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Continuous maps in the Bohr Topology

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Abstract. The Bohr topology of an Abelian group G is the initial topology on G with respect to the family of all homomorphisms of G into the circle group. The group G equipped with the Bohr topology is denoted by $G^{\#}$. It was an open question of van Douwen whether for any two discrete abelian groups G and H of the same cardinality the topological *spaces* $G^{\#}$ and $H^{\#}$ are homeomorphic. A negative solution to van Douwen's problem was given independently by Kunen [19] and by Watson and the author [9, 10]. In both cases infinite dimensional vector spaces V_p over the finite field $\mathbb{Z}_p$ were used to show that there is no homeomorphism between $V_p^{\#}$ and $V_q^{\#}$ for $p \neq q$ and $|V_p| = |V_q|$. More precisely, it was shown that every continuous map $V_p^{\#} \to V_q^{\#}$ is constant on an infinite subset of V_p hence cannot be a homeomorphism. Motivated by this phenomenon we establish in this paper the "typical" behavior of a continuous map $f : V_2^{\#} \to H^{\#}$ (and discuss without proofs the more general case $f : V_p^{\#} \to H^{\#}$). The specific choice of $p = 2$ permits to consider V_2 as the set of all finite subsets of an infinite set B (the base of V_2). A special attention will be paid to the restriction of f to the doubletons and the four element subsets of B.

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1. Introduction

The *Bohr compactification* of an abelian group G is a compact group $\mathsf{b}G$ that contains G as a dense subgroup such that every homomorphism of G into a compact group K extends to a continuous homomorphism of the group $\mathsf{b}G$ in K. The *Bohr topology* of the group G is the topology of G induced by the Bohr compactification $r_G : G \to \mathsf{b}G$. This is precisely the initial topology on G with respect to the family of all homomorphisms of G into the circle group. The Bohr compactification and the Bohr topology can be defined analogously for

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an *arbitrary topological* group G. In such a case *continuous* homomorphisms defined on G should be considered. Moreover, if G is not abelian, then the circle group should be replaced by the unitary groups $U(n)$ (where for every natural n the group $U(n)$ consists of all unitary $n \times n$ matrices over the field $\mathbb{C}$ of complex numbers). In general r_G need not be injective, the groups with injective Bohr compactification r_G are known (according to J. von Neumann [26]) as *maximally almost periodic*, related to the fact that these are precisely the groups G such that the almost periodic complex valued functions $G \to \mathbb{C}$ separate the points of G. Locally compact abelian groups are maximally almost periodic. For such a group G we denote by G^+ the group G equipped with the Bohr topology. In this paper we consider only the special case when G is an Abelian group equipped with the *discrete* topology, so following van Douwen [25] we write $G^\#$ in place of G^+ for a discrete group G.

By a classical theorem of Glicksberg [14] G and G^+ have the same compact sets for every locally compact abelian group G. This fact was extended to other compactness-like properties (pseudocompactness, real compactness, Lindelöf property etc.) in [2], [5], [6], [23] (see also [22] for a related result). Hernández [16] proved that also the Lebesgue covering dimension is preserved by the passage $G \mapsto G^+$. On the other hand, this correspondence strongly fails to preserve normality — Trigos [24] proved that $G^\#$ is not normal when G is an arbitrary discrete uncountable abelian group. Answering a question of van Douwen [25] Comfort, Hernández and Trigos ([2, Theorem 3.3]) proved that for a discrete abelian group G the group $G^\#$ is real-compact if and only if $|G|$ is not Ulam-measurable (i.e., when G itself is real-compact). This result is extended in [2, Theorem 3.8] for locally compact abelian groups, namely a LCA group G is real-compact, iff G^+ is real-compact iff G^+ is topologically complete.

The present article cannot certainly cover even a small part of the recent research on Bohr topology (cf. [11, 12, 20]). The main topic here is the continuity (in the Bohr Topology) of maps between abelian groups motivated by the following question set by van Douwen [25] (cf. [1, 515. Question 3F.3.]).

Problem 1.1. *Let G and H be discrete abelian groups with $|G| = |H|$. Are $G^\#$ and $H^\#$ homeomorphic as topological spaces?*

The answer to this question turned out to be negative, i. e., there are two groups of the same cardinality with non-homeomorphic Bohr topologies. Counter-examples, based on different ideas, were given independently and around the same time in [19, 9]. The groups in the counterexample in [9, 10] are uncountable (see Theorem 1.2 below). An example in the countable case was produced by Kunen [19] (see Theorem 5.6).

Throughout the paper κ will be a fixed infinite cardinal number. For a countable abelian group K denote by G_K the direct sum of κ many copies of K (i.e., G_K is the group of functions from κ to K with finite support). For a natural $m > 1$ denote by $\mathbb{Z}_m$ the cyclic group of order m.In the case when

$K = \mathbb{Z}_m$ we write G_m instead of G_K (i.e., G_m will be the direct sum of κ many copies of $\mathbb{Z}_m$).

Theorem 1.2. [10] *If $\kappa > 2^{2^{\mathfrak{c}}} = \beth_3$, then there is no continuous 1-1 function from $G_2^\#$ into $G_3^\#$.*

By means of an elegant inductive argument making use of Ramsey ultrafilters, Kunen [19] established a partition theorem for nets in vector spaces over finite fields and proved that *for distinct primes p and q there is no continuous 1-1 function from $G_p^\#$ into $G_q^\#$ for $\kappa = \omega$* ([19, Th.4.1]). Actually, it was shown both in [19, 10] that every continuous map $G_2^\# \to G_3^\#$ is constant on an *infinite subset* of G_2 hence cannot be a homeomorphism. The proof in [10] based on a combinatorial lemma that allows for an easy application of the elementary convergence properties of the groups $G_m^\#$. We show here that an appropriate modification of the argument from [10] works with *arbitrary* target group H instead of $G_3^\#$ (cf. 4.9 and 5.1 for the counterpart of the combinatorial lemma, and Lemma 2.1-2.7 for the convergence properties of the groups $H^\#$). Namely, here is the precise counterpart of 1.2:

Theorem 1.3. *If $\kappa > \beth_3$ and there exists a continuous finitely many-to-one map $G_2^\# \to H^\#$, then H contains an infinite Boolean subgroup.*

Roughly speaking, the Bohr-continuous 1-1 maps can "measure" the Boolean subgroups. The basic idea of the proof of this theorem is to follow the behaviour of a continuous map $\pi : G_2^\# \to H^\#$ when restricted to the subspace $\mathcal{D}_\kappa$ of $G_2^\#$ consisting of 0 and all doubletons (when G_2 is identified with $[\kappa]^{<\omega}$). Making substantial use of the values of π on the *four-elements sets* of κ one proves that for some infinite $Z \subseteq \kappa$ either π vanishes on $\mathcal{D}_\kappa$, or π sends $\mathcal{D}_\kappa$ injectively into some Boolean subgroup of H (Theorem 5.3, with proof given in §5.4). The important point is that *the restriction $\pi|_{\mathcal{D}_\kappa}$ alone cannot help in detecting non-homeomorphisms.* Indeed, *every infinite abelian group G admits a continuous 1-1 map $\mathcal{D}_{|G|} \to G^\#$* (actually *an embedding, if G is uncountable*, cf. Theorem 3.3).

Of course, Theorem 1.3, as well as Kunen's paper [19] leave many open questions. For example: is Theorem 1.3 true for $\kappa = \omega$ (i.e., must an abelian group H necessarily have infinite Boolean subgroups whenever there exists a continuous 1-1 map $(\bigoplus_\omega \mathbb{Z}_2)^\# \hookrightarrow H^\#$)? In particular, is $G_2^\#$ homeomorphic to $\mathbb{Z}^\#$ for $\kappa = \omega$? Actually, it is not known whether $G_2^\#$ admits a continuous 1-1 map into $\mathbb{Q}^\#$ for $\kappa = \omega$.

It will be nice to classify, up to homeomorphism, all spaces $G^\#$ with G discrete abelian group of a given cardinality (e.g., $G = \mathbb{Z}$ or $G = G_2$). A possibility for classification was conjectured in [19] ($G^\#$ and $H^\#$ are Bohr homeomorphic iff there exist finite index subgroups $G' \le G$ and $H' \le H$ with $G' \cong H'$). Comfort, Hernández and Trigos-Arrieta [4] showed recently that this classification program fails by proving that $\mathbb{Q}^\#$ and $\mathbb{Z}^\# \times (\mathbb{Q}/\mathbb{Z})^\# = (\mathbb{Z} \times (\mathbb{Q}/\mathbb{Z}))^\#$ are homeomorphic. Note that by Theorem 1.3 the powers $\mathbb{Q}^\kappa$ and

$(\mathbb{Z} \times (\mathbb{Q}/\mathbb{Z}))^\kappa$ are not homeomorphic when equipped with their Bohr topologies and $\kappa > \beth_3$ (cf. Corollary 5.8).

The homeomorphism result of Comfort, Hernández and Trigos-Arrieta [4] was obtained in the framework of another question of van Douwen. If H has finite index in G, then $H^\#$ is clopen in $G^\#$, hence is a retract of $G^\#$. In the groups G_p every subgroup splits off algebraically, hence it is a topological direct summand in $G_p^\#$. This phenomenon led van Douwen [25] to pose also the following natural question

Proposition 1.4. [25, Question 4.12] *Is every (countable) subgroup H of a group $G^\#$ a retract of $G^\#$?*

and its natural generalization

Proposition 1.5. [25, Question 4.13] *Is every countable closed* subset *of $G^\#$ a retract of $G^\#$?*

A negative answer to Question 1.5 was given by Gladdines [13].

In §5.3 we comment without proofs some further contributions towards the non-homeomorphism problem obtained in [7]. It turns out that Theorem 1.3 remains true for $G_p^\#$, where p is an arbitrary prime number and $\kappa > \beth_{2p-1}$ (cf. Theorem 5.10). In particular, if $G^\#$ and $H^\#$ are homeomorphic as topological spaces and are sufficiently large, then they have similar properties related to torsion (see Theorem 1.6). Contrary to Kunen's result in the case of bounded torsion abelian groups, reasonable non-homeomorphism theorems seem hard to be realized with *countable* groups in the case of non-torsion groups. For example, if $G^\#$ and $H^\#$ are Bohr homeomorphic, then H torsion-free does not yield G has at most finitely many torsion elements (as $\mathbb{Q}^\#$ and $(\mathbb{Z} \times (\mathbb{Q}/\mathbb{Z}))^\#$ are homeomorphic and the torsion subgroup of the latter group is countably infinite). The following notion is especially meaningful for uncountable groups. An abelian group H is said to be *almost torsion-free* if H has finite p-rank for every prime p (clearly, the torsion part of an almost torsion-free group is countable, but there are countable groups that are not almost torsion-free group). Roughly speaking, the torsion part of a group that is Bohr homeomorphic to an almost torsion-free group cannot be large:

Theorem 1.6. *If G is an abelian group with $|t(G)| > \beth_\omega$, then there exist no continuous finitely many-to-one map $\pi : G^\# \to H^\#$, with H almost torsion-free.*

Consequently, if $G^\#$ and $H^\#$ are homeomorphic and H is almost torsion-free, then $|t(G)| \leq \beth_\omega$ (cf. Corollary 5.11). Theorem 1.6 immediately follows from Theorem 5.10.

More applications of our results (including a new proof of Gladdines' theorem), as well as complete proofs of those announced in §5.3 will be given in [7, 8].

The paper is organized as follows. A detailed description of the topology of the groups $G_K^\#$ in terms of convergent nets is given in §2. The key notion of splitting net is given in §2.2. In §3 many examples of continuous and discontinuous maps in the Bohr topology are given. Then we show in §§4, 5 that every continuous map $\pi : G_2^\# \to H^\#$ can be "straightened", i.e., appropriate restrictions of π are among the "typical" sample maps considered in §3.

Notation and terminology. The symbols $\mathbb{N}$ and $\mathbb{Z}$ are used for the set of positive integers and the group of integers, respectively. The symbol $\mathfrak{c}$ stands for the cardinality of the continuum, so $\mathfrak{c} = 2^{\aleph_0}$. The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals $\mathbb{R}$ and carries its usual compact topology. For $d, n \in \mathbb{N}$, the fact that d divides n abbreviates to $d|n$.

We consider here only Abelian groups, so only additive notation is used. The symbol 0 stands for the neutral element of an Abelian group G. We write $H \leq G$ if H is a subgroup of G. If α is an ordinal, we use G^α and $\bigoplus_\alpha G$ to denote the direct product and direct sum of α copies of the group G, respectively.

Let G be an abelian group. The cyclic subgroup of G generated by $b \in G$ is denoted by $\langle b \rangle$. For every $n \in \mathbb{N}$, we put $G[n] = \{g \in G : ng = 0\}$.We denote by $t(G)$ the torsion subgroup of G, by $r(G)$ the free rank of G and by $r_p(G)$ (for a prime p) the p-rank of G (this is the dimension of $G[p]$ over the field $\mathbb{Z}/p\mathbb{Z}$).

Let K be a countable abelian group. We use the convention that a finite function σ from some finite set F of ordinals in κ into $K \setminus \{0\}$ is to be identified with the function f from κ into K defined by $f(\alpha) = 0$ when α is not in F and by $f(\alpha) = \sigma(\alpha)$ otherwise. We set $\operatorname{supp} f = F$. In particular, we keep this convention and notation for the group $G_m = \oplus_\kappa \mathbb{Z}_m$ (in this case one has finite functions σ from some finite set $F \subseteq \kappa$ into $\{1, 2, \ldots, m-1\}$).

2. Properties of the Bohr convergence in $G_K^\#$

It follows directly from the definition of the Bohr topology, that a net $x_d \to 0$ in $G^\#$ iff the net $\chi(x_d) \to 0$ in $\mathbb{T}$ for every character $\chi : G \to \mathbb{T}$. Moreover, a map $f : G^\# \to H^\#$ is continuous iff the composition $\chi \circ f : G^\# \to \mathbb{T}$ is continuous for every character $\chi : H \to \mathbb{T}$.

Since the image of every homomorphism $G_m \to \mathbb{T}$ is contained in $\mathbb{Z}_m$, a typical subbasic open set U_ζ around 0 in $G_m^\#$ is given by a function $\zeta : \kappa \to m$ and is defined by $U_\zeta = \{f \in G_m : f\zeta = 0\}$ where the multiplication is the inner product as vectors. The characteristic function $\kappa \to m$ of a set $A \subseteq \kappa$, will be denoted by ζ_A. It has constant value 1 on the *support* A of ζ_A. Since every homomorphism $G_m \to \mathbb{Z}_m$ is a finite linear combination of characteristic functions, to check that a net n_d converges to 0 in $G_m^\#$ it suffices to check that $\zeta_A(n_d) \to 0$ for every $A \subseteq \kappa$.

2.1. Convergence in $G_2^\#$.

The group G_2 is actually $[\kappa]^{<\omega}$ equipped with the operation symmetric difference. This makes the topology of $G_2^\#$ extremely transparent. Here is the description of the nets converging to 0 in $G_2^\#$ following directly from the definition of Bohr topology:

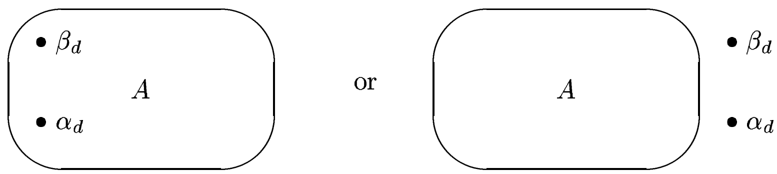
Lemma 2.1. *An arbitrary net $\{n_d : d \in D\}$ converges to 0 in $G_2^\#$ if and only if for any $A \subset \kappa$, there is $d' \in D$ such that $|\operatorname{supp} n_d \cap A|$ is even for all $d > d'$.*

In particular, if the net $\{n_d : d \in D\}$ converges to 0 in $G_2^\#$, then $|\operatorname{supp} n_d|$ is even for all $d > d'$.

