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New common fixed point theorems for multivalued maps

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Abstract

Common fixed point theorems for a new class of multivalued maps are obtained, which generalize and extend classical fixed point theorems of Nadler and Reich and some recent Suzuki type fixed point theorems.

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1. Introduction

Let (X, d) be a metric space and $CL(X)$ the family of all nonempty closed subsets of X. $(CL(X), H)$ equipped with the generalized Hausdorff metric H defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

where $A, B \in CL(X)$ and $d(x, K) = \inf_{z\in K} d(x, z)$, is called the generalized hyperspace of X.

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For any nonempty subsets A, B of $X, d(A, B)$ denotes the gap between the subsets A and B, while

$$\rho(A,B) = \sup\{d(a,b) : a \in A, b \in B\},$$
$$BN(X) = \{A : \varnothing \neq A \subseteq X \text{ and the diameter of } A \text{ is finite}\}.$$

As usual, we write $d(x, B)$ (resp. $\rho(x, B)$) for $d(A, B)$ (resp. $\rho(A, B)$) when $A = \{x\}$. For $x, y \in X$, we follow the following notation, where S and T are maps to be defined specifically in a particular context:

$$M(Sx,Ty) = \left\{d(x,y), \frac{d(x,Sx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Sx)}{2}\right\}.$$

Recently Suzuki [23] obtained a forceful generalization of the famous Banach contraction theorem. Subsequently, a number of new fixed point theorems have been established and some applications have been discussed (see, for instance, [1, 5, 6, 7, 8, 9, 10, 13, 16, 20, 21, 22, 24]).

The following result is essentially due to Kikkawa and Suzuki [8] (see also [22]) which generalizes the classical multivalued contraction theorem due to Nadler [11] (see also [2, 12, 14, 18]).

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to CL(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$d(x,Tx) \leq (1+r)d(x,y) \quad \textit{implies} \quad H(Tx,Ty) \leq rd(x,y).$$

Then there exists $z \in X$ such that $z \in Tz$.

The following generalization of Theorem 1.1 is due to Singh and Mishra [20].

Theorem 1.2. *Let X be a complete metric space and $T : X \to CL(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$d(x,Tx) \leq (1+r)d(x,y) \quad \textit{implies} \quad H(Tx,Ty) \leq rM(Tx,Ty).$$

Then there exists $z \in X$ such that $z \in Tz$.

The following general common fixed point theorem is due to Sastry and Naidu [19].

Theorem 1.3. *Let X be a complete metric space and S, T maps from X to itself. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$d(Sx,Ty) \leq r \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty)+d(y,Sx)}{2}\right\}. \tag{1.1}$$

Then S and T have a unique common fixed point.

For an excellent discussion on several special cases and variants of Theorem 1.3, one may refer to Rus [18]. The generality of Theorem 1.3 may be appreciated from the fact that the condition (1.1) in Theorem 1.3 cannot be replaced by a slightly more general condition:

$$d(Sx,Ty) \leq r \max\{d(x,y), d(x,Sx), d(y,Ty), d(x,Ty), d(y,Sx)\}. \tag{1.2}$$

See [19, Ex. 5]. Notice that the condition (1.2) with $S = T$ is Ćirić's quasi-contraction [4]. We remark that, in Rhoades' comprehensive comparison of contractive conditions [15], the condition (1.2) with $S = T$ is considered the most general contraction for a self-map of a metric space.

A particular case of our main result (cf. Theorem 2.1) generalizes Theorems 1.1 and 1.2. Some other special cases are also discussed.

2. Main Results

We shall need the following lemma essentially due to Nadler, Jr. [11] (see also [2], [3], [16, p. 4], [16, 17], [18, p. 76]).

Lemma 2.1. *If $A, B \in CL(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$.*

Theorem 2.2. *Let X be a complete metric space and let S and T maps from X to $CL(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$\min\{d(x, Sx), d(y, Ty)\} \leq (1 + r)d(x, y) \quad \textit{implies} \quad H(Sx, Ty) \leq rM(Sx, Ty).$$

Then there exists an element $u \in X$ such that $u \in Su \cap Tu$.

Proof. Obviously $M(Sx, Ty) = 0$ iff $x = y$ is a common fixed point of S and T. So we may assume that $M(Sx, Ty) > 0$.

