if and only if $U(\mu) \leq U(\nu)$) which is continuous or at least upper semicontinuous in the topology induced on $M'(X)$ by the weak topology τ^w. We recall that a complete preorder $\preceq$ on a subset $M'(X)$ of $M(X)$ is said to be *upper* (*lower*) *semicontinuous* if $\{\nu \in M'(X) : \mu \preceq \nu\}$ ($\{\nu \in M'(X) : \nu \preceq \mu\}$) is a closed set for every $\mu \in M'(X)$. Further, a complete preorder $\preceq$ is said to be *continuous* if it is both upper and lower semicontinuous.

3. Continuous representations

In the following theorem, we are concerned with the existence of a continuous or at least upper semicontinuous utility function for a complete preorder on an arbitrary set $M'(X)$ of upper-continuous capacities.

Theorem 3.1. *Let (X, d) be a compact metric space. Then the following statements hold:*

(i) *Every upper semicontinuous complete preorder $\preceq$ on every subset $M'(X)$ of $M(X)$ admits an upper semicontinuous utility function U;*
(ii) *Every continuous complete preorder $\preceq$ on every subset $M'(X)$ of $M(X)$ admits a continuous utility function U.*

Proof. From Zhou [7, Theorem 3], the space $(M(X), \tau^w)$ is a compact metric space, and therefore it is in particular a separable metric space (see e.g. Engelking [3, Theorem 4.1.18]). Then every subset $M'(X)$ of $M(X)$ can be metrized as a separable metric space, and therefore as a second countable metric space (see e.g. Engelking [3, Corollary 4.1.16]). If $\preceq$ is any upper semicontinuous complete preorder on $(M'(X), \tau^w_{M'(X)})$, then $\preceq$ admits an upper semicontinuous utility function U by Rader's theorem (see Rader [6, Theorem 1]). If $\preceq$ is any continuous complete preorder on $(M'(X), \tau^w_{M'(X)})$, then $\preceq$ admits a continuous utility function U by Debreu's theorem (see Debreu [1, Proposition 3]). So the proof is complete. □

For an additive capacity μ on Σ, the condition of upper-continuity is equivalent to the condition of countable additivity. Therefore, since the space $\Delta(X)$ of all countably additive probability measures on (X, Σ) is contained in $M(X)$, Theorem 3.1 generalizes Theorem 1 in Grandmont [5] in case that a compact metric space is considered.

Given a compact metric space (X, d), and a complete preorder $\preceq$ on a subset $M'(X)$ of $M(X)$, containing the set D of all probability measures on the measurable space (X, Σ) which are concentrated (i.e., $D = \{p \in \Delta(X) : p = p_x$ for some $x \in X\}$ with p_x the probability measure assigning probability 1 to the Borel set $\{x\}$), we can consider the complete preorder $\preceq^X$ on X which is *induced by the complete preorder* $\preceq$ *on* $M'(X)$, in the sense that, for every $x, y \in X$, $x \preceq^X y$ if and only if $p_x \preceq p_y$. The following corollary to the previous theorem concerns the representability of $\preceq^X$ by means of a continuous or at least upper semicontinuous utility function u on (X, d).

Corollary 3.2. *Let (X, d) be a compact metric space. Then the following statements hold:*

(i) *For every upper semicontinuous complete preorder $\preceq$ on every subset $M'(X)$ of $M(X)$ containing D, the induced complete preorder $\preceq^X$ admits an upper semicontinuous utility function u;*
(ii) *For every continuous complete preorder $\preceq$ on every subset $M'(X)$ of $M(X)$ containing D, the induced complete preorder $\preceq^X$ admits a continuous utility function u.*

Proof. Given an upper semicontinuous complete preorder $\preceq$ on any subset $M'(X)$ of $M(X)$ containing D, by the previous theorem there exists an upper semicontinuous utility function U for $\preceq$. We claim that the real-valued function u on X defined by

$$u(x) = U(p_x) \ \ (x \in X)$$

is an upper semicontinuous utility function for the induced complete preorder $\preceq^X$ on X. It is straightforward to show that u is a utility function for $\preceq^X$. Indeed, we have

$$x \preceq^X y \Leftrightarrow p_x \preceq p_y \Leftrightarrow U(p_x) \leq U(p_y) \Leftrightarrow u(x) \leq u(y)$$

for every $x, y \in X$. In order to prove that u is upper semicontinuous (i.e., the set $\{x \in X : \alpha \leq u(x)\}$ is closed for every real number α), consider any real number α, any point $x \in X$, and any sequence $\{x_n\} \subseteq X$ converging to x such that $\alpha \leq u(x_n)$ for every n. Since the sequence $\{p_{x_n}\} \subseteq D$ converges to $p_x \in D$, U is upper semicontinuous, and from the definition of U we have $\alpha \leq U(p_{x_n})$ for every n, it must be $\alpha \leq U(p_x) = u(x)$, and therefore the conclusion follows. This consideration finishes the first part of the proof.

If $\preceq$ is a complete preorder on any subset $M'(X)$ of $M(X)$ containing D, by the previous theorem there exists a continuous utility function U for $\preceq$. Then, by analogous considerations it can be shown that the function u defined above is a continuous utility function for $\preceq^X$. So the proof is complete. $\square$

It is almost immediate to check that the statements named (i) in the previous theorem and corollary are still valid if we replace the terms *"upper semicontinuous complete preorder"* and *"upper semicontinuous utility function"* by the terms *"lower semicontinuous complete preorder"* and respectively *"lower semicontinuous utility function"*. Indeed, one can replace functions U and u by $-U$ and respectively $-u$, and then apply the previous results by considering the *dual complete preorders* $\preceq_d$ and $\preceq_d^X$ defined by $[\mu \preceq_d \nu \Leftrightarrow \nu \preceq \mu]$ and respectively $[x \preceq_d^X y \Leftrightarrow y \preceq^X x]$.

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