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Unions of chains of subgroups of a topological group

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Abstract. We consider the following problem: If a topological group G is the union of an increasing chain of subgroups and certain cardinal invariants of the subgroups in the chain are known, what can be said about G? We prove that if the index of boundedness of each subgroup is strictly less than λ for some infinite cardinal λ, then the index of boundedness of G is at most λ. We also prove that if both the index of boundedness and the pseudocharacter of each subgroup in the chain are at most λ and G is countably compact, then $|G| \leq 2^\lambda$. Finally, we show that the last assertion is not valid in general, not even for pseudocompact groups.

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1. Introduction

Let G be a topological group and suppose that $G = \bigcup\{G_\alpha : \alpha \in \kappa\}$, where $G_\alpha \leqslant G_\beta < G$ if $\alpha < \beta < \kappa$. What can be said about G if the values of some cardinal functions on the subgroups G_α's are known? In Section 2 we prove that if the index of boundedness of each subgroup in the chain is strictly less than λ for some infinite cardinal λ, then the index of boundedness of G is at most λ. We also prove that if both the character and the index of boundedness of every G_α are strictly less than λ, then the weight of G does not exceed 2^λ. We show that the hypotheses in the last result could be weakened, but only if the length of the chain has cofinality $\neq \lambda^+$.

It is well known that every topological group G satisfies $|G| \leq 2^{ib(G)\cdot\psi(G)}$ (see Theorem 4.6 in [11]). In Section 3 we prove an "increasing chain" version of this result for countably compact groups and construct an example to show that this generalisation is not valid in general, not even for pseudocompact groups.

The symbols χ, ψ, d, nw and w will denote, as usual, the character, pseudocharacter, density, network weight and weight of a space. We will write

$H \leqslant G$ when H is a subgroup of G and $H \lhd G$ when H is a normal subgroup of G. The cardinality of the continuum will be denoted by $\mathfrak{c}$, that is, $\mathfrak{c} = 2^{\aleph_0}$.

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2. Index of boundedness

Let τ be any infinite cardinal. A topological group is said to be *τ-bounded* if for every neighborhood U of the identity in G there exists a subset $K \subseteq G$ with $|K| \leq \tau$ such that $K \cdot U = G$. The *index of boundedness* of a topological group G is defined as the least cardinal τ such that G is τ-bounded. It is easy to see that every subgroup of a τ-bounded group is τ-bounded and that homomorphic images of τ-bounded groups are τ-bounded (see [4]). The index of boundedness of a topological group G will be denoted by $ib(G)$. Before proving the main results of this section, we present three simple facts about this cardinal function. The first of them strenghtens a little a result by Guran ([4]).

Proposition 2.1. *If H is a dense subgroup of G, then $ib(H) = ib(G)$.*

Proof. It is clear that $ib(H) \leq ib(G)$, so it suffices to show that, for any τ, if H is τ-bounded, so is G. Let U be any open neighborhood of the identity e in G and choose an open symmetric neighborhood V of e in G such that $V^2 \subseteq U$. Put $W = V \cap H$. Since H is τ-bounded, there exists a subset $K \subseteq H$ such that $|K| \leq \tau$ and $K \cdot W = H$. We claim that $K \cdot U = G$. Indeed, if $x \in G$, then $x \in \overline{H}$ and hence $xV \cap H \neq \varnothing$. So there exists $k \in K$ such that $xV \cap kW \neq \varnothing$. Therefore $x \in kW \cdot V^{-1} \subseteq kV^2 \subseteq kU$ and we have that $G = K \cdot U$. □