Let $\varepsilon > 0$ be such that $\beta = r + \varepsilon < 1$. Let $u_0 \in X$ and $u_1 \in Tu_0$. By Lemma 2.1, their exists $u_2 \in Su_1$ such that

$$d(u_2, u_1) \leq H(Su_1, Tu_0) + M(Su_1, Tu_0).$$

Similarly, their exists $u_3 \in Tu_2$ such that

$$d(u_3, u_2) \leq H(Tu_2, Su_1) + \varepsilon M(Tu_2, Su_1).$$

Continuing in this manner, we find a sequence $\{u_n\}$ in X such that

$$u_{2n+1} \in Tu_{2n}, u_{2n+2} \in Su_{2n+1}$$

and

$$d(u_{2n+1}, u_{2n}) \leq H(Tu_{2n}, Su_{2n-1}) + M(Tu_{2n}, Su_{2n-1}),$$
$$d(u_{2n+2}, u_{2n+1}) \leq H(Su_{2n+1}, Tu_{2n}) + \varepsilon M(Su_{2n+1}, Tu_{2n}).$$

Now, we show that for any $n \in N$,

$$d(u_{2n+1}, u_{2n}) \leq \beta d(u_{2n-1}, u_{2n}). \tag{2.1}$$

Suppose if $d(u_{2n-1}, Su_{2n-1}) \geq d(u_{2n}, Tu_{2n})$, then

$$\min\{d(u_{2n-1}, Su_{2n-1})d(u_{2n}, Tu_{2n})\} \leq (1 + r)d(u_{2n-1}, u_{2n}).$$

Therefore by the assumption,

$$\begin{aligned} d(u_{2n+1}, u_{2n}) &\le H(Su_{2n-1}, Tu_{2n}) \\ &\le rM(Su_{2n-1}, Tu_{2n}) \\ &\le rM(Su_{2n-1}, Tu_{2n}) + \varepsilon M(Su_{2n-1}, Tu_{2n}) \\ &= \beta M(Su_{2n-1}, Tu_{2n}) \\ &= \beta \max \Big\{ d(u_{2n-1}, u_{2n}), \frac{d(u_{2n-1}, Su_{2n-1}) + d(u_{2n}, Tu_{2n})}{2}, \\ &\qquad \frac{d(u_{2n-1}, Tu_{2n}) + d(u_{2n}, Su_{2n-1})}{2} \Big\} \\ &\le \beta \max d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1}). \end{aligned}$$

This yields (2.1).
Suppose, if $d(u_{2n}, Tu_{2n}) \ge d(u_{2n-1}, Su_{2n-1})$, then

$$\min\{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} \le (1 + r)d(u_{2n-1}, u_{2n}).$$

Therefore by the assumption,

$$\begin{aligned} d(u_{2n+1}, u_{2n}) &\le H(Su_{2n-1}, Tu_{2n}) \\ &\le rM(Su_{2n-1}, Tu_{2n}) \\ &\le rM(Su_{2n-1}, Tu_{2n}) + \varepsilon M(Su_{2n-1}, Tu_{2n}) \\ &= \beta M(Su_{2n-1}, Tu_{2n}) \end{aligned}$$

$$\begin{aligned} &= \beta \max \Big\{ d(u_{2n-1}, u_{2n}), \frac{d(u_{2n-1}, Su_{2n-1}) + d(u_{2n}, Tu_{2n})}{2}, \\ &\qquad \frac{d(u_{2n-1}, Tu_{2n}) + d(u_{2n}, Su_{2n-1})}{2} \Big\} \\ &\le \beta \max\{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1})\}. \end{aligned}$$

This prove (2.1). In an analogous manner, we show that

$$d(u_{2n+2}, u_{2n+1}) \le \beta d(u_{2n+1}, u_{2n}). \tag{2.2}$$

We conclude from (2.1) and (2.2) that for any $n \in N$,

$$d(u_{n+1}, u_n) \le \beta d(u_n, u_{n-1}).$$

Therefore $\{u_n\}$ is a Cauchy sequence and has a limit in X. Call it u.
Since $u_n \to u$, there exists $n_0 \in N$ (natural numbers) such that

$$d(u, u_n) \le \frac{1}{3} d(u, y) \quad \text{for } y \ne u \text{ and all } n \ge n_0.$$

Then as in [23, p. 1862],

$$\begin{aligned}(1+r)^{-1}d(u_{2n-1},Su_{2n-1}) &\leq d(u_{2n-1},Su_{2n-1})\\ &\leq d(u_{2n-1},u_{2n})\\ &\leq d(u_{2n-1},u)+d(u,u_{2n})\\ &\leq \frac{2}{3}d(y,u)\\ &= d(y,u)-\frac{1}{3}d(y,u)\\ &\leq d(y,u)-d(u_{2n-1},u)\\ &\leq d(u_{2n-1},y).\end{aligned}$$

