On the existence of best proximity points for generalized contractions

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ABSTRACT

In this article we establish the existence of a unique best proximity point for some generalized non-self contractions on a metric space in a simpler way using a geometric result. Our results generalize some recent best proximity point theorems and several fixed point theorems proved by various authors.

1. INTRODUCTION

Fixed point theory plays an important role in supplying a uniform treatment for solving equations of the form $Tx = x$ where T is a self mapping defined on a subset of a metric space, partially ordered metric space, topological vector space or some suitable space. Given two non-empty subsets A and B of a metric space (X, d), consider a non-self mapping $T : A \to B$. The mapping T is said to be a $k-contraction$ if $d(Tx, Ty) \le kd(x, y)$ hold $\forall\ x, y \in A$ and for some $k \in [0, 1)$. If T is a self map, that is, if $A = B$ and A is complete, then the famous Banach contraction principle implies that T has a unique fixed point in A. As this principle has applications in various fields, many generalizations of this principle have appeared in the literature (see [12, 13]) by generalizing

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the contractive condition used by Banach. In 1975, Matkowski [2] used the following contractive condition:

$$(1.1) \qquad d(Tx, Ty) \leq \varphi(d(x,y)), \quad \forall \ x, y \in A,$$

where φ is a function from $\mathbb{R}_+$, the set of all nonnegative reals, into $\mathbb{R}_+$ such that φ is nondecreasing and satisfies $lim_{n \to \infty} \varphi^n(t) = 0$ for any positive t. Matkowski [2] proved that T has a unique fixed point if T is a self map and A is complete. On the other hand, Rhoades [3] in 2001 gave an existence result of unique fixed point for mappings satisfying the following contractive condition:

$$(1.2) \qquad d(Tx, Ty) \leq d(x,y) - \psi(d(x,y)), \ \forall \ x, y \in A,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, continuous function with $\psi^{-1}(0) = \{0\}$ and $\lim_{t \to \infty} \psi(t) = \infty$ (If A is bounded, then the infinity condition can be omitted). Rhoades [3] proved that T has a unique fixed point if T is a self map and A is complete. Next, we present a brief discussion about best proximity point.

It is clear that $T(A) \cap A \neq \varnothing$ is a necessary (but not sufficient) condition for the existence of a fixed point for the map $T : A \to B$. If the necessary condition fails, then $d(x, Tx) > 0$, for all $x \in A$. This means that the mapping $T : A \to B$ does not have any fixed point, that is, the equation $Tx = x$ has no solution. From this point of view, we think of a point x in A which is closest to Tx in some sense. Best approximation and best proximity point results are being studied in this direction. The well-known best approximation theorem due to Ky Fan [14] states that if M is a non-empty compact convex subset of a normed linear space E and $S : M \to E$ is a continuous function, then there exists a point $x \in M$ such that $\|x - Sx\| = d(Sx, M) = inf\{\|Sx - a\| : \ a \in M\}$. Such an element $x \in M$ satisfying $\|x - Sx\| = d(Sx, M)$ is called a best approximant.

On the other hand, though a best approximant acts as an approximate solution of the equation $Sx = x$, such element is not an optimal solution in the sense that the distance between x and Sx is minimum. Naturally for given subsets A and B of a metric space and a mapping $T : A \to B$ one can think of finding a point $x^* \in A$ such that $d(x^*, Tx^*) = min\{d(x, Tx) : x \in A\}$. As $d(x, Tx) \geq dist(A, B) = inf\{d(a, b) : a \in A, \ b \in B\} \ \forall \ x \in A$, then an optimal solution of $min\{d(x, Tx) : x \in A\}$ is one for which the value $dist(A, B)$ is attained. A point $x^* \in A$ is said to be a best proximity point for the function $T : A \to B$ if $d(x^*, Tx^*) = dist(A, B)$. So a best proximity point of the map T is an approximate solution of the equation $Tx = x$ which is optimal in the sense that distance between x and Tx is minimum. It is clear that all best proximity point theorems work as a natural generalization of fixed point theorems if T is a self-map. For some interesting best proximity point results one can refer to [6, 7, 9, 10]. Some applications of best proximity point results can be found in [15, 16]. Recently V. Sankar Raj [4] obtained the following best proximity point theorem for mappings satisfying (1.2).