Proposition 2.2. *If $N \lhd G$, then $ib(G) = ib(N) \cdot ib(G/N)$*

Proof. Note that $ib(N) \leq ib(G)$ and $ib(G/N) \leq ib(G)$. Let τ be any infinite cardinal and suppose that both N and G/N are τ-bounded. Given an open neighborhood U of the identity e in G, choose an open neighborhhod V of e such that $V^2 \subseteq U$ and put $W = V \cap N$. Then there exist sets $K \subseteq N$ and $L \subseteq G$, both of cardinality at most τ such that $N = K \cdot W$ and $G/N = L^* \cdot V^*$ where, for any $A \subseteq G$, $A^* = \{Nx : x \in A\}$. We claim that $G = L \cdot K \cdot U$. In order to see this, observe that given any $g \in G$, there exist $l \in L$ and $v \in V$ such that $Ng = Nl \cdot Nv$. Therefore $g \in Nlv = lNv \subseteq L \cdot N \cdot V = L \cdot K \cdot W \cdot V \subseteq L \cdot K \cdot V^2 \subseteq L \cdot K \cdot U$. Since $|L \cdot K| \leq \tau$, we conclude that $ib(G) \leq \tau$. □

It has recently come to the author's notice that a similar result to 2.3 has been independently obtained in [7].

Proposition 2.3. *If G is a topological group with $ib(G) = \tau$ and $\lambda < \tau$ is an infinite cardinal, then there exists a subgroup $H \leqslant G$ such that $|H| = ib(H) = \lambda$.*

Proof. Consider any infinite $\lambda < \tau$. Then, since $ib(G) > \lambda$, there exists a neighborhood U of the identity in G such that $K \cdot U \neq G$ for every $K \subseteq G$

such that $|K| \leq \lambda$. We can therefore define, by recursion, a sequence of points of G, $\{x_\alpha : \alpha < \lambda\}$, in such a way that, $x_\alpha \notin x_\beta \cdot U$ if $\beta < \alpha < \lambda$. Let H be the subgroup of G generated by $\{x_\alpha : \alpha < \lambda\}$. Then $|H| = \lambda$ and so $ib(H) \leq \lambda$. Suppose $ib(H) = \mu < \lambda$ and let V be an open symmetric neighborhood of the identity in G such that $V^2 \subseteq U$. Then there exists $F \subseteq H$ with $|F| \leq \mu$ and such that $H = F \cdot (V \cap H)$. Hence, for every $\alpha < \lambda$, there exist $g_\alpha \in F$ and $v_\alpha \in V$ such that $x_\alpha = g_\alpha \cdot v_\alpha$. Since $|F| < \lambda$, we can find $\beta < \alpha < \lambda$ such that $g_\beta = g_\alpha = g$. Therefore $x_\beta = g \cdot v_\beta$ and $x_\alpha = g \cdot v_\alpha$. This implies that $g = x_\beta \cdot v_\beta^{-1}$ and, consequently,

$$x_\alpha = x_\beta \cdot v_\beta^{-1} \cdot v_\alpha \in x_\beta \cdot V^{-1} \cdot V \subseteq x_\beta \cdot V^2 \subseteq x_\beta \cdot U.$$

This contradicts the manner in which the sequence $\{x_\alpha : \alpha < \lambda\}$ was defined. It follows that $ib(H) = |H| = \lambda$. □

We now consider a topological group G which can be written as the union of an increasing chain of subgroups. The next two results give bounds for the index of boundedness of G in terms of the index of boundedness of the subgroups in the chain.

Lemma 2.4. *Let G be a topological group, $G = \bigcup\{G_\alpha : \alpha \in \kappa\}$, where $G_\alpha \leqslant G_\beta < G$ if $\alpha \leq \beta < \kappa$ and $ib(G_\alpha) \leq \lambda$ for every $\alpha \in \kappa$. Then $ib(G) \leq \lambda \cdot \kappa$.*

Proof. Let U be any neighborhood of the identity of G and, for every $\alpha \in \kappa$, let $K_\alpha \subseteq G_\alpha$ be such that $|K_\alpha| \leq \lambda$ and $K_\alpha \cdot (U \cap G_\alpha) = G_\alpha$. Then, clearly $K = \bigcup\{K_\alpha : \alpha \in \kappa\}$ has cardinality at most $\lambda \cdot \kappa$ and $G = K \cdot U$. □

Theorem 2.5. *If G is a topological group, $G = \bigcup\{G_\alpha : \alpha \in \kappa\}$, where $G_\alpha \leqslant G_\beta < G$ if $\alpha \leq \beta < \kappa$ and $ib(G_\alpha) < \lambda$ for every $\alpha \in \kappa$, then $ib(G) \leq \lambda$.*