Therefore

$$d(u_{2n-1},Su_{2n-1}) \leq (1+r)d(u_{2n-1},y). \tag{2.3}$$

Now either $d(u_{2n-1},Su_{2n-1}) \leq d(y,Ty)$ or $d(y,Ty) \leq d(u_{2n-1},Su_{2n-1})$. In either case, by (2.3) and the assumption,

$$\begin{aligned}d(u_{2n},Ty) &\leq H(Su_{2n-1},Ty)\\ &\leq rM(Su_{2n-1},Ty).\\ &\leq r\max\left\{d(u_{2n-1},y),\frac{d(u_{2n-1},Su_{2n-1})+d(y,Ty)}{2},\right.\\ &\qquad\left.\frac{d(u_{2n-1},Ty)+d(y,Su_{2n-1})}{2}\right\}.\end{aligned}$$

Making $n\to\infty$,

$$\begin{aligned}d(u,Ty) &\leq r\max\left\{d(u,y),\frac{d(u,u)+d(y,Ty)}{2},\frac{d(u,Ty)+d(y,u)}{2}\right\}\\ &\leq r\max\left\{d(u,y),\frac{d(u,Ty)+d(u,y)}{2}\right\}.\end{aligned}\tag{2.4}$$

It is clear from (2.4) that

$$d(u,Ty) \leq rd(u,y). \tag{2.5}$$

Now we show that

$$H(Su,Ty) \leq r\max\left\{d(u,y),\frac{d(u,Su)+d(y,Ty)}{2},\frac{d(u,Ty)+d(y,Su)}{2}\right\} \tag{2.6}$$

Assume that $y\neq u$. Then for every $n\in N$, there exists $z_n\in Ty$ such that

$$d(u,z_n) \leq d(u,Ty)+\frac{1}{n}d(y,u).$$

So we have by (2.5),

$$\begin{aligned} d(y,Ty) &\le d(y,z_n) \\ &\le d(y,u)+d(u,z_n) \\ &\le d(y,u)+d(u,Ty)+\frac{1}{n}d(y,u) \\ &\le d(y,u)+rd(u,y)+\frac{1}{n}d(u,y) \\ &= \left(1+r+\frac{1}{n}\right)d(y,u). \end{aligned}$$

Hence

$$d(y,Ty) \le (1+r)d(y,u). \tag{2.7}$$

Now either $d(u,Su) \le d(y,Ty)$ or $d(y,Ty) \le d(u,Su)$.
So in either case by (2.7) and the assumption, $H(Su,Ty) \le rM(Su,Ty)$, which is (2.6).
Now taking $y = u_{2n}$ in (2.6), we have

$$\begin{aligned} d(Su,u_{2n+1}) &\le H(Su,Tu_{2n}) \\ &\le r\max\left\{d(u,u_{2n}), \frac{d(u,Su)+d(u_{2n},u_{2n+1})}{2},\right. \\ &\qquad \left.\frac{d(u,u_{2n+1})+d(u_{2n},Su)}{2}\right\}. \end{aligned}$$

Passing to the limit this obtains $d(Su,u) \le \frac{r}{2}d(Su,u)$. So $u \in Su$, as Su is closed.

In an analogous manner, we can show that $u \in Tu$. □

Corollary 2.3. *Let X be a complete metric space and $S,T : X \to X$. Assume there exists $r \in [0,1)$ such that for every $x,y \in X$,*

$$\min\{d(x,Sx),d(y,Ty)\} \le (1+r)d(x,y) \quad \textit{implies} \quad d(Sx,Ty) \le rM(Sx,Ty).$$

Then S and T have a unique common fixed point.

Proof. It comes from Theorem 2.2 that S and T have a common fixed point. The uniqueness of the common fixed point follows easily. □

Corollary 2.4. *Theorem* 1.2.

Corollary 2.5 ([20])**.** *Let X be a complete metric space and $T : X \to X$. Assume there exists $r \in [0,1)$ such that for every $x,y \in X$,*

$$d(x,Tx) \le (1+r)d(x,y) \quad \textit{implies} \quad d(Tx,Ty) \le rM(Tx,Ty).$$

Then T has a unique fixed point.

Proof. It comes from Corollary 2.3 when $S = T$. □

Now we give an application of Corollary 2.3.

Theorem 2.6. *Let $P, Q : X \to BN(X)$. Assume there exists $r \in [0,1)$ such that for every $x, y \in X$,*

$$\min\{\rho(x, Px), \rho(y, Qy)\} \leq (1+r)d(x,y) \tag{2.8}$$

implies

$$\rho(Px, Qy) \leq r \max\left\{d(x,y), \frac{\rho(x,Px)+\rho(y,Qy)}{2}, \frac{d(x,Qy)+d(y,Px)}{2}\right\} \tag{2.9}$$

Then there exsits a unique point $z \in X$ such that $z \in Pz \cap Qz$.