Theorem 1.1 ([4, Theorem 3.1])**.** *Let A, B be two non-empty closed subsets of a complete metric space (X, d) such that the pair (A, B) has the P-property*

and $A_0 \neq \varnothing$ and $T : A \to B$ be a mapping such that $T(A_0) \subseteq B_0$ and it satisfies(1.2). Then there exists a unique x^* in A such that $d(x^*, Tx^*) = dist(A, B)$.

1.1. **Our contribution.** In this paper we prove the existence of a unique best proximity point for mappings satisfying the contractive condition (1.1) and for mappings satisfying a condition which is a weaker form of condition (1.2) (where ψ is assumed to be either continuous or nondecreasing and the infinity condition is not needed). Our result enables us prove the above Theorem 1.1 under weaker assumptions. In addition, our theorem includes the generalization of Banach's contraction principle due to Matkowski [2, Theorem 1.2] and help us to improve [3, Theorem 1] by Rhoades.

2. Preliminaries

In this section we give some definitions and results which are useful and related to context of our results.

Let A and B be two non-empty subsets of a metric space (X, d). Throughout this article we denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = dist(A, B) \text{ for some } y \in B\}$$
$$B_0 = \{y \in B : d(x, y) = dist(A, B) \text{ for some } x \in A\}.$$

For the sufficient conditions for the non-emptiness of A_0 and B_0, one can refer to [11].

Let (A, B) be a pair of two non-empty subsets of a metric space (X, d) with $A_0 \neq \varnothing$. Then the pair (A, B) is said to have the P-property [4] if and only if

$$\left. \begin{array}{l} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{array} \right\} \quad \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is easy to check that for a non-empty subset A of (X, d), the pair (A, A) has the P-property.

Example 2.1 ([4]). Let A and B be two non-empty closed convex subsets of a real Hilbert space H, then the pair (A, B) has the P-property.

Example 2.2 ([8]). Let A and B be two nonempty bounded closed convex subsets of a uniformly convex Banach space X, the pair (A, B) has the P-property.

3. Main results

We begin this section with the following two auxiliary results.

Lemma 3.1. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\psi^{-1}(0) = \{0\}$ and ψ is either nondecreasing or continuous. Then, for any bounded sequence $\{t_n\}$ of positive reals, $\psi(t_n) \to 0$ implies $t_n \to 0$.*

Proof. Let $\{t_n\}$ be a bounded sequence of positive reals such that $\psi(t_n) \to 0$.

Let us assume that ψ is nondecreasing. Suppose $t_n \not\to 0$. Then there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $t_{n_k} \geq \delta$ for some $\delta > 0$ and $\forall\, k \in \mathbb{N}$. As ψ is nondecreasing, so $\psi(t_{n_k}) \geq \psi(\delta) \quad \forall\, k$, which is a contradiction.

Let ψ be continuous. Suppose that the sequence $\{t_n\}$ is not convergent. Then $\overline{lim}\, t_n \neq \underline{lim}\, t_n$. This implies that there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $t_{n_k} \to t_0 > 0$ which implies that $\psi(t_{n_k}) \to \psi(t_0) > 0$, a contradiction. Hence $t_n \to \overline{lim}\, t_n = \underline{lim}\, t_n = 0$. $\qquad\square$

Lemma 3.2. *Let A and B be two non-empty subsets of a metric space and $\psi :$ $\mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\psi^{-1}(0) = \{0\}$ and for any bounded sequence $\{t_n\}$ of positive reals, $\psi(t_n) \to 0$ implies $t_n \to 0$. Suppose that $T : A \to B$ be a mapping such that*

$$d(Tx, Ty) \leq d(x,y) - \psi(d(x,y)) \ \forall\, x, y \in A.$$

Then, for every $\epsilon > 0$, there exist $\delta > 0$ and $\gamma \in (0, \epsilon)$ such that for all $x, y \in A$, $d(x,y) < \epsilon + \delta$ implies $d(Tx, Ty) \leq \gamma$.