This gives the following immediate corollary for nets of doubletons converging to 0 from [10]:

Corollary 2.2. *For a net $S = \{(\alpha_d, \beta_d) : d \in D\}$ of doubletons of κ the following are equivalent:*

- *S converges to 0 in the Bohr topology of G_2;*
- *for any $A \subset \kappa$, there is $d' \in D$ such that for all $d > d'$, either $(\alpha_d, \beta_d) \subset A$ or $(\alpha_d, \beta_d) \cap A = \varnothing$.*



Note 1. Clearly, a net $\{n_d : d \in D\}$ of κ that converges to 0 in the Bohr topology of G_2 is *free*, i.e., for every $\alpha \in \kappa$ there exists $d_0 \in D$ such that $\alpha \notin \operatorname{supp} n_d$ for $d > d_0$ (just take A to be the complement to the singleton $\{\alpha\}$). Consequently, for every finite set $F \subseteq \kappa$ there exists $d_0 \in D$ such that $F \cap \operatorname{supp} n_d = \varnothing$ for $d > d_0$.

In the sequel we consider doubletons (α, β) and four-element subsets (quadruples) $(\alpha, \beta, \gamma, \delta)$ of κ which will be identified as above with elements of G_2. In such a case we always assume that $\alpha < \beta < \gamma < \delta$. Taking into account that the Bohr topology is a group topology and the equalities

$$(\alpha, \beta, \gamma, \delta) = (\alpha, \beta) + (\gamma, \delta) = (\alpha, \gamma) + (\beta, \delta)$$

we get (applying Lemma 2.2):

Lemma 2.3. ([10]) *Let $(\alpha, \beta, \gamma, \delta)$ be a net of four-element sets of κ.*

(1) *If α and β are fixed and the corresponding net (γ, δ) converges in the Bohr topology to 0, then the net $(\alpha, \beta, \gamma, \delta)$ converges in the Bohr topology to (α, β).*
(2) *If the corresponding nets (α, γ) and (β, δ) converge in the Bohr topology to 0, then the net $(\alpha, \beta, \gamma, \delta)$ converges in the Bohr topology to* 0.

The following lemma is needed to generate converging quadruples in $G_2^\#$.

Lemma 2.4. *Let $Z \subseteq \kappa$ be an infinite set of ordinals.*

(a) *If $\gamma, \delta \in Z$ are fixed and there are infinitely many elements of Z less than γ, then there is a net of finite functions $(\alpha, \beta, \gamma, \delta)$ where all the ordinals come from Z and where the corresponding net (α, β) converges in the Bohr topology to 0.*

(b) *For every a partition $Z = Z' \cup Z''$ into infinite disjoint subsets Z', Z'' there is a net $(\alpha, \beta, \gamma, \delta)$ such that $\alpha, \gamma \in Z'$, $\beta, \delta \in Z''$ and the corresponding nets (α, γ) and (β, δ) converge to 0 in $G_2^\#$ (and consequently, $(\alpha, \beta, \gamma, \delta) \to 0$). Conversely, if a net $(\alpha, \beta, \gamma, \delta) \to 0$ with $\alpha, \gamma \in Z'$ and $\beta, \delta \in Z''$, then (α, γ) and (β, δ) converge to 0.*

Proof. (a) Take as index set I the set of all finite families $F = \{A_1, \ldots, A_n\}$ of subsets of Z ordered by inclusion. Then choose the α, β for a particular index F in such a way that $\alpha < \beta < \gamma$ and (α, β) lies entirely inside or outside each A_i, $i = 1, \ldots, n$.

(b) This can be arranged as in the proof of (a). Use finite families F of sets $A \subset Z_0$ ordered by inclusion as the index set I. Then, for any particular F, choose infinite $W' \subset Z'$ and $W'' \subset Z''$ so that each of W' and W'' lie entirely inside or outside each $A \in F$. Then choose $\alpha < \beta < \gamma < \delta$ so that $\alpha, \gamma \in W'$ and $\beta, \delta \in W''$. ☐

Although convergent nets we consider here have most often limit 0, we give for completeness the following general property that permits to isolate the limit function as a "part of the net" (cf. item (b) below):

Lemma 2.5. *Let $\{n_d : d \in D\} \to \nu$ be a net in $G_K^\#$. Then:*

(a) *for some tail of the net* $\operatorname{supp} \nu \subseteq \operatorname{supp} n_d$*;*
(b) *for some tail of the net $\nu \subseteq n_d$ if $G_K = G_m$;*
(c) *if* $\operatorname{supp} \nu \neq \varnothing$*, then for some tail of the net* $\operatorname{supp} \nu$ *is an initial segment of* $\operatorname{supp} n_d$ *if $G_K = G_m$ and $\kappa = \omega$.*

Proof. (a) Pick $\lambda \in A = \operatorname{supp} \nu$ and note that $n_d(\lambda)$ converges to $\nu(\lambda)$. This means that $n_d(\lambda) \neq 0$ for every large $d \in D$. So $\lambda \in \operatorname{supp} n_d$ for those d. Since A is finite, we can find a tail that works for all $\lambda \in A$. To prove (b) note that if $G_K = G_m$ we have $n_d(\lambda) = \nu(\lambda)$ for every $\lambda \in A$ and every sufficiently large $d \in D$.

(c) Now let $\kappa = \omega$ and assume that $\nu \neq 0$ and let α_0 be the minimal element of $\operatorname{supp} \nu$. Let B be the (finite) set of all $\alpha < \alpha_0$. By Observation 1 $\operatorname{supp} n_d \cap B = \varnothing$ for some tail of the net. Thus $\operatorname{supp} \nu$ is an initial segment of $\operatorname{supp} n_d$ for some tail of the net. ☐

An important property of $G_2^\#$ is that for every infinite $Z \subseteq \kappa$ the closure $\overline{[Z]^4}$ contains $[Z]^2$. Indeed, take any pair $\alpha < \beta$ in Z. Now choose a net $\gamma < \delta$ in Z with $\beta < \gamma$ such that (γ, δ) converges to 0 in $G_2^\#$. Then the net $(\alpha, \beta, \gamma, \delta) \to (\alpha, \beta)$, and $(\alpha, \beta, \gamma, \delta) \in [Z]^4$, thus $(\alpha, \beta) \in \overline{[Z]^4}$. Now we prove it in general.

Lemma 2.6. *The following holds in $G_2^\#$.*

(a) *$[\kappa]^k \subseteq \overline{[\kappa]^{k+2}}$ for every k.*
(b) *$\nu \in \overline{[\kappa]^k}$ iff $|\operatorname{supp} \nu| \leq k$ and has the same parity with k.*

In particular, $0 \in \overline{[\kappa]^k}$ iff k is even.

Proof. (a) It suffices to note that for every fixed $x = (a_1, \dots, a_k) \in [\kappa]^k$ and every convergent net of doubletons $(u, v) \to 0$ with $a_k < u < v$ one has $x = \lim(a_1, \dots, a_k, u, v)$.

(b) Let us prove first that $0 \in \overline{[\kappa]^k}$ iff k is even. Assume first that $0 \in \overline{[\kappa]^k}$ and fix a net in $[\kappa]^k$ with $n_d \to 0$. Then k must be even, as noted immediately after Lemma 2.1. The inverse implication follows from (a).

To prove now the first assertion in (b) note that if $n_d \to \nu$ with $n_d \in [\kappa]^k$, then $\operatorname{supp} \nu \subseteq \operatorname{supp} n_d$ by Lemma 2.5 and $m_d = n_d - \nu \to 0$. Let $s = |\operatorname{supp} \nu|$, so that $m_d \in [\kappa]^{k-s}$, hence the first part of the proof implies that $k - s$ is even. □

Let us resume our observations in the following:

Lemma 2.7. *Set* $[G_2]^{odd} = \bigcup_{n=0}^{\infty} [\kappa]^{2n+1}$ *and* $[G_2]^{ev} = \bigcup_{n=0}^{\infty} [\kappa]^{2n}$. *Then:*

(a) $[G_2]^{odd}$ *and* $[G_2]^{ev}$ *are clopen subsets of* $G_2^{\#}$*;*

(b) $[\kappa]^{\leq n}$ *is a closed subset of* $G_2^{\#}$ *and for every* n *the closure of* $[\kappa]^n$ *is* $[\kappa]^{\leq n} \cap [G_2]^{odd}$*, if* n *is odd, and* $[\kappa]^{\leq n} \cap [G_2]^{ev}$*, if* n *is even.*

(c) *if* $\pi : G_2^{\#} \to H^{\#}$ *is a continuous map that vanishes on* $[Z]^4$ *for some infinite subset* Z *of* κ*, then* π *vanishes on* $[Z]^2$ *too.*

Proof. (a) and (b) follow from Lemma 2.6.

(c) To see that π vanishes also on $[Z]^2$ note that $[Z]^2$ is contained in the closure of $[Z]^4$. Now, by the continuity of π we can conclude that $\pi(\alpha, \beta) = 0$ for every $\alpha < \beta$ in Z. □

2.2. **Splitting of convergent nets.** Now we describe the converging nets in the Bohr topology for arbitrary groups. This was done in [10] for $G_m^{\#}$ using splitting of nets. Now we extend this notion to nets in arbitrary abelian groups in the following way.

Definition 2.8. *Let* $\{n_d : d \in D\}$ *be a net in an abelian group* G*. We say that* n_d *splits into a sum of nets* $\{m_d^{(i)} : d \in D\}$, $i = 1, 2, \dots, n$*, if:*

1) $n_d = \sum_{i=1}^{n} m_d^{(i)}$ *for every* d*;*
2) *the family of subgroups* $\langle m_d^{(i)} : d \in D \rangle$, $i = 1, 2, \dots, n$*, is independent (i.e., their sum is direct).*

A relevant case is $G = G_m$, or just any group $G_K = \bigoplus_\kappa K$, where K is a (countable) abelian group. Now the support $\operatorname{supp}(x)$ is defined for every element x of such a group, so that we usually ensure (and ask) the stronger condition $\operatorname{supp} m_d^{(i)} \cap \operatorname{supp} m_e^{(j)} = \varnothing$ for $i \neq j$ and every $d, e \in D$ instead of 2). Note that this property (all "cross-intersections" are empty) implies the weaker property "all supports $\operatorname{supp} m_d^{(i)}$ in n_d are pairwise disjoint for every fixed $d \in D$". In the sequel, with only exception of 2.9 and 2.10, we work in groups of the form G_K and we intend splitting in this *stronger* sense with supports.

The next theorem gives the most important property of splitting nets in the general context.

Theorem 2.9. *Assume $n_d \to 0$ and there is a splitting $n_d = \sum_{i=1}^{n} m_d^{(i)}$. Then $m_d^{(i)} \to 0$ for every i.*

The proof of the theorem follows immediately from the next proposition proved in [7]:

Proposition 2.10. [7] *Let G be an abelian group and let H_0, H_1 be subgroups of G with $H_0 \cap H_1 = 0$. If for $\nu = 0,1$ $\{h_d^\nu : d \in I\}$ is a net in H_ν such that the net $\{h_d^0 + h_d^1 : d \in I\}$ converges to 0 in $G^\#$, then also h_d^ν converges to 0 in $G^\#$ for $\nu = 0,1$.*

Let us consider now some examples. By Lemma 2.9 a non-trivial convergent net $n_d \to 0$ of doubletons in $G_m^\#$ cannot split. For nets of quadruples in $G_2^\#$ we have:

Corollary 2.11. *Let $Z \subseteq \kappa$ and let $Z = Z' \cup Z''$ be a partition of Z. Then for a net $(\alpha,\beta,\gamma,\delta)$ such that $\alpha,\gamma \in Z'$ and $\delta,\beta \in Z''$ the following are equivalent:*

(a) (α,γ) and (β,δ) converge to 0 in $G_2^\#$,

(b) $(\alpha,\beta,\gamma,\delta)$ converges to 0 in $G_2^\#$.

Conversely, if $n_d = (\alpha,\beta,\gamma,\delta) \to 0$ is a non-trivial convergent net $G_2^\#$ admitting a splitting $n_d = m_d + r_d$, then there exists $Z \subseteq \kappa$ and a partition $Z = Z' \cup Z''$ of Z, such that one of the three possibilities holds on a cofinal part of the net:

- (a) *("overlapping" supports)*
  - (a$_1$) $m_d = (\alpha,\gamma) \subseteq Z'$ *and* $r_d = (\beta,\delta) \subseteq Z''$;
  - (a$_2$) $r_d = (\alpha,\gamma) \subseteq Z'$ *and* $m_d = (\beta,\delta) \subseteq Z''$;
- (b) *("disjoint" supports)*
  - (b$_1$) $m_d = (\alpha,\beta) \subseteq Z'$ *and* $r_d = (\gamma,\delta) \subseteq Z''$;
  - (b$_2$) $r_d = (\alpha,\beta) \subseteq Z'$ *and* $m_d = (\gamma,\delta) \subseteq Z''$;
- (c) *("nested" supports)*
  - (c$_1$) $m_d = (\alpha,\delta) \subseteq Z'$ *and* $r_d = (\beta,\gamma) \subseteq Z''$;
  - (c$_2$) $r_d = (\alpha,\delta) \subseteq Z'$ *and* $m_d = (\beta,\gamma) \subseteq Z''$.

Proof. The equivalence of (a) and (b) follows from Lemma 2.3 and Theorem 2.9 since $(\alpha,\beta,\gamma,\delta) = (\alpha,\gamma) + (\beta,\delta)$ is a splitting.

To prove the second part let $Z' = \bigcup_{d\in D} \operatorname{supp} m_d$, $Z'' = \bigcup_{d\in D} \operatorname{supp} r_d$ and $Z = Z' \cup Z''$. Then $Z' \cap Z'' = \varnothing$. As $m_d \to 0$ and $r_d \to 0$ by Theorem 2.9, we conclude that both m_d and r_d are doubletons on a tail of the net. Clearly, for every *fixed* quadruple $(\alpha,\beta,\gamma,\delta)$ one of the following six cases must hold,

- (a$_1$) $\min\operatorname{supp} m_d < \min\operatorname{supp} r_d < \max\operatorname{supp} m_d < \max\operatorname{supp} r_d$;
- (a$_2$) $\min\operatorname{supp} r_d < \min\operatorname{supp} m_d < \max\operatorname{supp} r_d < \max\operatorname{supp} m_d$;
- (b$_1$) $\max\operatorname{supp} m_d < \min\operatorname{supp} r_d$;
- (b$_2$) $\max\operatorname{supp} r_d < \min\operatorname{supp} m_d$;
- (c$_1$) $\min\operatorname{supp} m_d < \min\operatorname{supp} r_d < \max\operatorname{supp} r_d < \max\operatorname{supp} m_d$;
- (c$_2$) $\min\operatorname{supp} r_d < \min\operatorname{supp} m_d < \max\operatorname{supp} m_d < \max\operatorname{supp} r_d$.

It is clear that at least one of this six cases holds on a cofinal part of the net. This ends the proof. □

According to Lemma 2.5, $\nu \subseteq n_d$ holds for some tail of a converging net $\{n_d : d \in D\} \to \nu$ in $G_m^\#$. This property permits to split the limit function ν as a part of the net. Indeed, $m_d = n_d - \nu \to 0$ and $\operatorname{supp} m_d \cap \operatorname{supp} \nu = \varnothing$ on a tail of the net. So that n_d splits into a sum $n_d = m_d + r_d$, where $r_d = \nu$ is constant. This means that every converging net splits into a constant *root* r_d (that coincides with the limit of the net n_d on a tail of the net), and a *moving part* m_d, that converges to 0. This splitting is unique (up to taking a tail) in the following stronger sense: whenever one has a splitting $n_d = m_d + r_d \to \nu$ where r_d has a constant support, then necessarily $r_d = \nu$ and $m_d \to 0$ is the moving part of n_d.