Proof. We can clearly assume, without loss of generality, that κ is a regular cardinal. If $\kappa \leq \lambda$ then, by Lemma 2.4, we have $ib(G) \leq \lambda \cdot \kappa = \lambda$. Let us suppose, therefore, that $\kappa > \lambda$. If $ib(G) > \lambda$, then take a neighbourhood U of the identity in G and a sequence of points of G $\{x_\gamma : \gamma \in \lambda\}$ such that $x_\gamma \notin x_\beta \cdot U$ if $\beta < \gamma < \lambda$. As κ is regular and $\lambda < \kappa$, there is $\alpha < \kappa$ such that $\{x_\gamma : \gamma \in \lambda\} \subseteq G_\alpha$. Then, with an argument analogous to the one used in the proof of Proposition 2.3, we conclude that G_α must fail to satisfy $ib(G_\alpha) < \lambda$. This contradiction proves that $ib(G) \leq \lambda$. □

It is well known that every topological group G satisfies $w(G) = ib(G) \cdot \chi(G)$ (see 4.1 in [11]). The next two results establish bounds for the weight of a group in terms of the character and index of boundedness og the subgroups in the chain.

Theorem 2.6. *Let G be a topological group, $G = \bigcup\{G_\alpha : \alpha \in \kappa\}$, where $G_\alpha \leqslant G_\beta < G$ if $\alpha \leq \beta < \kappa$. If $ib(G_\alpha) \cdot \chi(G_\alpha) < \lambda$ for every $\alpha < \kappa$, then $w(G) \leq 2^\lambda$.*

Proof. We know that $nw(G_\alpha) \leq w(G_\alpha) = ib(G_\alpha) \cdot \chi(G_\alpha) < \lambda$ for every $\alpha \in \kappa$. Since every topological group is Hausdorff, we can apply Corollary 1.1 in [10] to conclude that $w(G) \leq 2^\lambda$. □

Theorem 2.7. *Let G be a topological group and suppose that $G = \bigcup\{G_\alpha : \alpha \in \kappa\}$, where $G_\alpha \leqslant G_\beta < G$ if $\alpha \leq \beta < \kappa$. If $ib(G_\alpha) \cdot \chi(G_\alpha) \leq \lambda$ for every $\alpha < \kappa$ and $cf(\kappa) \neq \lambda^+$, then $w(G) \leq 2^\lambda$.*

Proof. We can clearly assume, without loss of generality, that κ is regular and $\kappa \neq \lambda^+$. If $\kappa \leq \lambda$, then $nw(G_\alpha) \leq w(G_\alpha) = ib(G_\alpha) \cdot \chi(G_\alpha) < \lambda$ for every $\alpha < \kappa$, so that, by 6.2 in [6], $nw(G) \leq \kappa \cdot \lambda = \lambda$. And since every topological group is regular, we can apply 2.7 of [6] to conclude that $w(G) \leq 2^{d(G)} \leq 2^{nw(G)} \leq 2^\lambda$. If $\kappa > \lambda^+$, since $w(G_\alpha) < \lambda^+$ for every $\alpha \in \kappa$, and κ is a regular cardinal greater than λ^+, any subspace H of G of cardinality at most λ^+ is contained in some G_α and has, therefore, weight $< \lambda^+$. Applying Theorem 1.1 of [10], we conclude that $w(G) < \lambda^+ \leq 2^\lambda$. □

The following example shows that the hypothesis $\kappa \neq \lambda^+$ in the previous theorem is essential. Let us recall that if $\mathcal{U}$ is an ultrafilter, the *character* of $\mathcal{U}$ is the minimal cardinality of a base of $\mathcal{U}$. The character of an ultrafilter $\mathcal{U}$ is denoted $\chi(\mathcal{U})$. By a result of K. Kunen (see [8]), for every infinite cardinal κ there exists a non-principal uniform ultrafilter on κ with character 2^κ.