Proof. Suppose that there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ and $\gamma \in (0, \epsilon_0)$ there exist $x, y \in A$ such that $d(x,y) < \epsilon_0 + \delta$ implies $d(Tx, Ty) > \gamma$. Let $\delta_n = \dfrac{1}{n^2}$ and $\gamma_n = \epsilon_0 \dfrac{n^2}{1+n^2} \ \forall\, n \in \mathbb{N}$, so there exist $\{x_n\}$ and $\{y_n\}$ in A such that

$$(3.1) \qquad d(x_n, y_n) < \epsilon_0 + \frac{1}{n^2} \quad and \quad d(Tx_n, Ty_n) > \epsilon_0 \frac{n^2}{1+n^2}$$

Now, we get

$$\begin{aligned} \epsilon_0 \frac{n^2}{1+n^2} < d(Tx_n, Ty_n) \ &\leq\ d(x_n, y_n) - \psi(d(x_n, y_n)) \\ &<\ \epsilon_0 + \frac{1}{n^2} - \psi(d(x_n, y_n)) \end{aligned}$$

which implies $\psi(d(x_n, y_n)) < \dfrac{\epsilon_0}{1+n^2} + \dfrac{1}{n^2}$. Thus $\psi(d(x_n, y_n)) \to 0$ as $n \to \infty$ and since $\{d(x_n, y_n)\}$ is bounded, by the given hypothesis, $d(Tx_n, Ty_n) \to 0$ as $n \to \infty$. Now by (3.1) $lim_{n\to\infty} d(Tx_n, Ty_n) \geq \epsilon_0$, which is a contradiction. $\quad\square$

Now we recall the following result of Hegedűs and Szilágyi [1, Lemma 1].

Lemma 3.3. *For a given subset D of $\mathbb{R}_+^2 = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\}$, the following statements are equivalent:*

   (i) for any $\epsilon > 0$, there exist $\delta > 0$ and $\gamma \in (0, \epsilon)$ such that for all $(t,u) \in D$, $t < \epsilon + \delta$ implies $u \leq \gamma$;

   (ii) there exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ where φ is continuous and nondecreasing with $\varphi(t) < t, \ \forall\, t > 0$ and $u \leq \varphi(t) \ \forall\, (t,u) \in D$.

The following theorem is our main result which gives sufficient conditions for the existence of a unique best proximity point for some generalized contractions.

Theorem 3.4. *Let A and B be two non-empty closed subsets of a complete metric space (X, d) such that the pair (A, B) has the P-property and $A_0 \neq \varnothing$ and $T : A \to B$ be a mapping such that $T(A_0) \subseteq B_0$, satisfying any one of the following contractive conditions:*

(I) $d(Tx, Ty) \leq \phi(d(x, y))$, $\forall\, x, y \in A$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and satisfies $lim_{n \to \infty} \phi^n(t) = 0$ for any $t > 0$.

(II) $d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))$, $\forall\, x, y \in A$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is either nondecreasing or continuous with $\psi^{-1}(0) = \{0\}$;

Then there exists a unique x^* in A such that $d(x^*, Tx^*) = dist(A, B)$. Moreover, if $x_0 \in A_0$ and x_n is defined by $d(x_n, Tx_{n-1}) = dist(A, B)\ \forall\, n \in \mathbb{N}$, then $x_n \to x^*$ as $n \to \infty$.