Example 2.12.

1) Let $\kappa = \omega$. According to the above lemma,
   a) if $(\alpha, \beta, \gamma) \to \nu$ is a non-trivial convergent net in $G_2^\#$, then $\nu = \alpha$ on a tail of the net and $(\beta, \gamma) \to 0$.
   b) if $(\alpha, \beta, \gamma, \delta) \to \nu$ is a non-trivial convergent net in $G_2^\#$, then either $\nu = 0$, or $\nu = (\alpha, \beta)$ and $(\gamma, \delta) \to 0$ on a tail of the net.
2) If $\kappa > \omega$, then 1) need not be true. Indeed, it is easy to arrange for a net $(\alpha, \beta, \gamma) \to \gamma$, where $\gamma \geq \omega$ is fixed and $(\alpha, \beta) \to 0$.

Definition 2.13. *A family $\{A_i\}_{i \in I}$ of subsets of some set X is called* weakly disjoint*, if for every $i, j \in I$ the sets A_i and A_j are either disjoint or coincide.*

Clearly, every family of singletons is weakly disjoint. The proof of the following lemma can be found in [10].

Lemma 2.14. *Let $m_d \to 0$ in $G_m^\#$. If $\{\operatorname{supp} m_d : d \in D\}$ is a weakly disjoint family, then $m_d = 0$ for some tail of the net.*

Proof. Assume that $\operatorname{supp} m_d \neq \varnothing$ for cofinally many $d \in D' \subseteq D$ and choose for those d the least point $p_d \in \operatorname{supp} m_d$. Since $\{\operatorname{supp} m_d : d \in D'\}$ form a weakly disjoint family, for $d \neq d'$ either m_d and $m_{d'}$ have disjoint supports, or $p_d = p_{d'}$. Now for $P = \{p_d : d \in D'\}$ and for every $d \in D'$ $\zeta_P(n_d) = n_d(p_d)$ is always $\neq 0$, so that cannot converge to 0 in $\mathbb{Z}_m$. Hence the subnet $\{n_d : d \in D'\}$ cannot converge to 0, a contradiction. □

Let $n_d = \sum_{i=1}^n m_d^{(i)}$ be a splitting net in $G_m^\#$. A component $m_d^{(i)}$ of n_d is called a *strongly moving component* of the net, if $\{\operatorname{supp} m_d^{(i)} : d \in D\}$ is a weakly disjoint family. It follows immediately from this lemma and Lemma 2.9 that if $n_d \to 0$, then all strongly moving components of the net vanish on some tail of the net. This fact will be used very often in the sequel. Obviously, it holds also for any group G of finite exponent m instead of just G_m.

Note 2. Following the proof the above lemma one can prove that it remains true also when G_m is replaced by a direct sum G_K of copies of an arbitrary abelian group K, but the hypothesis "weakly disjoint" should be streightened with the following *additional* request: when the supports of m_d and $m_{d'}$ are not disjoint, then the values of m_d and $m_{d'}$ at the minimum element of the common support must coincide (accordingly, one must strengthen the definition of strongly moving component). Indeed, it suffices to note that now the non-zero value $\zeta_P(n_d) = n_d(p_d)$ is *constant.* In the main applications of the lemma this stronger restraint is fulfilled.

2.3. **Convergence in $G_m^\#$.** Since $G_m^\#$ is a topological group, the convergence in $G_m^\#$ is described by the convergent nets $n_d \to 0$. Clearly, every convergent net $n_d \to 0$ in $G_m^\#$ satisfies the obvious necessary condition

(S) $\zeta_\kappa(n_d) = 0$ on a tail of the net.

Apart from this obvious necessary condition, for $m > 2$ the convergence in $G_m^\#$ cannot be so easily reduced, except some special cases described below, to simple computations on the supports as in $G_2^\#$. For a further simplification, assume that $|\operatorname{supp} n_d| = l$ is constant for some tail of the net.

It is natural to start with nets taken in the subset $[\kappa]^l$ of $G_m^\#$ equipped with the induced topology, here we write $x \in [\kappa]^l$ iff x is the characteristic function of $\operatorname{supp} x$ and $|\operatorname{supp} x| = l$. For nets $n_d \in [\kappa]^l$ condition (S) is equivalent to $m|l$. Consequently, a simple modification of the proof of Lemma 2.6 shows that $\nu \in \overline{[\kappa]^l}$ iff $|\operatorname{supp} \nu| \leq l$ and has the same rest as l modulo m, whenever $\nu \in [\kappa]^s$ for some s. This is why, from now on we shall consider $m|l$ as a blanket condition.

Assume $n_d \in [\kappa]^m$. In such a case $n_d \to 0$ in $G_m^\#$ iff the following condition holds:

(C) for every subset A of κ there exists an index d_0 such that either $\operatorname{supp} n_d \subseteq A$ or $\operatorname{supp} n_d \cap A = \varnothing$ for all n_d with $d \geq d_0$.

This describes completely the topology of $\{0\} \cup [\kappa]^m$, since there are no other non-trivial convergent to 0 nets there. For $r = 0, 1, \ldots m-1$ set $B_r = \bigcup_{n=0}^\infty [\kappa]^{mn+r}$. Then B_r are clopen subsets of $G_m^\#$ and give a partition of the subspace $[\kappa]^{<\omega}$ of $G_m^\#$ consisting of all finite characteristic functions (compare with Lemma 2.7).

For the future use in this paper, the spaces $\{0\} \cup [\kappa]^{km}$ are sufficient. In fact, only the cases $k = 1, 2$ will be of primary interest to us. In the sequel we shall briefly comment another natural case.

The image of a net n_d in $[\kappa]^l$ under the multiplication by a constant k has the obvious property: every n_d takes a constant value k on $\operatorname{supp} n_d$ that does not depend on d. In case $k|m$, such nets are obtained also from nets n_d in $[\kappa]^l \subseteq G_{m/k}^\#$ via the natural embeddings $i : G_{m/k}^\# \hookrightarrow G_m^\#$ (note that if $k < m$ then $i(n_d)$ is not a characteristic function in G_m). This naturally introduces the following condition that strengthens (S) under the assumption of $m|l$:

(A) for some tail of the net n_d is constant (say, $k \in \mathbb{Z}_m$) on $\operatorname{supp} n_d$ that does not depend on d.

Let us note that for a net n_d in $[\kappa]^l$ (without the assumption $m|l$) (A) along with (C) implies $n_d \to 0$. On the other hand, the conjunction of (S) and (C) yields $n_d \to 0$ for an arbitrary net n_d in $G_m^\#$.

Note 3. Now we discuss the relations between (A) and (C) for a net $n_d \to 0$ not necessarily in $[\kappa]^l$.

(1). For a converging net $n_d \to 0$ (A) does not imply (C) (unlike the case when n_d are characteristic functions). Indeed, take a splitting net $(\alpha, \beta, \gamma, \delta) \to 0$ in $G_2^{\#}$ as in a) of Corollary 2.11 (see also Lemma 2.4). Define $n_d \in G_4$ as the finite function with support $(\alpha, \beta, \gamma, \delta)$ and constant value 2. Now for every $A \subseteq \kappa$ the number $l = |\operatorname{supp} n_d \cap A|$ is even, so that $\zeta_A(n_d) = 2l = 0$ in $\mathbb{Z}_4$. Hence (S) holds and $n_d \to 0$. Let us note now that (C) fails.

(2). Now take a net $(\alpha, \beta, \gamma, \delta) \to 0$ in $G_2^{\#}$ so that (C) holds. Define n_d to be the finite function $\{(\alpha, 1), (\beta, 2), (\gamma, 3), (\delta, 2)\}$ in G_4. Then $n_d \to 0$ in $G_4^{\#}$ and (C) holds, but (A) fails.

(3). When $m = p$ is prime, then (A) implies (C) for nets $n_d \to 0$ in $G_m^{\#}$ with $|\operatorname{supp} n_d| = m$.

Now we give several examples of converging nets that do not satisfy (A).

Example 2.15. Here we consider small m.

(1). Let now $n_d \to 0$ be a non-trivial convergent a net in $G_3^{\#}$ with $|\operatorname{supp} n_d| \leq 3$. Then (C) holds and one of the following occurs:

a) $|\operatorname{supp} n_d| = 2$ for some tail of the net and n_d always takes two values on $\operatorname{supp} n_d$.

b) $|\operatorname{supp} n_d| = 3$ for some tail of the net and n_d is constant on $\operatorname{supp} n_d$.

Obviously, every net with these properties converges to 0 in $G_3^{\#}$. (For b) note that only constant characters count in the test of Bohr convergence.) Now assume that $n_d \to 0$ non-trivially. Then on some tail of the net $|\operatorname{supp} n_d|$ has a constant value $k \leq 3$. If $k = 2$, then taking again constant characters ζ_A for the test of Bohr convergence we conclude that n_d takes two values on $\operatorname{supp} n_d$. Assume now that $k = 3$. Taking the constant character ζ_κ for the test of Bohr convergence we conclude that n_d cannot take distinct values (as the sum of those of them computed at all $\lambda \in \operatorname{supp} n_d$ must be zero).

If $n_d \to 0$ in $G_3^{\#}$ without the property (A), then $k = \operatorname{supp} n_d$ need not be divisible by 3. Just take a splitting net $(\alpha, \beta, \gamma, \delta) \to 0$ in $G_2^{\#}$ as above and define $n_d = \{(\alpha, 1), (\beta, 1), (\gamma, 2), (\delta, 2)\}$ if the splitting is $(\alpha, \gamma) + (\beta, \delta)$.

(2). Let now $m = 4$. It is easy to check that a net n_d with $|\operatorname{supp} n_d| = 4$ converges to 0 if one of the following conditions hold:

i) (C) and one of the following:
- n_d has a constant value on $\operatorname{supp} n_d$;
- n_d has values 1,1,3,3;
- n_d has values 1,2,2,3;

ii) n_d has a constant value 2 on $\operatorname{supp} n_d$ and $\operatorname{supp} n_d \to 0$ in $G_2^{\#}$.

iii) $\operatorname{supp} n_d \to 0$ in $G_2^\#$ and $n_d = m_d + r_d$ splits (necessarily, with supports of size 2), and on each one of the supports the functions takes either values 1, 3, or values 2,2.

3. Examples of continuous and discontinuous maps

We pay special attention to the doubletons in $G_2^\#$, so for $Z \subseteq \kappa$ we denote by $\mathcal{D}_Z$ the subspace $\{0\} \cup [Z]^2$ of $G_2^\#$. It was proved in [15] that $[\kappa]^2$ is a discrete subset of $G_2^\#$. By Lemma 2.4 0 belongs to the closure of $[\kappa]^2$ in $G_2^\#$ (see also [15]) and $\mathcal{D}_Z$ is a closed set. Hence $\mathcal{D}_Z$ is not discrete whenever Z is an infinite subset of κ.

Example 3.1. Here we consider two examples of sample maps $\mathcal{D}_\kappa \to G_3^\#$ that will play an important role in the sequel. The second is continuous, while the first one is not.

(a). The map $\lambda : \mathcal{D}_\kappa \to G_3^\#$ defined by

$$\lambda(0) = 0 \text{ and } \lambda(\alpha, \beta) = \{(\alpha, 1), (\beta, 1)\} \text{ for every } \alpha < \beta \text{ in kappa}$$

is not continuous. Indeed, if $(\alpha_d, \beta_d) \to 0$ in $G_2^\#$, then $m_d = \lambda(\alpha_d, \beta_d)$ does not converge to 0 in $G_3^\#$. (Fix $\alpha \in \kappa$ and take $A = \kappa \setminus \{\alpha\}$ and $\chi = \zeta_Z$. Then the convergence to 0 in $G_2^\#$ yields (by Lemma 2.2) that $\alpha_d, \beta_d \in A$ for all sufficiently large $d \in D$. Then $\chi(m_d) = \chi(\alpha_d, 1) + \chi(\beta_d, 1) \neq 0$.) This argument shows that the restriction of λ to $\mathcal{D}_Z$ is discontinuous whenever Z is an infinite subset of κ. Obviously, 3 can be replaced by any natural number $m > 2$.

(b). On the other hand, the map $\mu : \mathcal{D}_\kappa \to G_3^\#$ defined by

$$\mu(0) = 0 \text{ and } \mu(\alpha, \beta) = \{(\alpha, 1), (\beta, -1)\} \text{ for every } \alpha < \beta \text{ in } \kappa$$

is continuous. In fact, μ is actually a homeomorphic embedding. Since $[\kappa]^2$ is discrete in both $G_2^\#$ and $G_3^\#$, it suffices to see that a net (α_d, β_d) converges to 0 in $G_2^\#$ iff the net $\mu(\alpha_d, \beta_d)$ converges to 0 in $G_3^\#$. Testing with homomorphisms $\chi : G_3 \to \mathbb{Z}_3$ that are constant on their support, one can easily see that the convergence of $\mu(\alpha_d, \beta_d)$ implies the convergence of (α_d, β_d) to 0. Assume $(\alpha_d, \beta_d) \to 0$ in $G_2^\#$. Let $\chi = \zeta_A : G_3 \to \mathbb{Z}_3$ be an arbitrary characteristic function. Then choose d_0 such that for all $d \geq d_0$ $(\alpha_d, \beta_d) \subseteq A$ or $(\alpha_d, \beta_d) \cap A = \varnothing$. This guarantees that $\chi(\mu(\alpha_d, \beta_d)) = 0$ for all $d \geq d_0$. Obviously, 3 can be replaced by any natural number $m > 1$ (see [19] for prime m). We shall keep the same notation $\mu : \mathcal{D}_\kappa \to G_m^\#$ for the general case.

(c). Now we fix a partition $\omega = Z_1 \cup Z_2$ (e.g., odd/even) and define a function $\widetilde{\lambda} : \mathcal{D}_\omega \to G_3^\#$ as follows: $\widetilde{\lambda}(0) = 0$ and

$$\widetilde{\lambda}(\alpha, \beta) = \begin{cases} 0 & \text{whenever } \alpha, \beta \text{ have the same parity} \\ \lambda(\alpha, \beta) & \text{whenever } \alpha, \beta \text{ have distinct parity.} \end{cases}$$

Since every convergent net $(\alpha, \beta) \to 0$ belongs definitely to $[Z_1]^2 \cup [Z_2]^2$, this function is continuous, even if it coincides with λ on a large part of $\mathcal{D}_\omega$.

Our aim is to show that the map μ is a "generic model" of a non-zero continuous map $\mathcal{D}_\kappa \to G_p^\#$ sending 0 to 0 (actually this holds for more general codomain, see Theorem 4.3 and Definition 4.2).

Example 3.2. Now we discuss continuity of maps defined on quadruples of ω.

(a). If $Z \subseteq \omega$ is infinite, then none of the following functions $\{0\} \cup [\kappa] \to G_3^\#$ sending 0 to 0 is continuous:

$$\begin{aligned} f_1(\alpha,\beta,\gamma,\delta) &= \lambda(\alpha,\gamma) \pm \lambda(\beta,\delta), & g_1(\alpha,\beta,\gamma,\delta) &= \mu(\alpha,\gamma) \pm \mu(\beta,\delta),\\ f_2(\alpha,\beta,\gamma,\delta) &= \lambda(\alpha,\beta) \pm \lambda(\gamma,\delta), & g_2(\alpha,\beta,\gamma,\delta) &= \mu(\alpha,\beta) \pm \mu(\gamma,\delta),\\ f_3(\alpha,\beta,\gamma,\delta) &= \lambda(\alpha,\delta) \pm \lambda(\beta,\gamma), & g_3(\alpha,\beta,\gamma,\delta) &= \mu(\alpha,\delta) \pm \mu(\beta,\gamma) \end{aligned}$$

for all $\alpha < \beta < \gamma < \delta$ in Z.