Proof. Choose $\lambda \in (0,1)$. Define single-valued maps $S, T : X \to X$ as follows. For each $x \in X$, let Sx be a point of Px which satisfies

$$d(x, Sx) \geq r^{\lambda}\rho(x, Px).$$

Similarly, for each $y \in X$, let Ty be a point of Qy such that

$$d(y, Ty) \geq r^{\lambda}\rho(y, Qy).$$

Since $Sx \in Px$ and $Ty \in Qy$,

$$d(x, Sx) \leq \rho(x, Px) \quad \text{and} \quad d(y, Ty) \leq \rho(y, Qy).$$

So (2.8) gives

$$\min\{d(x,Sx), d(y,Ty)\} \leq \min\{\rho(x,Px), \rho(y,Qy)\} \leq (1+r)d(x,y), \tag{2.10}$$

and this implies (2.9). Therefore

$$\begin{aligned} d(Sx, Ty) &\leq \rho(Px, Qy) \\ &\leq r.r^{-\lambda} \max\left\{r^{\lambda}d(x,y), \frac{r^{\lambda}\rho(x,Px)+r^{\lambda}\rho(y,Qy)}{2},\right. \\ &\qquad \left.\frac{r^{\lambda}d(x,Qy)+r^{\lambda}d(y,Px)}{2}\right\} \\ &\leq r^{1-\lambda} \max\left\{d(x,y), \frac{d(x,Sx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Sx)}{2}\right\}. \end{aligned}$$

So (2.10), viz., $\min\{d(x,Sx), d(y,Ty)\} \leq (1+r')d(x,y)$ imlpies

$$d(Sx, Ty) \leq r' \max\left\{d(x,y), \frac{d(x,Sx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Sx)}{2}\right\},$$

where $r' = r^{1-\lambda} < 1$.

Hence by Corollary 2.3, S and T have a unique point $z \in X$ such that $Sz = Tz = z$. This implies $z \in Pz \cap Qz$. □

The following result show that Theorem 2.6 is a generalization of the result of Singh and Mishra [20, Theorem 3.6].

Corollary 2.7. *Let $P : X \to BN(X)$. Assume there exists $r \in [0,1)$ such that*

$$\rho(x, Px) \leq (1+r)d(x,y)$$

implies

$$\rho(Px, Py) \leq r \max\left\{d(x,y), \frac{\rho(x,Px)+\rho(y,Py)}{2}, \frac{d(x,Py)+d(y,Px)}{2}\right\}.$$

Then there exists a unique point z in X such that $z \in Pz$.

Proof. It comes from Theorem 2.6 when $Q = P$. □

We remark that Corollaries 2.5 and 2.7 generalize fixed point theorems from [11, 14, 18] and others.

Now we give two examples to show the generality of our results.

Example 2.8. Let $X = \{(0,0),(4,0),(0,4),(4,5),(5,4)\}$ and d be defined by

$$d[(x_1,x_2),(y_1,y_2)] = |x_1 - y_1| + |x_2 - y_2|.$$

Let S and T be such that

$$S(x_1,x_2) = \begin{cases}(x_1,0) & \text{if } x_1 \leq x_2\\ (0,x_2) & \text{if } x_1 > x_2\end{cases} \text{ and } \quad T(x_1,x_2) = \begin{cases}(0,x_1) & \text{if } x_1 \leq x_2\\ (0,x_2) & \text{if } x_1 > x_2\end{cases}$$

Then maps S and T do not satisfy (1.1) of Theorem 1.3 (e.g. $(x,y) = ((4,5),(5,4))$). However, S and T satisfy all the hypotheses of Corollary 2.3.

Example 2.9. Let $X = \{(1,1),(4,1),(1,4),(4,5),(5,4)\}$ and d be defined by

$$d[(x_1,x_2),(y_1,y_2)] = |x_1 - y_1| + |x_2 - y_2|$$

Let T be such that

$$T(x_1,x_2) = \begin{cases}(x_1,1) & \text{if } x_1 \leq x_2\\ (1,x_2) & \text{if } x_1 > x_2\end{cases}$$

Then T satisfies all the hypotheses of Corollary 2.5, but does not satisfy Ciric's quasi-contraction, viz. (1.2) with $S = T(e.g. x = (4,5), y = (5,4))$.

We close this paper with the following.

Question 2.10. Can we replace "$H(Sx,Ty) \leq rM(Sx,Ty)$" in Theorem 2.1 by the following:

$$(2.11) \quad H(Sx,Ty) \leq r \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty)+d(y,Sx)}{2}\right\}.$$

We remark that (2.11) with $S = T$ is the Ciric's generalized contraction [3] for $T : X \to CL(X)$.

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