Proof. Since A_0 is non-empty, let $x_0 \in A_0$. As $T(x_0) \in T(A_0) \subseteq B_0$, by definition of B_0 there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A, B)$. Again, as $T(x_1) \in T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = dist(A, B)$. Repeating this process, we can obtain a sequence $\{x_n\} \subseteq A_0$ such that

$$(3.2) \qquad d(x_{n+1}, Tx_n) = dist(A, B), \quad \forall\, n \in \mathbb{N}.$$

Assume that $x_{n+1} \neq x_n\ \forall\, n$, otherwise there is nothing to prove. By the P-property of (A, B) it is clear that

$$(3.3) \qquad d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n), \quad \forall\, n \in \mathbb{N}.$$

To prove that $\{x_n\}$ is a Cauchy sequence, we shall first claim that there exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ where φ is nondecreasing such that $\varphi(t) < t$, $\forall\, t > 0$, $\varphi(0) = 0$ and $lim_{n \to \infty} \varphi^n(t) = 0$ for any $t > 0$ and $d(Tx, Ty) \leq \varphi(d(x, y))$ for any $x, y \in A$.

Let us assume that T satisfies condition (I). Clearly $\phi(t) < t$ for $t > 0$. Indeed, if there exists $t_0 > 0$ with $\phi(t_0) \geq t_0$, then $\phi^n(t_0) \geq t_0\ \forall\, n \in \mathbb{N}$ as ϕ is increasing, a contradiction. Also note that $\phi(0) = 0$. Therefore our claim is true by taking $\varphi = \phi$.

Suppose T satisfies (II). Then by applying Lemma 3.1 and Lemma 3.2 we see that for any $\epsilon > 0$, there exist $\delta > 0$ and $\gamma \in (0, \epsilon)$ such that for all $x, y \in A$, $d(x, y) < \epsilon + \delta$ implies $d(Tx, Ty) \leq \gamma$.

Now applying Lemma 3.3 to the set $D = \{(d(x, y), d(Tx, Ty)) : x, y \in A\}$, we see that there exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that φ is continuous and nondecreasing with $\varphi(t) < t$, $\forall\, t > 0$ and $d(Tx, Ty) \leq \varphi(d(x, y))$ for any $x, y \in A$. Clearly $lim_{n \to \infty} \varphi^n(t) = 0$ for any $t > 0$. Indeed if there exists $t_0 > 0$ such that $lim_{n \to \infty} \varphi^n(t_0) = \beta \neq 0$, then $\beta = lim_{n \to \infty} \varphi(\varphi^{n-1}(t_0)) = \varphi(\beta) < \beta$, a contradiction. Therefore our claim is true. Now,

$$d(x_2, x_3) = d(Tx_1, Tx_2) \leq \varphi(d(x_1, x_2)) \leq \varphi^2(d(x_0, x_1)).$$

By induction we get $d(x_n, x_{n+1}) \leq \varphi^n((d(x_0, x_1))\ \ \forall\, n \in \mathbb{N}$. From the hypothesis it is clear that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Thus for a given $\epsilon > 0$ there

exists $N \in \mathbb{N}$ such that

(3.4) $$d(x_n, x_{n+1}) \leq \epsilon - \varphi(\epsilon) \quad \forall\, n \geq N.$$

Denoting a ball with center x and radius ϵ by $B[x, \epsilon]$, we will show the following relations

(a) $T(B[x_N, \epsilon] \cap A) \subseteq B[Tx_{N-1}, \epsilon]$;
(b) $y \in B[Tx_{N-1}, \epsilon]$ with $d(x, y) = dist(A, B)$, $x \in A_0 \Rightarrow x \in B[x_N, \epsilon] \cap A$.

If $x \in B[x_N, \epsilon] \cap A$, then

$$
\begin{aligned}
d(Tx, Tx_{N-1}) &\leq d(Tx, Tx_N) + d(Tx_N, Tx_{N-1}) \\
&\leq \varphi(d(x, x_N)) + d(x_{N+1}, x_N) \\
&\leq \varphi(\epsilon) + \epsilon - \varphi(\epsilon) \leq \epsilon
\end{aligned}
$$

and hence (a) follows.