To prove that f_1 is not continuous, fix a partition $Z = Z' \cup Z''$ into infinite disjoint subsets Z', Z'' and find a net $(\alpha,\beta,\gamma,\delta)$ such that $\alpha, \gamma \in Z'$, $\beta, \delta \in Z''$ and the corresponding nets (α,γ) and (β,δ) converge to 0 in $G_2^\#$, so that by Lemma 2.3, the net $(\alpha,\beta,\gamma,\delta)$ Bohr-converges to 0 in $G_2^\#$. This can be arranged by (b) of Lemma 2.4. Assume f_1 is continuous at 0. Then $f_1(\alpha,\beta,\gamma,\delta)$ converges to 0 in $G_2^\#$. By the choice of the sets Z' and Z'' the supports of $\lambda(\alpha,\gamma)$ and $\lambda(\beta,\delta)$ are always disjoint. Hence we can apply Theorem 2.9 to claim that $\lambda(\beta,\delta)$ converges to 0 in $\mathcal{D}_{Z''}$. This contradicts (a) of Example 3.2. Analogous argument works for f_2 and f_3. To finish with, we note that the functions g_1, g_2 and g_3 are among the functions f_1, f_2 and f_3.

(b). In the notation of (c) of Example 3.1 we extend the function $\widetilde{\lambda}$ to $[\omega]^4$ in the following way: $\widetilde{\lambda}(\alpha,\beta,\gamma,\delta) =$

$$\begin{cases} 0 & \text{whenever even number of } \alpha,\ \beta,\ \gamma,\ \delta \text{ are even}\\ (\alpha,1)-(\beta,1)+(\gamma,1)+(\delta,1) & \text{whenever odd number of } \alpha,\ \beta,\ \gamma,\ \delta \text{ are even and } \alpha,\ \beta \text{ have the same parity,}\\ (\alpha,1)+(\beta,1)-(\gamma,1)+(\delta,1) & \text{whenever odd number of } \alpha,\ \beta,\ \gamma,\ \delta \text{ are even and } \gamma,\ \delta \text{ have the same parity.} \end{cases}$$

To see that $\widetilde{\lambda} : \overline{[\kappa]^4} \to G_3^\#$ is continuous consider a converging net $(\alpha,\beta,\gamma,\delta) \to \nu$. If $\nu = 0$ then the second and the third case cannot occur on any tail of the net, since in such a case, if say $\delta \in Z_2$ is the only even cardinal, then for the character $\chi = \zeta_{Z_2}$ the value $\chi(\alpha,\beta,\gamma,\delta) = 1$ cannot converge to 0. Therefore, the value of $\widetilde{\lambda}$ at any tail of the net is 0. Hence $\widetilde{\lambda}(\alpha,\beta,\gamma,\delta) \to 0$. Now assume that $\nu \neq 0$. This is possible only if $\nu = (\alpha,\beta)$ on a tail of the net and $(\gamma,\delta) \to 0$. Thus γ and δ have the same parity on a tail of the net. Therefore, we have two possibilities for the fixed α,β. If α,β have the same parity, then $\widetilde{\lambda}(\alpha,\beta,\gamma,\delta) = 0$ and we are done. If α,β have different parity, then $\widetilde{\lambda}(\alpha,\beta,\gamma,\delta)$

is defined to be $(\alpha, 1) + (\beta, 1) - (\gamma, 1) + (\delta, 1)$ according to the third case. Since $(\alpha, 1)+(\beta, 1) = \widetilde{\lambda}(\alpha, \beta)$ and since $(\delta, 1)-(\gamma, 1) = -\mu(\gamma, \delta) \to 0$ when $(\gamma, \delta) \to 0$, we conclude that $\widetilde{\lambda}(\alpha, \beta, \gamma, \delta) \to \widetilde{\lambda}(\alpha, \beta)$. This proves the continuity of $\widetilde{\lambda}$.

(c). A more careful analysis of the above argument shows that one could argue with a function f defined in an easier way, by noting that the set A of all quadruples satisfying the condition $\alpha \equiv \beta \equiv \gamma \not\equiv \delta$ or $\alpha \equiv \beta \equiv \delta \not\equiv \gamma$ is closed (clopen) and discrete in $\overline{[\kappa]^4}$ (where "$\equiv$" is the equivalence relation on Z corresponding to the partition $Z = Z' \cup Z''$). Hence one can define f in an arbitrary way on A, while can take f to vanish on the complement of A in $\overline{[\kappa]^4}$. This idea can be extended to the case of arbitrary even k as follows. Take a splitting of ω in $2k$ infinite sets Z_i and let A be the set of all $2k$-tuples in $[\omega]^{2k}$ that meet each Z_i in precisely one element. Then no net in A can converge, so that again A is closed and discrete and we can define f on $\overline{[\kappa]^{2k}}$ in a similar way.

Let us note that for all continuous functions $f : \overline{[\kappa]^{2k}} \to G_3^{\#}$ (with $k > 1$) we have considered so far there exists an infinite $Z \subseteq \kappa$ such that the restriction on $\overline{[Z]^{2k}}$ is constant.

3.1. **The spaces $\mathcal{D}_\kappa$ can be embedded in every group $G^{\#}$.** Our observations from the last examples can be extended to a more general fact:

Proposition 3.3. *Let G be an abelian group, let κ be an infinite cardinal and let $f : \kappa \to G$ be an arbitrary function. Then the map $\mu_f : \mathcal{D}_\kappa \to G^{\#}$ defined by $\mu_f(0) = 0$ and $\mu_f(\alpha, \beta) = f(\alpha) - f(\beta)$ is:*

(a) *continuous;*
(b) *an embedding, if the family of cyclic subgroups $\{\langle f(\alpha)\rangle : \alpha < \kappa\}$ is independent.*

Proof. (a) Let us see first that μ_f is continuous. Since the only non-trivial convergent nets in $\mathcal{D}_\kappa$ have the form $(\alpha, \beta) \to 0$, we have to prove that the net

$$x_{\alpha\beta} = \mu_f(\alpha, \beta) = f(\alpha) - f(\beta)$$

converges to 0 in $G^{\#}$ whenever the net $(\alpha, \beta) \to 0$ in $\mathcal{D}_\kappa$. Let $\zeta : G \to \mathbb{T}$ be a character and let $a_\alpha = \zeta(f(\alpha))$ for every $\alpha \in \kappa$. We have to prove that the net

$$\zeta(x_{\alpha\beta}) = a_\alpha - a_\beta \to 0$$

in $\mathbb{T}$ when the net $(\alpha, \beta) \to 0$ in $\mathcal{D}_\kappa$. Let $\varepsilon > 0$, $m > 1/\varepsilon$ and

$$A_n = \varphi(\{z \in \mathbb{R} : \frac{n-1}{m} \le z \le \frac{n}{m}\}),$$

where $n = 1, 2, \ldots, m$ and $\varphi : \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the canonical exponential map. Let

$$B_n = \{\alpha < \kappa : a_\alpha \in A_n\}.$$

Then there exists an index d_0 such that for all members of the net with index $\ge d_0$ either $(\alpha, \beta) \subseteq A_n$ or is disjoint with A_n, for $n = 1, 2, \ldots, m$. Since $\bigcup_{n=1}^m A_n$ is a cover of $\mathbb{T}$, there exists some $n = 1, 2, \ldots, m$ such that both

$a_\alpha, b_\alpha \in A_n$ for all members of the net with index $\geq d_0$. Then $a_\alpha - b_\alpha$ belongs to the ε-ball at 0 in $\mathbb{T}$ for all (α, β) on a tail of the net. Therefore μ_f is continuous.

(b) Assume that the family of cyclic subgroups $\{\langle f(\gamma)\rangle : \gamma < \kappa\}$ is independent so that the subgroup H generated by all $f(\gamma)$, $\gamma < \kappa$, is the direct sum $\bigoplus_{\gamma<\kappa}\langle f(\gamma)\rangle$. We have to prove that if

$$\mu_f(\alpha, \beta) \to \nu \text{ in } G^\# \text{ for some net } (\alpha, \beta) \text{ in } G_2^\#,$$

then

$$(\alpha, \beta) \text{ is convergent in } G_2^\# \text{ and } \mu_f(\lim(\alpha, \beta)) = \nu.$$

Clearly, $\mu_f(\alpha, \beta) \to \nu$ also in the topological subgroup $H^\#$ of $G^\#$. Moreover, we can replace H by the bigger group $G_K = \bigoplus_\kappa K$, where K is a countable abelian group containing an isomorphic copy of each cyclic group $\langle f(\alpha)\rangle$ $(\alpha < \kappa)$. With respect to this group G_K the support of $\mu_f(\alpha, \beta)$ coincides with (α, β). Therefore, by Lemma 2.5 and the hypothesis $\mu_f(\gamma, \beta) \to \nu$ we can conclude that $\operatorname{supp}\nu \subseteq (\alpha, \beta)$ for a tail of the net. This gives three possibilities depending on the size of $\operatorname{supp}\nu$.

Case 1. $|\operatorname{supp}\nu| = 2$, so that (α, β) coincides with the constant net $\operatorname{supp}\nu$ on a tail of the net. Then the net (α, β) trivially converges in $\mathcal{D}_\kappa$ and we are done.

Case 2. $|\operatorname{supp}\nu| = 0$, so $\nu = 0$. This case will be settled below.

Case 3. $|\operatorname{supp}\nu| = 1$, so $\operatorname{supp}\nu$ is a singleton. Assume without loss of generality that $\operatorname{supp}\nu = \{\alpha\}$. Then the only non-zero value a of ν taken at α may coincide with $f(\alpha)$ or not. If $a \neq f(\alpha)$, we replace the function f by the modified function f' that differs from f only at α, by setting

$$f'(\alpha) = f(\alpha) - a.$$

Then

$$\mu_{f'}(\alpha, \beta) = \mu_f(\alpha, \beta) - \nu \to 0$$

in $G^\#$ for our net (α, β), so Case 2 applies. We are left therefore with the case $a = f(\alpha)$. Now $\mu_f(\alpha, \beta) \to \nu$ gives $f(\beta) \to 0$ for the net (α, β) in $G_2^\#$. Since α is fixed now, we have always $\beta > \alpha$. This case will be settled below along with Case 2. To this end we need to define first appropriate characters of G_K.

Fix a symmetric neighbourhood U of 0 in $\mathbb{T}$ that contains no subgroups of $\mathbb{T}$ beyond $\{0\}$. Then for every non-zero element $x \in G_K$ choose a character $\chi_x : G_K \to \mathbb{T}$ such that $\chi_x(x) \neq 0$. Then $\langle \chi_x(x)\rangle \not\subseteq U$. Let this be witnessed by the non-zero element $n_x\chi_x(x)$, where $n_x \in \mathbb{Z}$. Let $\eta_x = n_x\chi_x$. Then for every $x \in \mathbb{T}, x \neq 0$ the character $\eta_x : G_K \to \mathbb{T}$ satisfies $\pm\eta_x(x) \notin U$. Fix now an arbitrary subset A of κ and define a character $\xi_A : H \to \mathbb{T}$ as follows:

$$\xi_A(f(\gamma)) = \eta_{f(\gamma)}(f(\gamma)) \text{ for } \gamma \in A$$

and

$$\xi_A(f(\gamma)) = 0 \text{ for } \gamma \in \kappa \setminus A$$

(this is possible since $\{\langle f(\gamma)\rangle : \gamma < \kappa\}$ is independent, in particular, $f(\gamma) \neq 0$ for every $\gamma < \kappa$). Now extend ξ_A to a character $\widetilde{\xi_A} : G_K \to \mathbb{T}$. Obviously,

$$(3.1) \qquad \widetilde{\xi_A}(f(\gamma)) \notin U \text{ for } \gamma \in A \text{ and } \widetilde{\xi_A}(f(\gamma)) = 0 \text{ for } \gamma \notin A.$$

For Case 2 take A arbitrary and note that when a doubleton (α, β) satisfies $|(\alpha, \beta) \cap A| = 1$, then $\widetilde{\xi_A}(f(\alpha) - f(\beta)) \notin U$ by (3.1). Therefore, $\mu_f(\alpha, \beta) \to 0$ yields $\widetilde{\xi_A}(\mu_f(\alpha, \beta)) \in U$, and consequently $|(\alpha, \beta) \cap A| \neq 1$, for a tail of the net. This proves that $(\alpha, \beta) \to 0$ in $\mathcal{D}_\kappa$.

In Case 3 we have $f(\beta) \to 0$. Take $A = \kappa \setminus \{\alpha\}$, so that always $\beta \in A$. Now (3.1) yields that this case cannot occur. □

In the sequel we denote by $\mathcal{D}^{(l)}_{Z,m}$ the subspace $\{0\} \cup [Z]^l$ of $G^\#_m$, where Z is a subset of κ. We abbreviate $\mathcal{D}^{(m)}_{Z,m}$ to $\mathcal{D}^{(m)}_Z$ and $\mathcal{D}^{(2)}_Z$ to $\mathcal{D}_Z$. If $n \leq m$ and $1 \leq i_1 < i_2 < \ldots < i_n \leq m$, then for the set $I = \{i_1, i_2, \ldots, i_n\}$ denote by p_I the projection $\mathcal{D}^{(m)}_\kappa \to \mathcal{D}^{(n)}_\kappa$ defined by $p_I(\alpha_1, \alpha_2, \ldots, \alpha_m) = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n})$ for every $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in [\kappa]^m$.

Proposition 3.4. *In the above notation:*

(a) $p_I : \mathcal{D}^{(m)}_\kappa \to \mathcal{D}^{(n)}_\kappa$ *is continuous for every set* $I = \{i_1, i_2, \ldots, i_n\}$.

(b) *For every integer* $m > 1$ *the space* $\mathcal{D}^{(m)}_\kappa$ *embeds into the power* $\mathcal{D}^{m-1}_\kappa$. *In particular,* $\mathcal{D}^{(m)}_\kappa$ *embeds into the group* $G^\#_2$.

Proof. (a) immediately follows from the properties of the Bohr topology discussed in §2.

(b) Now consider the diagonal map $j : \mathcal{D}^{(m)}_\kappa \to \mathcal{D}^{m-1}_\kappa$ of all projections p_{I_k}, where the doubletons $I_k = \{\alpha_k, \alpha_{k+1}\}$ are taken for $k = 1, \ldots, m-1$. Continuity of j follows from the continuity of all p_{I_k}. If n_d is a net in $\mathcal{D}^{(m)}_\kappa$ such that $p_{I_k}(n_d) \to 0$ in $\mathcal{D}_\kappa$ for all doubletons $I_k = \{\alpha_k, \alpha_{k+1}\}$, $k = 1, \ldots, m-1$, then $n_d \to 0$ in $\mathcal{D}^{(m)}_\kappa$. Indeed, take a subset A of κ. Then for some tail of the net each pair α_k, α_{k+1} is either contained in A or disjoint with A. Since the adjacent pairs I_k and I_{k+1} have a common element (namely, α_{k+1}), it follows that I_k and I_{k+1} are placed similarly w.r.t. A (i.e., either both contained in A, or both disjoint with A). It is clear now that this holds for *all* pairs I_k. This implies that for the same tail of the net $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ is either contained in A, or disjoint with A. This proves that $n_d \to 0$ in $\mathcal{D}^{(m)}_\kappa$. This proves that $\mathcal{D}^{(m)}_\kappa$ embeds into the power $\mathcal{D}^{m-1}_\kappa$. The conclusion follows from the obvious isomorphism $G^{m-1}_2 \cong G_2$. □

This proposition shows that $\mathcal{D}^{(3)}_\kappa$ embeds into $\mathcal{D}^2_\kappa$. It can be shown that $\mathcal{D}^{(3)}_\omega$ cannot be embedded into $\mathcal{D}_\omega$ ([21]).

Instead of the collection of doubletons $I_k = \{\alpha_k, \alpha_{k+1}\}$, $k = 1, \ldots, m-1$ one can build the embedding in the above proposition using the edges of any *connected* graph with m edges on the set of vertices $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ (e.g., the collection of doubletons $J_k = \{\alpha_k, \alpha_m\}$, $k = 1, \ldots, m-1$).