Let $y \in B[Tx_{N-1}, \epsilon]$ with $d(x, y) = dist(A, B)$, $x \in A_0$. Now by (3.2), $d(x_N, Tx_{N-1}) = dist(A, B)$. Therefore by using the P-property of (A, B) we have $d(x_N, x) = d(Tx_{N-1}, y)$ and hence (b) follows.

From (3.4), it is clear that $x_{N+1} \in B[x_N, \epsilon] \cap A$ and then by (a), we get $Tx_{N+1} \in B[Tx_{N-1}, \epsilon]$. From (3.2), $d(x_{N+2}, Tx_{N+1}) = dist(A, B)$ with $x_{N+2} \in A_0$. Therefore (b) implies $x_{N+2} \in B[x_N, \epsilon] \cap A$. Again by (a), we have $Tx_{N+2} \in B[Tx_{N-1}, \epsilon]$ and from (3.2), $d(x_{N+3}, Tx_{N+2}) = dist(A, B)$ with $x_{N+3} \in A_0$. Again (b) implies that $x_{N+3} \in B[x_N, \epsilon] \cap A$. Continuing this process we can conclude that

$$x_{N+m} \in B[x_N, \epsilon] \cap A, \quad \forall\, m \in \mathbb{N}.$$

Hence $\{x_n\}$ is a Cauchy sequence. As A is closed, there exists an element $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$. As $\varphi(t) < t$ for $t > 0$, we have $d(Tx, Ty) \leq d(x, y) \,\forall\, x, y \in A$ which implies that T is continuous in A. Therefore $Tx_n \to Tx^*$. From the continuity of the distance function we conclude that $d(x_n, Tx_n) \to d(x^*, Tx^*)$. Since $d(x_{n+1}, Tx_n) = dist(A, B) \,\forall\, n$, we have $d(x^*, Tx^*) = dist(A, B)$.

If x_1 and x_2 are two best proximity points of T, by the P-property of (A, B) we have $d(x_1, x_2) = d(Tx_1, Tx_2)$. Then,

$$
\begin{aligned}
d(x_1, x_2) = d(Tx_1, Tx_2) &\leq \varphi(d(x_1, x_2)) \\
&< d(x_1, x_2) \quad [\text{ since } \varphi(t) < t, \ \forall\, t > 0]
\end{aligned}
$$

which implies that $x_1 = x_2$. $\qquad\square$

Since, for any nonempty subset A of X, the pair (A, A) has the P-property, we can deduce the following result, as a corollary from the above theorem, by taking $A = B$.

Corollary 3.5. *Let (X, d) be a complete metric space and A be a nonempty closed subset of X. Let $T : A \to A$ be a self-map satisfying condition (II). Then T has a unique fixed point x in A.*

Remark 3.6. If the mapping satisfies conditions (II), it follows from Theorem 3.4 that the assumptions of Theorem 1.1 on the function ψ can be weakened. The following example illustrates that Theorem 3.4 generalizes Theorem 1.1.

Example 3.7. Let X be the set consists of the interval $[0, 1]$ together with the natural numbers $2, 3, 4, \cdots$. Let $d : X \times X \to \mathbb{R}$ such that

$$
\begin{aligned}
d(x, y) &= |x - y| \quad \text{if } x = y \text{ or both } x, y \in [0, 1], \\
&= x + y \quad \text{ if one of } x, y \notin [0, 1].
\end{aligned}
$$

Then (X, d) is a complete metric space (see [5, Remarks 3]). Let $A = [0, 1] \cup \{3, 5, 7, \cdots\}$ and $B = [0, 1] \cup \{2, 4, 6, \cdots\}$ be two subsets of X. Define the map $T : A \to B$ by

$$
\begin{aligned}
T(x) &= x - \frac{1}{2}x^2 \quad \text{if } x \in [0, 1], \\
&= x - 1 \quad \text{ if } x = 3, 5, 7, \cdots.
\end{aligned}
$$

Now, for $x, y \in [0, 1]$ with $x \neq y$,

$$
d(Tx, Ty) = \left| (x - y)(1 - \frac{1}{2}(x + y)) \right| \leq d(x, y)(1 - \frac{1}{2}d(x, y)),
$$

if $x \in \{3, 5, \cdots\}$ and $y \in A$ with $x \neq y$,

$$
d(Tx, Ty) = Tx + Ty \leq x + y - 1 = d(x, y) - 1.
$$

Thus, if we consider the map $\psi : [0, \infty) \to [0, \infty)$ by

$$
\begin{aligned}
\psi(t) &= \frac{1}{2}t^2 \quad 0 \leq t \leq 1 \\
&= 1 \quad 1 < t < \infty,
\end{aligned}
$$

then $d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)) \ \forall \ x, y \in A$ where ψ is nondecreasing with $\psi^{-1}(0) = \{0\}$. It is easy to check that A, B are closed subsets of X and (A, B) has the P-property. Also, $A_0 = B_0 = [0, 1]$ and $T(A_0) \subseteq B_0$. Thus, all the assumptions of Theorem 3.4 hold and note that $x^* = 0$ is the unique best proximity point.

Suppose that T satisfies (1.2) for some φ where $\lim_{t \to \infty} \varphi(t) = \infty$ as A is unbounded. Consider a sequence $\{t_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ where $t_n = d(0, 2n + 1)$ for $n \geq 1$. Since $t_n \to \infty$ and $\lim_{t \to \infty} \varphi(t) = \infty$, $\varphi(t_n) \to \infty$. Now, for $n \geq 1$,

$$
\begin{aligned}
(2n + 1) - 1 &= T(0) + T(2n + 1) \\
&= d(T(0), T(2n + 1)) \\
&\leq d(0, 2n + 1) - \varphi(d(0, 2n + 1)) = (2n + 1) - \varphi(t_n).
\end{aligned}
$$

Hence, $\varphi(t_n) \leq 1 \ \forall \ n \geq 1$, a contradiction. Thus Theorem 1.1 cannot be used to give the existence of the solution x^*.

Remark 3.8. As a corollary we get [2, Theorem 1.2] due to Matkowski (see also [12, p. 15]), from Theorem 3.4, by considering $A = B$ when the mapping satisfies (I).

Remark 3.9. As (II) includes (1.2), Corollary 3.5 is a generalized version of [3, Theorem 1] due to Rhoades and the following example justifies that.

Example 3.10. Let (X, d) be the metric space as in Example 3.7 and $A = X$. Define the mapping $T : A \to A$ by

$$
\begin{aligned}
T(x) &= x - \frac{1}{2}x^2 && \text{if } x \in [0, 1], \\
&= x - 1 && \text{if } x = 2, 3, 4, \cdots .
\end{aligned}
$$

Similar to Example 3.7, it is easy to check that

$$
\begin{aligned}
d(Tx, Ty) &\leq d(x,y)(1 - \frac{1}{2}d(x,y)) && \text{if } x, y \in [0, 1], \\
&\leq d(x,y) - 1 && \text{if } x \in \{2, 3, \cdots\} \text{ and } y \in A \text{ with } x \neq y.
\end{aligned}
$$

We see that $d(Tx, Ty) \leq d(x,y) - \psi(d(x,y)) \ \forall \ x, y \in A$, where ψ is the function as in Example 3.7. Thus Corollary 3.5 guarantees the existence of unique fixed point of T and note that $T(0) = 0$. Similar to Example 3.7, it is easy to verify that T does not satisfy (1.2). Thus [3, Theorem 1] cannot be applied to get the fixed point.

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