Example 3.5.

(a). Let G be an abelian group, let κ be an infinite cardinal and let $f:\kappa\to G$ be an arbitrary function. Let $1\leq i<j\leq m$ be fixed. Then by the above propositions the map $\mathcal{D}_\kappa^{(m)}\to G^\#$ sending 0 to 0 and $(\alpha_1,\dots,\alpha_m)\mapsto f(\alpha_i)-f(\alpha_j)$ is continuous as a composition of the continuous maps $p_{\{i,j\}}$ and μ_f.

(b). Let now $m>1$ and let G be an abelian group. In analogy with item (a) and Proposition 3.3 consider arbitrary functions $f_1,\dots,f_{m-1}:\kappa\to G$. Define the function $\mu_{f_1,\dots,f_{m-1}}:\mathcal{D}_\kappa^{(m)}\to G^\#$ by

$$\mu_{f_1,\dots,f_{m-1}}(0)=0$$

and

$$\mu_{f_1,\dots,f_{m-1}}(\alpha_1,\dots,\alpha_m)=\sum_{i=1}^{m-1}f_i(\alpha_i)-f_i(\alpha_{i+1}).$$

Then the net $\mu_{f_1,\dots,f_{m-1}}(\alpha_1,\dots,\alpha_m)$ converges to 0 in $G^\#$ whenever the net $(\alpha_1,\dots,\alpha_m)$ converges to 0 in $\mathcal{D}_\kappa^{(m)}$. This proves, as above, that the function $\mu_{f_1,\dots,f_{m-1}}$ is continuous (this follows also from the fact that $\mu_{f_1,\dots,f_{m-1}}$ is a sum of $m-1$ functions of the form considered in (a)). Moreover, if the family of subgroups $\{\langle f_i(\alpha)\rangle:\alpha<\kappa, i=1,\dots,m-1\}$ is independent, then $\mu_{f_1,\dots,f_{m-1}}$ is an embedding. For $m=2$ this is item (b) of Proposition 3.3, for $m=3$, $\kappa=\omega$ and $G=G_2$ this was mentioned in [19]). To see that $\mu_{f_1,\dots,f_{m-1}}$ is an embedding under this condition, set $H_i=\langle f_i(\alpha):\alpha<\kappa\rangle$ and note that by Proposition 3.3 $\mu_{f_i}:\mathcal{D}_\kappa\hookrightarrow H_i$ is an embedding for every $i=1,\dots,m-1$, while $H_1^\#\times\ldots\times H_{m-1}^\#$ embeds into $G^\#$. Now the composition of this embedding with the product of the embeddings μ_{f_i} defines an embedding $j:\mathcal{D}_\kappa^{m-1}\hookrightarrow G^\#$. To conclude we need only to mention that the composition of j with the embedding $\mathcal{D}_\kappa^{(m)}\hookrightarrow\mathcal{D}_\kappa^{m-1}$ from Proposition 3.3 gives precisely $\mu_{f_1,\dots,f_{m-1}}$.

(c). Generalizing the idea used in the previous item one can build more general functions $\mathcal{D}_\kappa^{(m)}\to G^\#$ as linear combinations of the "elementary" functions from item (a). More precisely, we define a function $\mu_{h,\mathcal{F}}:\mathcal{D}_\kappa^{(m)}\to G^\#$ by choosing first a family $\mathcal{F}$ of $k=\frac{m(m-1)}{2}$ functions $f_{ij}:\kappa\to G$ for every $1\leq i<j\leq m$ and an arbitrary homomorphism $h:G^k\to G$. Now let

$$\mu_{h,\mathcal{F}}(0)=0 \text{ and } \mu_{h,\mathcal{F}}(\alpha_1,\dots,\alpha_m)=h(\dots,f_{ij}(\alpha_i)-f_{ij}(\alpha_j),\dots).$$

By Proposition 3.3 each function $(\alpha_1,\dots,\alpha_m)\mapsto f_{ij}(\alpha_i)-f_{ij}(\alpha_j)$ is continuous as a map $\mathcal{D}_\kappa^{(m)}\to G^\#$. Consequently, their diagonal map $\mathcal{D}_\kappa^{(m)}\to(G^\#)^k$ is continuous. Now the composition with the continuous homomorphism $h:(G^\#)^k=(G^k)^\#\to G^\#$ gives the desired map $\mu_{h,\mathcal{F}}$.

As far as continuous functions $\mathcal{D}_{\kappa,2}^{(m)}\to G^\#$ with $m>2$ are concerned, the situation radically changes. Here some groups G are not eligible as a codomain.

Taking some function $f : \kappa \to G_2$ and choosing an arbitrary even natural m, one can prove that the function

$$\lambda_{f,m} : \mathcal{D}^{(m)}_{\kappa,2} \to G_2^{\#}$$

defined by $\lambda_{f,m}(0) = 0$ and $\lambda_{f,m}(\alpha_1, \ldots, \alpha_m) = \sum_{i=1}^{m} f(\alpha_i)$ is continuous (since $\lambda_{f,m}$ extends in an obvious way to a homomorphism $G_2 \to G_2$ and every homomorphism is continuous in the Bohr topology). We show later that for every continuous map

$$\pi : \mathcal{D}^{(4)}_{\kappa,2} \to H^{\#} \text{ with } \pi(0) = 0$$

and $\kappa > \beth_3$ there exists an infinite set $Z \subseteq \kappa$ and a function $f : Z \to H[2]$ such that π coincides with $\lambda_{f,m}$ when restricted to $\mathcal{D}^{(4)}_{Z,2}$ (Theorem 5.3).

Theorem 3.6. *Let G be an infinite abelian group of cardinality κ and $m > 1$ be an integer.*

(a) *If G contains an infinite direct sum of non-trivial cyclic subgroups (in particular, when $\kappa > \omega$), then $\mathcal{D}^{(m)}_{\kappa}$ embeds into $G^{\#}$.*

(b) *There always exists a continuous 1-1 map $\pi : \mathcal{D}^{(m)}_{\omega} \to G^{\#}$.*

Proof. (a) It is easy to see that in this case G contains a direct sum H of κ many cyclic subgroups. Hence we can find an embedding of $\mathcal{D}^{(m)}_{\kappa}$ into $H^{\#}$ applying item b) of the previous example. (Just take any function $f : \kappa \to H$ such that $f(\kappa)$ is a base of H.)

(b) The only case when G does not contain an infinite direct sum of non-trivial cyclic subgroups is when G has finite rank (i.e., $r(G) < \infty$ and $\sup_p r_p(G) < \infty$), so that κ is countable. Then G has either an infinite cyclic subgroup $C \cong \mathbb{Z}$, or an infinite co-cyclic subgroup $C = \mathbb{Z}(p^{\infty})$ for some prime p. In either case one can easily define a function $f : \omega \to C$ such that $\mu_f : \mathcal{D}_{\omega} \to C$ is injective. For $C \cong \mathbb{Z}$ set $f(n) = n!$, for $C = \mathbb{Z}(p^{\infty})$ set $f(n) = \varphi(p^{-n!}) \in \mathbb{Z}(p^{\infty})$. This settles the case $m = 2$ (continuity of μ_f follows from Proposition 3.3, injectivity can be proved in the line of the general argument given below for $m > 2$).

Let us see now that the above construction for $\mathcal{D}_{\omega}$ works also for $\mathcal{D}^{(m)}_{\omega}$ with $m > 2$. It remains to build a continuous injective map

$$\pi : \mathcal{D}^{(m)}_{\omega} \to \mathbb{Z}^{\#}$$

and a continuous injective map

$$\pi_p : \mathcal{D}^{(m)}_{\omega} \to \mathbb{Z}(p^{\infty})^{\#} \text{ for every prime } p.$$

For the function $\omega \to \mathbb{Z}$ defined as above by $f(n) = n!$, let $f_{ij} : \omega \to G$ be defined by $f_{ij} = 0$ when $j < m$ and $f_{im} = f$ otherwise. Now with $H : \mathbb{Z}^{m(m-1)/2} \to \mathbb{Z}$ defined by $h(\ldots, x_{ij}, \ldots) = x_{1m} + x_{2m} + \ldots + x_{m-1m}$, take $\pi = \mu_{h,\mathcal{F}}$, i.e.,

$$\pi(n_1, n_2, \ldots, n_m) = (n_1! - n_m!) + (n_2! - n_m!) + \ldots + (n_{m-1}! - n_m!).$$

Continuity of π is granted by item (c) of the above example. Let us see that π is injective. Assume that $\pi(n_1, n_2, \ldots, n_m) = \pi(n'_1, n'_2, \ldots, n'_m)$. Then

$$(n_1! - n_m!) + (n_2! - n_m!) + \ldots + (n_{m-1}! - n_m!) \\ = (n'_1! - n'_m!) + (n'_2! - n'_m!) + \ldots + (n'_{m-1}! - n'_m!),$$

hence

(3.2) $n_1! + n_2! + \ldots + n_{m-1}! + (m-1)n'_m! = n'_1! + n'_2! + \ldots + n'_{m-1}! + (m-1)n_m!$.

Now we are going to use the fact that a representation of a natural number M as a sum of factorials of the form

$$M = a_1 k_1! + a_2 k_2 + \ldots + a_m k_m!, \tag{3.3}$$

with $1 < k_1 < \ldots < k_m$ and $0 \leq a_i < k_i$ for $i = 1, \ldots, m$ is unique. We shall refer to the natural coefficients a_i as *digits*. Let M be the natural number defined by the equal sums in (3.2). If $n'_m \notin \{n_1, n_2, \ldots, n_{m-1}\}$, then the uniqueness of representations (3.3) yield that necessarily $n_m \notin \{n'_1, n'_2, \ldots, n'_{m-1}\}$, since M has $m-1$ digits equal to 1 and one digit equal to $m-1$. Moreover, $m-1 > 1$ yields $n_m = n'_m$ and $\{n_1, n_2, \ldots, n_{m-1}\} = \{n'_1, n'_2, \ldots, n'_{m-1}\}$. Consequently, $(n_1, n_2, \ldots, n_{m-1}, n_m) = (n'_1, n'_2, \ldots, n'_{m-1}, n'_m)$. Let us see now that the case $n'_m \in \{n_1, n_2, \ldots, n_{m-1}\}$ cannot occur. Indeed, in that case M has a presentation with one digit m, the other $m-1$ digits equal to 1. Since the other presentation must have the same distribution of digits, also $n_m \in \{n'_1, n'_2, \ldots, n'_{m-1}\}$. Now, assume that $n_m = n'_i$ and $n'_m = n_k$ with $1 \leq i, k < m$. Then necessarily $n_m = n'_i < n'_m = n_k$, a contradiction.

Fix an arbitrary prime p. To define the map $\pi_p : \mathcal{D}_\omega^{(m)} \to \mathbb{Z}(p^\infty)^\#$ set $f(n) = \varphi(p^{-n!}) \in \mathbb{Z}(p^\infty)$. Then $\pi_p = \mu_{f_1, f_2, \ldots, f_{m-1}}$, where $f_k = (-1)^k f$. Then continuity of π_p follows from item (b) of the above example. Let us prove that π_p is injective. Assume that $\pi_p(n_1, n_2, \ldots, n_m) = \pi_p(n'_1, n'_2, \ldots, n'_m)$. Then

$$(p^{-n_1!} - p^{-n_2!}) - (p^{-n_2!} - p^{-n_3!}) + \ldots + (-1)^m (p^{-n_{m-1}!} - p^{-n_m!}) \\ = (p^{-n'_1!} - p^{-n'_2!}) - (p^{-n'_2!} - p^{-n'_3!}) + \ldots + (-1)^m (p^{-n'_{m-1}!} - p^{-n'_m!})$$

hence

$$p^{-n_1!} - 2p^{-n_2!} + 2p^{-n_3!} + \ldots + (-1)^{m-1} 2p^{-n_{m-1}!} + (-1)^m p^{-n_m!} \\ = p^{-n'_1!} - 2p^{-n'_2!} + 2p^{-n'_3!} + \ldots + (-1)^{m-1} 2p^{-n'_{m-1}!} + (-1)^m p^{-n'_m!}.$$

This gives

(3.4)

$p^{-n_1!} + 2(p^{-n'_2!} + p^{-n_3!} + \ldots) + p^{-n''_m!} = p^{-n'_1!} + 2(p^{-n_2!} + 2p^{-n'_3!} + \ldots) + p^{-n'''_m!}$,

where $\{n''_m, n'''_m\}$ coincides with $\{n_m, n'_m\}$ and $n''_m = n_m$ if m is odd, $n''_m = n'_m$ if m is even. Now we use the fact that every rational r has a unique representation of the form

$$r = a_1 p^{-1} + \ldots + a_m p^{-m},$$

with $0 \leq a_i < p$ for $i = 1, \ldots, m$. Arguing as before with uniqueness of this representation we deduce from (3.4) that $n_1 = n_1'$. After canceling $p^{-n_1!} = p^{-n_1'!}$ we can conclude that $n_2 = n_2'$ etc. □

We do not know whether $\mathcal{D}_\omega$ can be *embedded* into $\mathbb{Z}^\#$ (seemingly the continuous injection μ_f given above is not an embedding). Of course, a positive answer to this question will immediately yield that $G_2^\#$ is not homeomorphic to $\mathbb{Z}^\#$. This will follow also from the negative answer to the following question

Problem 3.7. *There exists no continuous injective map* $\pi : \mathcal{D}_\omega^{2,4} \to \mathbb{Z}^\#$.

The same questions remain for $\mathbb{Z}(p^\infty)^\#$.

4. Straightening of a continuous map $\pi : G_2^\# \to H^\#$ over doubletons

We shall see that for large κ, every continuous map

$$\pi : \{0\} \cup [\kappa]^4 \to G_3^\# \text{ with } \pi(0) = 0$$

necessarily sends $[Z]^4$ to 0 for some infinite $Z \subseteq \kappa$. This remains true when G_3 is replaced by G_m for all odd m and more generally, by any abelian group H with finite 2-rank (Section 5.2).

On the other hand, according to Proposition 3.3 one cannot claim that every continuous map $\pi : \mathcal{D}_\kappa \to G_3^\#$ necessarily sends some doubleton to 0. We shall prove that for every continuous map $\pi : \mathcal{D}_\kappa \to G_m^\#$ with $\pi(0) = 0$ there exists a large set $Z \subseteq \kappa$ such that either π vanishes on $\mathcal{D}_Z$ (i.e., $\pi(\alpha, \beta) = 0$ for all $(\alpha, \beta) \in [Z]^2$), or π coincides on $\mathcal{D}_Z$ with μ (up to a composition with a homeomorphism of $G_m^\#$, cf. Theorem 4.3). For more flexible terminology we introduce the following:

Definition 4.1. *Two maps*

$$\pi, \pi' : G \to H$$

are equivalent (we write $\pi \sim \pi'$ to denote this), if there exists a homeomorphism $\eta : H^\# \to H^\#$ such that $\pi' = \eta \circ \pi$.

Clearly, if

$$\pi, \pi' : G^\# \to H^\# \text{ and } \pi \sim \pi',$$

then π is 1-1, (resp., continuous, a homeomorphism) iff π' has the same property. Therefore, for the purposes of this paper we can always replace π by π'. Since every map $\pi : G \to H$ is equivalent to a map

$$\pi' : G \to H \text{ with } \pi'(0) = 0,$$

we shall consider from now on maps π with $\pi(0) = 0$.

We adopt the above notation also for partial maps, in particular for maps

$$\pi, \pi' : \mathcal{D}_A \to H, \text{ where } A \subseteq \kappa.$$

As above, it suffices to consider maps π with $\pi(0) = 0$.

4.1. **The straightening set of a map $\pi : G_2^\# \to V^\#$.** Now assume that the target group V has the form $V = \bigoplus_\kappa K$, where K is either a cyclic group, or a countable field. For every $i < \kappa$ fix a non-zero element e_i of the i-th copy K_i of K, such that e_i is a generator of K_i when K is cyclic. Then for the base $B = \{e_i : i < \kappa\}$ define (as before)

$$\mu_B : \mathcal{D}_\kappa \to V$$

by $\mu_B(i,j) = e_i - e_j$ for every $i < j < \kappa$ and put $\mu_B(0) = 0$.

Definition 4.2. *With V and B as above, a subset $Z \subseteq \kappa$ is a* straightening set *for a continuous map*

$$\pi : \mathcal{D}_\kappa \to V^\#$$

with respect to the base B *if $\pi \sim r\mu_B$ for some $r \in \mathbb{N}$.*

Clearly, $\mu_B \sim \mu_{B'}$ for any two bases B, B' of V. This is why we shall omit the subscript $_B$ when no confusion is possible. Similarly, we shall speak of straightening set *without any specification of a base* in the target group V.

We use this terminology also for continuous maps

$$\pi : G_2^\# \to V^\#.$$

In other words, Z is a straightening set of π if π coincides on $\mathcal{D}_Z$ with $r\mu$ (up to a composition with a homeomorphism of $V^\#$). Note that when $r = 0$ then π vanishes on $[Z]^2$.

For an infinite cardinal λ the $\beth_n$-function is defined by $\beth_0(\lambda) = \lambda$ and $\beth_{n+1}(\lambda) = 2^{\beth_n(\lambda)}$ for every natural n. As usual, we abbreviate $\beth_n(\omega)$ to $\beth_n$ and let $\beth_\omega = \sup_{n<\omega} \beth_n$ [18].

Theorem 4.3. *Let V be vector space of dimension κ over a countable field K and let*

$$\pi : \mathcal{D}_\kappa \to V^\#$$

be a continuous map with $\pi(0) = 0$.

(a) *If $\kappa > \mathfrak{c}$, then there exists an infinite straightening set of π.*
(b) *If λ is an infinite cardinal and $\kappa > \beth_6(\lambda)$, then there exists a straightening set of π of size $> \lambda$.*

Clearly, this theorem implies that:

(a) if $\kappa > \mathfrak{c}$, then every continuous finite-to-one map $\pi : G_2^\# \to G_m^\#$ admits an infinite set $S \subseteq \kappa$ on which $\pi \sim \mu$;
(b) if λ is an infinite cardinal and $\kappa > \beth_6(\lambda)$ then every continuous finite-to-one map $\pi : G_2^\# \to G_m^\#$ admits a set $S \subset \kappa$ of size $> \lambda$ on which $\pi \sim \mu$.

To prove Theorem 4.3 we make use of the Main Lemma. To this end we need some preliminary discussion in order to motivate Definition 4.4:

a) every function $f : \kappa \to \kappa$ admits a subset S of κ of size κ such that $f \restriction_S$ is either constant or injective; and

b) every function $f : \kappa \to [\kappa]^{<\omega}$, for uncountable κ, admits a subset S of κ of size κ such that $f \restriction_S$ is a Δ-system [18].

We aim to show that this phenomenon extends to functions defined on k-tuples of a subset $S \subseteq \kappa$ (see Lemma 4.9). Let G be a direct sum of copies of a fixed group K, so that every non-zero element $x \in G$ determines uniquely a non-empty finite set $\mathrm{supp}(x)$ of indices i such that "x takes non-zero value at i". For the sake of completeness set $\mathrm{supp}(0) = \varnothing$. Two elements $x, y \in G$ are *order isomorphic* if the (unique) order isomorphism

$$\varphi : \mathrm{supp}\, x \to \mathrm{supp}\, y$$

"commutes", in the obvious way, with the finite functions x and y (i.e., $y(\varphi(i)) = x(i)$ for every $i \in \mathrm{supp}\, x$).

Definition 4.4. *For an infinite set $S \subseteq \kappa$ and positive $k \in \mathbb{N}$ call a function*

$$f : [S]^k \to G \text{ (with } G \text{ as above)}$$

standard, *if either f is constant, or for $A \neq A' \in [S]^k$*

(a) $\mathrm{supp}\, f(A) \cap \mathrm{supp}\, f(A') = \varnothing$, *and*
(b) *there exists an order isomorphism between the finite functions $f(A)$ and $f(A')$.*

In particular, (b) implies that all supports of $f(A)$ have the same size.

The standard functions present a certain prototype of a "base" for continuous maps $G_2^{\#} \to G^{\#}$ in an appropriate sense (see §4.3). In Claim 4.7 we show an "independence" property of standard 1-variable functions with respect to convergence to 0 (if $\tau_1(\alpha) + \tau_2(\beta) \to 0$ for every net (α, β) converging to 0 in $G_2^{\#}$, then τ_1, τ_2 are linearly dependent on a cofinite subset of κ).

Main Theorem. *Let λ, κ and π be as in Theorem 4.3 and $V = G_K$ where K is a countable group. Then there exists a infinite set $S \subset \kappa$ (of size $> \lambda$ if $\kappa > \beth_6(\lambda)$ holds), and a standard function $\tau : S \to V$, such that*

$$\pi(\alpha, \beta) = \tau(\alpha) - \tau(\beta) \text{ for all } \alpha < \beta \text{ in } S \tag{4.5}$$

(i.e., π coincides with μ_τ of $[S]^2$).

Proof of Theorem 4.3. Apply the Main Lemma to get a subset $S \subset \kappa$ of size $> \lambda$ and a standard function $\tau : S \to V$ such that (4.5) holds. If τ is constant we are done as π vanishes on $[S]^2$. Assume that τ is not constant. We show now that S is a straightening set of π.

Since the function τ is standard and all supports $\mathrm{supp}\, \tau(\alpha)$ have size $n > 0$, the family $M = \{\tau(\alpha)\}_{\alpha \in S}$ is independent in V. Then, the subspace $H = \langle M \rangle$ of V is isomorphic to $\oplus_S K$ as a vector space. Fix the unique isomorphism

$$\iota : H \to \oplus_S K$$

(of vector spaces over K) that sends $\tau(\alpha)$ to the element $e_\alpha \in \oplus_S K$ of the canonical base of V for every $\alpha \in S$. Note that $V = H \oplus L$ for some subgroup $L \cong V/H$ of V. We can assume without loss of generality that $|S| < \kappa$. Then

$$|H| = |S| < \kappa = |V|.$$

Therefore, $|L| = |V|$, so that $L \cong V$. Obviously, V splits also as

$$V = \left(\bigoplus_S K \right) \oplus L', \quad \text{with} \quad L' = \bigoplus_{\kappa \setminus S} K \cong V.$$

Hence $L \cong L'$. Therefore, there exists an isomorphism

$$\iota' : V^\# \to V^\#$$

extending ι, i.e., $\iota'(\alpha) = e_\alpha$ for every $\alpha \in S$. This isomorphism is automatically topological with respect to the Bohr topology. Now the composition $\pi' = \iota \circ \pi$ coincides with μ on $[S]^2$, hence S is a straightening set of π. □

Note 4. *Let us note that this proof works also for $K \cong \mathbb{Z}$. For $K = \mathbb{Z}_m$ a finite non-simple cyclic group, i.e., $G_m = \bigoplus \mathbb{Z}_m$, the following slight modification is needed. First, we restrict further S so that all elements $\tau(\alpha)$, for $\alpha \in S$, have the same order k. Then the proof proceeds as above with only difference due to the fact that the group H is now isomorphic to $\bigoplus_S \mathbb{Z}_k$, so the above argument proves that $\pi \sim r \cdot \mu$, where $r = m/k$.*

According to Example 3.5 there exists a continuous injective map $\mathcal{D}_{|G|} \to G^\#$ for every infinite group G.

We shall see later that if H is torsion free with $|H| > \beth_3$, then every abelian group $G^\#$ homeomorphic to $H^\#$ must have few involutions, i.e., $r_2(G) < |G|$. More precisely, if $\kappa > \beth_3$ then every continuous map $G_2^\# \to H^\#$ into a torsion-free group has an infinite fiber, so cannot be a homeomorphism (hence, if H is torsion free and $\pi : G^\# \to H^\#$ is a continuous 1-1 map, then $r_2(G) \leq \beth_3$).

4.2. ***e*-Straightening sets.** The definition of a straightening set is somewhat too restrictive as far as the codomain is concerned. According to the Main Lemma, for all codomains G_K one can "straighten" a continuous map

$$\pi : \mathcal{D}_\kappa \to G_K^\#$$

to get a restriction $\pi|_{\mathcal{D}_S}$ of the form μ_τ for some (standard) function $\tau : S \to G_K$. This suggests the following more flexible notion that works for *all* codomains.

Definition 4.5. *A subset $Z \subseteq \kappa$ is a* e-straightening set *for a continuous map*

$$\pi : \mathcal{D}_\kappa \to G^\#$$

if there exist an abelian group H that contains G as a subgroup and a function $f : Z \to H$, such that $\pi = \mu_f$ when restricted to $\mathcal{D}_Z$.

Since every function μ_f is continuous by Proposition 3.3, this is the limit we can push the generality.

Theorem 4.6. *Let G be an abelian group and let*

$$\pi : \mathcal{D}_\kappa \to G^\#$$

be a continuous map with $\pi(0) = 0$. *If* $\kappa > \mathfrak{c}$ *(resp.,* $\kappa > \beth_6(\lambda)$ *for some infinite cardinal* λ*), then there exists an infinite e-straightening set (resp., e-straightening set of size* $> \lambda$*) of* π.

To get a proof of this version of the straightening set theorem we only need to fix a divisible group

$$H \cong \oplus_{|G|}(\mathbb{Q} \times \mathbb{Q}/\mathbb{Z})$$

containing G and to apply the Main Lemma to the continuous map

$$\pi : \mathcal{D}_\kappa \to H^\#$$

to get a subset S of κ of size $> \lambda$ and a function $\tau : S \to H$ such that $\pi(\alpha, \beta) = \tau(\alpha) - \tau(\beta)$ for all $\alpha < \beta$ in S. Thus $\pi = \mu_\tau$.

4.3. **Proof of the Main Lemma.** The proof of the Main Lemma is based on two essential points. The first one is the following "independence" property anticipated above (for $V = G_3$ this is Claim 8 from [10], for a proof of the present version see [7]).

Claim 4.7. *Let* κ *be an infinite cardinal and let* V *be a group as in the Main Lemma.*

(i) *If* $\tau : \kappa \to V$ *is a standard function and* $\tau(\alpha) \to 0$ *for some net* α *in* κ, *then* τ *vanishes.*

(ii) *If* $\tau_1 : \kappa \to V$, $\tau_2 : \kappa \to V$ *are disjoint standard functions such that*

$$n_{\alpha,\beta} = \tau_1(\alpha) + \tau_2(\beta) \to 0$$

for every net (α, β) *converging to* 0 *in* $G_2^\#$, *then there exists a cofinite subset* $S \subseteq \kappa$ *where* $\tau_2 = -\tau_1$.

Note 5. *Actually, the following more general result is proved in* [7]. *Let* $m > 1$ *and let* V *be as above. Assume*

$$\tau_i : \kappa \to V, \; i = 1, \ldots, m$$

are disjoint standard functions such that

$$n_{\alpha_1,\ldots,\alpha_m} = \sum_{i=1}^{m} \tau_i(\alpha_i) \to 0 \textit{ for every net } (\alpha_1, \ldots, \alpha_m) \to 0 \textit{ in } G_m^\#,$$

then there exists a cofinite subset S *of* κ *such that*

$$\sum_{i=1}^{m} \tau_i(\alpha) = 0 \textit{ for every } \alpha \in S.$$

The second ingredient in the proof of our Main Lemma is a combinatorial lemma proved in [10] in a simpler version. The following definition is needed in the combinatorial lemma.

Definition 4.8. *Let $V = \bigoplus_\kappa K$, where K is a countable group. Two functions*

$$f_1 : [S]^{k_1} \to V \text{ and } f_2 : [S]^{k_2} \to V$$

are disjoint *if*

$$\operatorname{supp} f_1(A_1) \cap \operatorname{supp} f_2(A_2) = \varnothing$$

for every $A_1 \in [S]^{k_1}$ and $A_2 \in [S]^{k_2}$ (with $A_1 \neq A_2$ in case $k_1 = k_2$).

Lemma 4.9. (Combinatorial Lemma I) *If $\kappa > \mathfrak{c}$ and π is* any *map from $[\kappa]^2$ to V, then there exists an infinite subset $S \subset \kappa$ and four pairwise disjoint standard functions τ_{ij} $(i, j = 0, 1)$, such that:*

(1) *the function $\tau_{00} : [S]^2 \to G_3$ is constant (i.e, $\tau_{00}(\alpha, \beta)$ does not depend on $\alpha, \beta \in [S]^2$);*
(2) *$\tau_{10} : S \to G_3$ and $\tau_{01} : S \to G_3$ are one-variable functions;*
(3) *$\tau_{11} : [S]^2 \to G_3$;*
(4) *$\pi(\alpha, \beta) = \tau_{00} + \tau_{10}(\alpha) + \tau_{01}(\beta) + \tau_{11}(\alpha, \beta)$ for all $\alpha < \beta$ in S.*

For a sketch of the proof consider the supports of $\pi(\alpha, \beta)$ and apply Erdös-Rado theorem [18] to the coloring of the set of doubletons (α, β) with countably many colors defined by the countably many (up-to order isomorphism) finite functions from κ to the *countable* group K. Since $\kappa > \mathfrak{c}$ there exists a homogeneous set $S_1 \subseteq \kappa$ of size $> \omega$, i.e., the induced coloring of $[S_1]^2$ is constant (in one color). This means that all finite functions $\pi(\alpha, \beta)$ are order isomorphic when $\alpha, \beta \in S_1$. In particular, all supports of $\pi(\alpha, \beta)$ have a fixed size n for $\alpha, \beta \in S_1$. If $n > 0$ for the map $[S_1]^2 \to [\kappa]^n$ defined by

$$(\alpha, \beta) \mapsto \operatorname{supp}(\pi(\alpha, \beta))$$

find (applying twice Erdös-Rado, cf. [10]) an infinite subset S_2 of S_1 such that for every $\alpha < \beta$ in S_2

$$\operatorname{supp}(\pi(\alpha, \beta)) = \sigma_{00} \cup \sigma_{10}(\alpha) \cup \sigma_{01}(\beta) \cup \sigma_{11}(\alpha, \beta), \tag{$*$}$$

such that

- σ_{00} is constant;
- all sets $\sigma_{10}(\alpha)$, $\alpha \in S_2$, are pairwise disjoint, have the same size and $\sigma_{10}(\alpha) \cap \sigma_{00} = \varnothing$;
- all sets $\sigma_{01}(\beta)$, $\beta \in S_2$, are pairwise disjoint, have the same size and $\sigma_{01}(\beta) \cap \sigma_{00} = \varnothing$;
- all sets $\sigma_{11}(\alpha, \beta)$, $\alpha < \beta$ from S_2, are pairwise disjoint, have the same size and $\sigma_{11}(\alpha, \beta) \cap \sigma_{00} = \varnothing$;
- if $\alpha \neq \beta$ in S_2, then $\sigma_{10}(\alpha) \cap \sigma_{01}(\beta) = \varnothing$;
- if $\alpha < \beta$ and $\alpha' \in S_2$, then $\sigma_{10}(\alpha') \cap \sigma_{11}(\alpha, \beta) = \varnothing$ and $\sigma_{01}(\alpha') \cap \sigma_{11}(\alpha, \beta) = \varnothing$.
- for any $i, j = 1, 2$ the positions of the elements of $\operatorname{supp} \sigma_{ij}(\alpha, \beta)$ in the n-tuple $\operatorname{supp} \pi(\alpha, \beta)$ do not depend on α, β (but depend on i, j).

Now just take $\tau_{ij}(\alpha, \beta)$ to be the restriction of $\pi(\alpha, \beta)$ to the set $\sigma_{ij}(\alpha, \beta)$. Note that $\tau_{00}(\alpha, \beta)$ must be constant since its domain σ_{00} is constant. It is easy to see that the functions τ_{10}, τ_{01} and τ_{11} are standard.

Note 6. If $\kappa > \beth_6(\lambda)$ for an infinite cardinal λ, then it is possible to get a set $S \subseteq \kappa$ of size $> \lambda$ in the combinatorial lemma. Indeed, the first application of Erdös-Rado's theorem, as above, gives an infinite subset $S_1 \subseteq \kappa$ of size $> \beth_5(\lambda)$ where all finite functions $\pi(\alpha, \beta)$ are order isomorphic, in particular have supports of the same size n. If $n > 0$, then the second applications of Erdös-Rado's theorem to the map $[S_1]^2 \to [\kappa]^n$ gives an infinite subset S_2 of S_1 of size $> \lambda$ where $(*)$ holds as above. Now one proceeds as above.

Example 4.10. The function $\mu : \mathcal{D}_\kappa \to G_m$ is not standard. Its presentation via the Main Lemma is $\mu(\alpha, \beta) = \tau(\alpha) - \tau(\beta)$, where $\tau(\alpha) = e_\alpha$ is standard.

Proof of Main Lemma. By Lemma 4.9 there exists an uncountable subset $S \subseteq \kappa$ of size $> \lambda$ satisfying 1–4 from the lemma. We shall carry out a computation of the functions τ_{ij} on S. More precisely, we show that:

$$\begin{aligned} &\tau_{00}(\gamma, \delta) = 0, \\ &\tau_{11}(\gamma, \delta) = 0 \text{ and} \\ &\quad \tau_{10}(\gamma) = -\tau_{01}(\gamma) \text{ for all but finitely many } \gamma, \delta \in S. \end{aligned}$$

In particular $\pi(\gamma, \delta) = \tau_{10}(\gamma) - \tau_{10}(\delta)$.

Apply Lemma 2.4 to get a net in which (γ, δ) converges to 0 in $\mathcal{D}_S$. By continuity of π also the net $\pi(\gamma, \delta)$ converges to 0 in $V^\#$. Now we exploit the fact that the functions τ_{ij} are standard and pairwise disjoint to prove that all $\tau_{11}(\gamma, \delta) = 0$. This entails that the supports $\tau_{11}(\gamma, \delta)$ are all disjoint from all the other τ_{ij}'s supports (and from each other), hence we can apply Lemma 2.14 to the Bohr-converging net

$$\pi(\gamma, \delta) = \tau_{00}(\gamma, \delta) + \tau_{10}(\gamma) + \tau_{01}(\delta) + \tau_{11}(\gamma, \delta) \to 0 \tag{4.6}$$

in which $\tau_{11}(\gamma, \delta)$ splits as a strongly moving component. So we deduce that $\tau_{11}(\gamma, \delta) = 0$ for a tail of the net. By item 3 in Lemma 4.9 all $\operatorname{supp} \tau_{11}(\gamma, \delta)$ are of the same size k. Hence $k = 0$, i.e., *all* $\tau_{11}(\gamma, \delta) = 0$.

To show $\tau_{00} = 0$, we note that the net (4.6) splits in $\tau_{00} + m_{(\gamma,\delta)}$, where

$$m_{(\gamma,\delta)} = \tau_{10}(\gamma) + \tau_{01}(\delta)$$

and $\tau_{00}(\gamma, \delta)$ does not depend on γ, δ. Thus Theorem 2.9 can be applied to conclude that $\tau_{00} = 0$. Now Claim 4.7 can be applied. This gives

$$\tau_{10}(\alpha) = -\tau_{01}(\alpha)$$

for all α from some cofinite subset S' of S.

When $\kappa > \beth_6(\lambda)$ by Observation 6 one can get a subset $S \subseteq \kappa$ of size $> \lambda$ satisfying 1–4 from the lemma. Then one concludes as above, by using an appropriate version of Claim 4.7 for $|S| > \lambda$. □

5. Straightening over $[\kappa]^3$ and $[\kappa]^4$

5.1. **Straightening of a map over the triples.** The following combinatorial lemma ensures the existence of a splitting analogous to that of Lemma 4.9 based on analogous splitting of n-tuple valued functions defined on triples.

Lemma 5.1. *If $\kappa \geq 2^{\mathfrak{c}}$ and π is* any *map from $[\kappa]^3$ to V, then there is an infinite set Z of π and 8 pairwise disjoint standard functions*

$$\sigma_{ijk} : [Z]^{i+j+k} \to G_3 \text{ defined for every } i, j, k = 0, 1$$

such that

$$\begin{aligned}\pi(\alpha, \beta, \gamma) = \sigma_{111}&(\alpha, \beta, \gamma)\\ &+ \sigma_{110}(\alpha, \beta) + \sigma_{101}(\alpha, \gamma) + \sigma_{011}(\beta, \gamma)\\ &+ \sigma_{100}(\alpha) + \sigma_{010}(\beta) + \sigma_{001}(\gamma)\\ &+ \sigma_{000}.\end{aligned}$$

for every $\alpha < \beta < \gamma$ in Z.

According to our convention, σ_{000} is a constant function.

In the sequel, for every subset $Z \subseteq \kappa$ the set $[Z] \cup [Z]^3$ will be equipped with the topology induced by $G_2^{\#}$. Let H be an abelian group. It can be proved as before, that for every function $f : \kappa \to H$ the map $\mu'_f : [Z] \cup [Z]^3 \to H$ defined by

$$\alpha \mapsto f(\alpha) \text{ and } (\alpha, \beta, \gamma) \mapsto f(\alpha) - f(\beta) + f(\gamma)$$

is continuous. In particular, when $H = \bigoplus_\kappa K$, where K is a field or a cyclic group, then for a base $B = \{e_\alpha : \alpha < \kappa\}$ the map $\mu' : [Z] \cup [Z]^3 \to V$ defined by

$$\mu'(\alpha) = e_\alpha \text{ and } \mu'(\alpha, \beta, \gamma) = e_\alpha - e_\beta + e_\gamma$$

is continuous. Now an argument similar to that of the Main Lemma proves that for every continuous map $\pi : [\kappa] \cup [\kappa]^3 \to V$ there exists some infinite Z and a standard function $\tau : S \to [V]$ so that $\pi = \mu'_\tau$

$$(\text{i.e., } \pi(\alpha) = \tau(\alpha) \text{ and } \pi(\alpha, \beta, \gamma) = \tau(\alpha) - \tau(\beta) + \tau(\gamma)).$$

Then, with a final step as in the proof of Theorem 4.3, one can show that $\pi \sim \mu'$ when both restricted to $[Z] \cup [Z]^3$. In case $V = G_m$ with m non-prime, one has to admit also a coefficient $r \in \mathbb{N}$, i.e., $\pi \sim r\mu'$. This is the *straightening theorem for triples.* We are not going to give rigorous definitions and formulations since we are not going to use this straightening theorem.

5.2. **Straightening over $[\kappa]^4$ and vanishing of $\pi(\alpha, \beta, \gamma, \delta)$.** Suppose

$$\pi : G_2^{\#} \to H^{\#}$$

is continuous for an arbitrary abelian group H. Composing π with an appropriate translation in the group H, we can assume without loss of generality that $\pi(0) = 0$. Our plan is to see, that $2\pi(\alpha, \beta, \gamma, \delta) = 0$ for all $\alpha < \beta < \gamma < \delta$ in some infinite subset Z of κ and consequently, $2\pi(\alpha, \beta) = 0$ for all doubletons (α, β) from Z since $[Z]^2$ is contained in the closure of $[Z]^4$ by Lemma 2.6 (see Definition 5.2 and Theorem 5.3 for more precise formulation). In other words, we show that over an appropriate Z, the restriction of π to

$$\mathcal{D}_{Z,2}^{(4)} = \{0\} \cup [Z]^4$$

factorizes through the inclusion $H[2] \hookrightarrow H$ and as such a map it is either constant or equivalent to the inclusion $[Z]^4 \hookrightarrow H[2]$. This result is much stronger than what we have seen before on doubletons. Indeed, by passing to limits it yields that also the restriction of π to $[Z]^2$ factorizes through the inclusion $H[2] \hookrightarrow H$ and as such a map it is equivalent to the inclusion $[Z]^2 \hookrightarrow H[2]$.

Here is the relevant definition:

Definition 5.2. *For a map $\pi : \mathcal{D}^{(4)}_{\kappa,2} \to H^{\#}$ with $\pi(0) = 0$ we say that $Z \subseteq \kappa$ is a* straightening set *of π if either π vanishes on $[Z]^4$, or π sends $[Z]^4$ injectively into $H[2]$, so that the restriction of π to $\mathcal{D}^{(4)}_{Z,2}$ is equivalent to the inclusion $\mathcal{D}^{(4)}_{Z,2} \hookrightarrow H[2]$, i.e.,*

$$\pi(\alpha, \beta, \gamma, \delta) = e_\alpha + e_\beta + e_\gamma + e_\delta$$

for some appropriate base $\{e_\alpha : \alpha \in Z\}$ of $H[2]$ and all $\alpha < \beta < \gamma < \delta$ from Z.

Theorem 5.3. *If $\kappa > \beth_3$ and K is a countable abelian group, then every continuous map*

$$\pi : \mathcal{D}^{(4)}_{\kappa,2} \to (\bigoplus_{\kappa} K)^{\#}$$

with $\pi(0) = 0$ admits an infinite straightening set.

A detailed scheme of the proof of this theorem, similar (to a certain extent) to the proof of Theorem 1.2 from [10], is given in §5.4.

Theorem 5.4. *Let G be an abelian group with $r_2(G) > \beth_3$ that admits a continuous finitely many-to-one map*

$$\pi : G^{\#} \to H^{\#}.$$

Then $r_2(H)$ is infinite (i.e., H contains an infinite Boolean subgroup).

Proof. As already noted, we can assume that the continuous map $\pi : G \to H$ satisfies $\pi(0) = 0$. Note that the divisible hull $D(H)$ is a direct sum of countable groups that are direct summands of the group $K = \mathbb{Q} \times \mathbb{Q}/\mathbb{Z}$. Hence, H is contained in the group $V = \oplus_\kappa K$. Then we can identify $G[2]$ with G_2 and apply Theorem 5.3 to the restriction

$$\pi : G[2]^{\#} \to V^{\#}$$

to claim that for some infinite Z in κ the values of π taken on the subset $[Z]^4$ of $G[2]$ belong to $V[2]$, and consequently to $H[2] = V[2] \cap H$. Since π was supposed to be injective, this proves that $r_2(H)$ is infinite. □

This theorem gives as immediate corollary Theorem 1.3 from the introduction. The next corollary of Theorem 5.3 provides a more precise form of Theorem 1.2 from the introduction:

Corollary 5.5. *Let $\pi : G_2^{\#} \to G_m^{\#}$ be a continuous map with $\pi(0) = 0$, odd m and let $\kappa > \beth_3 = 2^{2^{\mathfrak{c}}}$. Then π is zero on $[Z]^2$ and $[Z]^4$ for some infinite set $Z \subseteq \kappa$.*

This yields that $G_n^{\#}$ and $G_m^{\#}$ are not homeomorphic for naturals m and n with distinct parity. Let us note that Kunen [19] proved Corollary 5.5 for $\kappa = \omega$ and arbitrary pair p, q of primes:

Theorem 5.6. [19, Th.4.1] *Let p and q be distinct primes and let $k \in \mathbb{N}$ satisfy $k > p$ and $p|k$. Then every continuous map*

$$\pi : \mathcal{D}_{\omega}^{(k,p)} \to G_q^{\#}$$

with $\pi(0) = 0$ is zero on $[Z]^k$ for some infinite set $Z \subseteq \omega$.

Corollary 5.7. *Let $\pi : G_2^{\#} \to G_m^{\#}$ be a continuous map with odd m. Then π is constant on some subset B of size κ of G_2.*

Proof. We can assume without loss of generality that $\pi(0) = 0$. One can present G_2 as a direct sum $G_2 = \bigoplus_{\rho<\kappa} H_\rho$, where each group $H_\rho \cong G_2$. Assume $\kappa > \beth_3$. Since the induced topology of H_ρ is the same as that of $H_\rho^{\#}$, we can claim with Corollary 5.5 that π vanishes on an infinite subset B_ρ of H_ρ for every $\rho < \kappa$. Therefore, π vanishes on $B = \bigcup_{\rho<\kappa} B_\rho$, and obviously $|B| = \kappa$. In the general case we can conclude using Kunen's theorem 5.6. □

It is interesting to note that even if the groups $(\mathbb{Q}^{\#})^\kappa$ and $[\mathbb{Z}^{\#} \times (\mathbb{Q}/\mathbb{Z})^{\#}]^\kappa$ are homeomorphic for every κ (as $\mathbb{Q}^{\#}$ and $\mathbb{Z}^{\#} \times (\mathbb{Q}/\mathbb{Z})^{\#}$ are homeomorphic [3]), one has

Corollary 5.8. *For $\kappa \geq \beth_3$ the groups $\mathbb{Q}^\kappa$ and $[\mathbb{Z} \times (\mathbb{Q}/\mathbb{Z})]^\kappa$ are not homeomorphic in the Bohr topology.*

Proof. Note that the group $H = \mathbb{Q}^\kappa$ is torsion-free, while $G = [\mathbb{Z} \times (\mathbb{Q}/\mathbb{Z})]^\kappa$ has $r_2(G) = 2^\kappa > \beth_3$. Hence by Theorem 5.4 there exists no continuous 1-1 map $G^{\#} \to H^{\#}$. □

Clearly, the above argument works for any pair of abelian groups G and H, such that H has no infinite Boolean subgroups and $G = L \times (\mathbb{Q}/\mathbb{Z})^\kappa$, with $\kappa = |G| = |H| \geq \beth_3$.

These results leave open the question for small κ:

Problem 5.9. *For which infinite cardinals κ does there exist a continuous 1-1 map from $G_2^{\#}$ to some $H^{\#}$ without infinite Boolean subgroups?*

It seems natural to conjecture that such a continuous 1-1 map does not exist for any infinite κ (see the stronger Conjecture SST below). The weaker conjecture that such 1-1 maps do not exist for $\kappa = \mathfrak{c}$ implies that $\mathbb{Q}^\omega$ and $[\mathbb{Z} \times (\mathbb{Q}/\mathbb{Z})]^\omega$ are not homeomorphic in the Bohr topology.

5.3. **Straightening over $\mathcal{D}_{\kappa,p}^{(k)}$.** It is possible to prove a version of Theorem 5.3 with $\mathcal{D}_{\kappa,p}^{(k)}$, $k > p$ and $p|k$, in place of $\mathcal{D}_{\kappa,2}^{(4)}$ and $V[p]$ in place of $V[2]$. This requires a still larger κ (namely, $\kappa > \beth_{k-1}$) and a careful definition of straightening set for maps

$$\pi : \mathcal{D}_{\kappa,p}^{(k)} \to V$$

(see [7]). As above one can deduce from this straightening theorem that there exists no continuous 1-1 map $G_p^\# \to H^\#$ when κ is sufficiently large and $r_p(H) < \infty$. More precisely:

Theorem 5.10. [7] *Let p be a prime number and let G be an abelian group with $r_p(G) > \beth_{2p-1}$ admitting a continuous finitely many-to-one map $G^\# \to H^\#$. Then H contains a copy of the group $\oplus_\omega \mathbb{Z}_p$, i.e., $r_p(H)$ is infinite.*

As a corollary we get the result anticipated in the introduction:

Corollary 5.11. *Let $G^\#$ and $H^\#$ be homeomorphic and assume that H is almost torsion-free. Then $|t(G)| \leq \beth_\omega$.*

These results give many pairs of groups G and H non-homeomorphic in the Bohr topology. For example, when H is almost torsion-free with $|H| = \kappa > \beth_{2m-1}$, L is arbitrary with $|L| \leq \kappa$, and $G = L \times (\bigoplus_\kappa \mathbb{Z}_m)$.

Corollary 5.12. *Let p be a prime number and let G, H be abelian groups such that the powers H^κ and G^κ are homeomorphic in the Bohr topology for some infinite cardinal $\kappa \geq \beth_{2p-1}$. Then $r_p(G) > 0$ iff $r_p(H) > 0$.*

Corollary 5.13. *If $\kappa \geq \beth_\omega$ and H^κ and G^κ are homeomorphic in the Bohr topology, then for every prime p, $r_p(G) > 0$ iff $r_p(H) > 0$. In particular, either both G, H are torsion-free, or both G, H have non-trivial torsion elements.*

Corollary 5.14. *If B_1 and B_2 are unital subrings of $\mathbb{Q}$, then TFAE:*

(a) *$B_1 = B_2$;*
(b) *$B_1 \cong B_2$ as abstract rings;*
(c) *$B_1 \cong B_2$ as abstract groups;*
(d) *there exists $\kappa \geq \beth_\omega$ such that for the underlying abelian groups B_1 and B_2 the powers $(B_1/\mathbb{Z})^\kappa$ and $(B_2/\mathbb{Z})^\kappa$ are homeomorphic in the Bohr topology.*

Denote by $\mathfrak{BF}$ the class of abelian groups G such that $G^\#$ admits a continuous 1-1 map into some torsion-free $H^\#$. Call an abelian group G *uniformly almost torsion-free* if there exists $n \in \mathbb{N}$ such that $r_p(G) \leq n$ for every prime p. It follows from the homeomorphism theorem in [4] that every uniformly torsion-free group G admits an embedding $G^\# \hookrightarrow H^\#$, where H is torsion-free. (Just note that there exists $n \in \mathbb{N}$ such that $t(G)$ is isomorphic to a subgroup of $(\mathbb{Q}/\mathbb{Z})^n$. Thus the divisible hull of G is isomorphic to a subgroup of $\mathbb{Q}^{(\alpha)} \times (\mathbb{Q}/\mathbb{Z})^n$, with $\alpha = r(G)$. Now $(\mathbb{Q}/\mathbb{Z})^n$ is Bohr-embeddable in $\mathbb{Q}^n$, thus G is Bohr-embeddable into $\mathbb{Q}^{(\alpha)} \times \mathbb{Q}^n$.) In other words, the class $\mathfrak{BF}$ contains the class $\mathfrak{UAF}$ of uniformly almost torsion-free groups. The following conjecture can help to describe better the class $\mathfrak{BF}$:

Conjecture SST (Strong Straightening Theorem). *For every prime number p and for every continuous map*

$$\pi : G_p^\# \to H^\# \text{ with } \pi(0) = 0$$

there exists an infinite set $Z \subseteq \omega$ such that either π vanishes on $\mathcal{D}_Z^{(p)}$ or $\pi(\mathcal{D}_Z^{(p)}) \subseteq H[p]$ and $\pi|_{\mathcal{D}_Z^{(p)}}$ is equivalent to the inclusion $i : \mathcal{D}_Z^{(p)} \hookrightarrow H[p]$.

It follows from the results in [19] that the Conjecture SST holds true when H is a bounded torsion group. Let us mention the following corollary of Conjecture SST: *if $(\bigoplus_\omega \mathbb{Z}_p)^\#$ admits a continuous finite-to-one map into a group $H^\#$, then $(\bigoplus_\omega \mathbb{Z}_p)^\#$ admits also a* topological group *embedding into $H^\#$.* Therefore, an abelian group G that admits a continuous finite-to-one map $G^\# \to H^\#$ into an almost torsion-free group H, is almost torsion-free itself. In particular, the class $\mathfrak{AF}$ of almost torsion-free groups contains the class $\mathfrak{BF}$, i.e., $\mathfrak{UAF} \subseteq \mathfrak{BF} \subseteq \mathfrak{AF}$. We do not know whether:

- (a) there exists an almost torsion-free abelian group G such that $G^\#$ does not admit a continuous 1-1 map into some torsion-free $H^\#$ (i.e., whether $\mathfrak{BF} \neq \mathfrak{AF}$);
- (b) there exists a (necessarily almost torsion-free) abelian group G such that $G^\#$ admits a continuous 1-1 map into some torsion-free $H^\#$ and G is not uniformly almost torsion-free (i.e., whether $\mathfrak{BF} \neq \mathfrak{UAF}$).

Conjecture SST can help to establish that $G_p^\#$ is not Bohr-homeomorphic to $G_{p^2}^\#$.

5.4. **Proof of Theorem 5.3.** The proof of Theorem 5.4 is based on the "independence" property given in Claim 4.7 and a counterpart of the Combinatorial Lemma 4.9 for 4-tuples. Here π splits into the sum of 16 standard functions σ_{ijkl} defined on $(i+j+k+l)$-tuples of ordinals where i,j,k,l are 0 or 1. So that σ_{0000} is a constant function, σ_{1000}, σ_{0100}, σ_{0010} and σ_{0001} are one-variable functions, etc., while σ_{1111} depends on 4-tuples. For any fixed multi-index $ijkl$ the function σ_{ijkl} is standard and these functions are pairwise disjoint. In particular, all supports of σ_{ijkl} have uniform size and the function σ_{ijkl} takes the same value at the minimum element of the support so that Lemma 2.14 can be applied. Since the functions σ_{ijkl} have supports of uniform size, it is not restrictive to assume that Z has type ω. For the same reason it suffices to find just one zero value of the function σ_{ijkl} in order to conclude it vanishes on $[Z]^{i+j+k+l}$.

In the sequel we shall take partitions of Z into a union of infinite disjoint sets Z' and Z'' as in Lemma 2.4. Then we find nets $(\alpha,\beta,\gamma,\delta)$ such that one of the following three possibilities are fulfilled:

- (A) $\alpha,\gamma \in Z'$, $\beta,\delta \in Z''$ and the corresponding nets (α,γ) and (β,δ) converge to 0 in $G_2^\#$, so that by Lemma 2.3, the net $(\alpha,\beta,\gamma,\delta)$ Bohr-converges to 0 in $G_2^\#$.
- (B) $\alpha,\delta \in Z'$, $\beta,\gamma \in Z''$ and the corresponding nets (α,δ) and (β,γ) converge to 0 in $G_2^\#$, so that by Lemma 2.3, the net $(\alpha,\beta,\gamma,\delta)$ Bohr-converges to 0 in $G_2^\#$.

(C) $\alpha, \beta \in Z'$, $\gamma, \delta \in Z''$ and the corresponding nets (α, β) and (γ, δ) converge to 0 in $G_2^{\#}$, so that by Lemma 2.3, the net $(\alpha, \beta, \gamma, \delta)$ Bohr-converges to 0 in $G_2^{\#}$.

This can be arranged by Lemma 2.4. Then, in all three cases, also $\pi(\alpha, \beta, \gamma, \delta)$ converges to 0 in $H^{\#}$ by continuity.

For all three types (A)–(C) of nets we have a splitting of π as in Definition 2.8:

(5.7) $\pi(\alpha, \beta, \gamma, \delta) = \sigma_{1111}(\alpha, \beta, \gamma, \delta) + m_{(\alpha,\beta,\gamma,\delta)} + n_{(\alpha,\beta,\gamma,\delta)} + k_{(\alpha,\beta,\gamma,\delta)} + \sigma_{0000}$,

where

$$\begin{aligned} m_{(\alpha,\beta,\gamma,\delta)} &= \sigma_{1110}(\alpha,\beta,\gamma) + \sigma_{1101}(\alpha,\beta,\delta) + \sigma_{1011}(\alpha,\gamma,\delta) + \sigma_{0111}(\beta,\gamma,\delta), \\ n_{(\alpha,\beta,\gamma,\delta)} &= \sigma_{1100}(\alpha,\beta) + \sigma_{0011}(\gamma,\delta) + \sigma_{1001}(\alpha,\delta) + \sigma_{0110}(\beta,\gamma) \\ &\quad + \sigma_{1010}(\alpha,\gamma) + \sigma_{0101}(\beta,\delta), \\ k_{(\alpha,\beta,\gamma,\delta)} &= \sigma_{1000}(\alpha) + \sigma_{0010}(\gamma) + \sigma_{0100}(\beta) + \sigma_{0001}(\delta). \end{aligned}$$

By Theorem 2.9, $\pi(\alpha, \beta, \gamma, \delta) \to 0$ gives:

(a) $\sigma_{1111}(\alpha, \beta, \gamma, \delta) \to 0$, consequently we conclude $\sigma_{1111}(\alpha, \beta, \gamma, \delta) = 0$ on a tail of the net (since this is a net with pairwise disjoint supports, cf. Lemma 2.14).
(b) $\sigma_{0000} = 0$ as a constant net converging to 0.
(c) $m_{(\alpha,\beta,\gamma,\delta)} \to 0$, $n_{(\alpha,\beta,\gamma,\delta)} \to 0$ and $k_{(\alpha,\beta,\gamma,\delta)} \to 0$.

With (a) and (b) we have

$$\sigma_{1111} = \sigma_{0000} = 0. \tag{5.8}$$

Now we consider step by step the consequences of the three limits 0 in (c).

5.4.1. *Step 1.* $m_{(\alpha,\beta,\gamma,\delta)} \to 0$ with $\alpha, \beta, \gamma, \delta$ as in (A) implies that the net splits, so that all four 3-variable functions vanish on Z by Lemma 2.14 (cf. Observation 2), i.e.,

$$\sigma_{1110} = \sigma_{1101} = \sigma_{1011} = \sigma_{0111} = 0 \text{ on } Z. \tag{5.9}$$

Step 2. Take again $\alpha, \beta, \gamma, \delta$ as in (A) and note that $n_{(\alpha,\beta,\gamma,\delta)} \to 0$ splits in sum of four nets:

$$n_{(\alpha,\beta,\gamma,\delta)} = s_{(\alpha,\beta,\gamma,\delta)} + \sigma_{0110}(\beta,\gamma) + \sigma_{1010}(\alpha,\gamma) + \sigma_{0101}(\beta,\delta),$$

where

$$s_{(\alpha,\beta,\gamma,\delta)} = \sigma_{1100}(\alpha,\beta) + \sigma_{0011}(\gamma,\delta) + \sigma_{1001}(\alpha,\delta).$$

Hence $s_{(\alpha,\beta,\gamma,\delta)} \to 0$ by Theorem 2.9 and $\sigma_{0110}(\beta,\gamma)$, $\sigma_{1010}(\alpha,\gamma)$ and $\sigma_{0101}(\beta,\delta)$ vanish (being strongly moving components) on a tail of the net by Lemma 2.14 (cf. Observation 2). By uniformity of the σ's this proves

$$\sigma_{0110} = \sigma_{1010} = \sigma_{0101} = 0 \text{ on } Z. \tag{5.10}$$

In order to eliminate the remaining 2-variable functions, take a partition of Z into a union of infinite disjoint sets Z' and Z'' and find a net $(\alpha,\beta,\gamma,\delta)$ of type (B). Then one has a splitting

$$s_{(\alpha,\beta,\gamma,\delta)} = \sigma_{1100}(\alpha,\beta) + \sigma_{0011}(\gamma,\delta) + \sigma_{1001}(\alpha,\delta) \to 0.$$

By Theorem 2.9 $\sigma_{1100}(\alpha,\beta) \to 0$, $\sigma_{0011}(\gamma,\delta) \to 0$ and $\sigma_{1001}(\alpha,\delta) \to 0$. As before we get

$$\sigma_{1100} = \sigma_{0011} = \sigma_{1001} = 0. \tag{5.11}$$

Step 3. To deal with the 1-variable functions take a partition and a net of type (A). Then $k_{(\alpha,\beta,\gamma,\delta)} \to 0$ splits in

$$k_{(\alpha,\beta,\gamma,\delta)} = [\sigma_{1000}(\alpha) + \sigma_{0010}(\gamma)] + [\sigma_{0100}(\beta) + \sigma_{0001}(\delta)],$$

so that $\sigma_{1000}(\alpha) + \sigma_{0010}(\gamma) \to 0$ and $\sigma_{0100}(\beta) + \sigma_{0001}(\delta) \to 0$. By Claim 4.7 the converging net $\sigma_{1000}(\alpha) + \sigma_{0010}(\gamma) \to 0$ gives

$$\sigma_{0010}(\gamma) = -\sigma_{1000}(\gamma) \tag{5.12}$$

for all γ from a cofinite subset Z_1' of Z'. Analogously, the converging net $\sigma_{0100}(\beta) + \sigma_{0001}(\delta) \to 0$ gives

$$\sigma_{0001}(\delta) = -\sigma_{0100}(\delta). \tag{5.13}$$

for all δ from a cofinite subset Z_1'' of Z''. Exchanging the roles of Z' and Z'' and taking another net $\alpha < \beta < \gamma < \delta$, we get a cofinite subset Z_2' of Z' such that (5.13) holds for all $\delta \in Z_2'$. Analogously, (5.12) holds for every $\gamma \in Z_2''$ from some cofinite subset Z_2'' of Z''. This produces a cofinite subset Z_1 of Z where both (5.12) and (5.13) hold. Assuming for simplicity that $Z_1 = Z$ from now on, i.e., both (5.12) and (5.13) hold on Z.

Summing up (5.8)–(5.13), we see that we are left with

$$\pi(\alpha,\beta,\gamma,\delta) = \sigma_{1000}(\alpha) + \sigma_{0100}(\beta) - \sigma_{1000}(\gamma) - \sigma_{0100}(\delta).$$

A further application of Theorem 2.9 with a partition of type (B) gives $\sigma_{1000}(\alpha) - \sigma_{0100}(\delta) \to 0$ when the net (α,δ) converges to 0 in $[Z']^2$. Applying Claim 4.7 we can find a cofinite subset Z_0 of Z' where

$$\sigma_{1000}(\alpha) = \sigma_{0100}(\alpha) \text{ for all } \alpha \in Z_0. \tag{5.14}$$

With (5.11)–(5.14) we are left with only one standard function $\tau = \sigma_{1000}$ of one variable, i.e.,

$$\pi(\alpha,\beta,\gamma,\delta) = [\tau(\alpha) + \tau(\beta)] - [\tau(\gamma) + \tau(\delta)] \text{ for all } \alpha,\beta,\gamma,\delta \in Z_0. \tag{5.15}$$

Step 4. To finish the proof take a partition of Z_0 of type (C). It is easy to check that one has a splitting as indicated in (5.15). Again by Theorem 2.9 we conclude that $\tau(\alpha) + \tau(\beta) \to 0$ when the net (α,β) converges to 0 in $[Z']^2$. By Claim 4.7 $\tau(\alpha) = -\tau(\alpha)$ must hold on a cofinite subset of Z'. Thus $2\tau(\alpha) = 0$ for cofinitely many $\alpha \in Z'$. This yields that τ takes values in $V[2]$ on a cofinite subset of Z'. By (5.7)–(5.15), the values of

$$\pi(\alpha,\beta,\gamma,\delta) = \tau(\alpha) + \tau(\beta) + \tau(\gamma) + \tau(\delta)$$

on $[Z']^4$ belong to $V[2]$. Now since all $\tau(\alpha)$ have the same size, they are either all zero (i.e., π vanishes on $[Z']^4$), or all non-zero. In the latter case $|H[2]| \geq |Z|$ (since the supports of $\tau(\alpha)$ are pairwise disjoint), so that we can find a "straightening" automorphism $\iota : H[2] \to H[2]$, such that $\iota(\tau(\alpha)) = e_\alpha$. This proves that $\pi|_{[Z']^4} \sim i : [Z']^4 \hookrightarrow H[2]$ and finishes the proof of Theorem 5.3.

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