Example 2.8. Let $\mathcal{U}$ be a non-principal ultrafilter on ω_1 with character 2^{ω_1} and such that every element of $\mathcal{U}$ has cardinality ω_1. Let G be the set of all finite subsets of ω_1. If we let $A \triangle B = (A \setminus B) \cup (B \setminus A)$ for any $A, B \in G$, then $(G, \triangle)$ is a boolean group with the empty set, $\varnothing$, as its identity. Now, for every $A \in G$ and every $F \in \mathcal{U}$, put $U_F^A = \{A \triangle S : S \subseteq F, |S| < \omega\}$. Then the family $\mathcal{B} = \{U_F^A : A \in G, F \in \mathcal{U}\}$ is a base for a Hausdorff group topology τ on G, (see [12]). So $(G, \triangle, \tau)$ is a topological group of cardinality ω_1 and can therefore be expressed as the union of an increasing chain of countable subgroups, $G = \bigcup\{G_\alpha : \alpha \in \omega_1\}$. We claim that, although $w(G) = 2^{\omega_1}$, $ib(G_\alpha) \cdot \chi(G_\alpha) \leq \omega$ for every $\alpha \in \omega_1$. Indeed, $ib(G_\alpha) \leq \omega$ for every $\alpha \in \omega_1$ because $|G_\alpha| \leq \omega$ for every $\alpha < \omega_1$. To see that $\chi(G_\alpha) \leq \omega$ for every $\alpha \in \omega_1$, let us consider the identity of G, the empty set $\varnothing$. For every set S, $\varnothing \triangle S = S$, so, given any $F \in \mathcal{U}$, $U_F^\varnothing = \{S \subset F : |S| < \omega\}$. Note that for any $\alpha \in \omega_1$, G_α is a countable family of finite subsets of ω_1 and hence $\bigcup G_\alpha$ is countable. Take $F \in \mathcal{U}$ such that $F \cap \bigcup G_\alpha = \varnothing$. Then $U_F^\varnothing \cap G_\alpha = \{\varnothing\}$, since otherwise we would have a non-empty $S \subseteq F$ such that $S \in G_\alpha$, contradicting our choice of F. This implies that each subgroup G_α of G is discrete, whence $\chi(G_\alpha) \leq \omega$. Finally we prove that $w(G) = \chi(\mathcal{U}) = 2^{\omega_1}$. In order to see this, consider any local base for the identity of G, $\mathcal{B} = \{U_\alpha : \alpha \in \kappa\}$ for G, where, for each $\alpha \in \kappa$, $U_\alpha = U_{F_\alpha}^\varnothing$ for some $F_\alpha \in \mathcal{U}$. Let $F \in \mathcal{U}$. If $S \subseteq F$ is finite, then $S \in U_F^\varnothing$, so there is $\alpha \in \kappa$ such that $S \in U_\alpha \subseteq U_F^\varnothing$. Since $U_\alpha \subseteq U_F^\varnothing$, for any finite $T \subseteq F_\alpha$, $T = \varnothing \triangle T \subseteq F$. Hence $F_\alpha \subseteq F$, and we conclude that $\kappa \geq \chi(\mathcal{U}) = 2^{\omega_1}$. In addition, $w(G) \leq |G| \cdot \chi(G) = 2^{\omega_1}$, whence $w(G) = 2^{\omega_1}$.

3. Bounds on the cardinality of groups

It is well known that every topological group G satisfies $|G| \leq 2^{ib(G) \cdot \psi(G)}$ (see Theorem 4.6 of [11]). In this section we prove an "increasing chain" version of this result when the group G is countably compact and we give an example to show that the result is not valid in general, not even for pseudocompact groups.

Theorem 3.1. *If G is a countably compact topological group, $G = \bigcup\{G_\alpha : \alpha \in \kappa\}$, where $G_\alpha \leqslant G_\beta < G$, if $\alpha \leq \beta < \kappa$ and $\psi(G_\alpha) \leq \omega$ for every $\alpha < \kappa$, then $|G| \leq 2^\omega$.*

Proof. We may clearly assume that κ is regular. Since G is countably compact, it is totally bounded (see [1]), and so, for every $\alpha \in \kappa$, we have that $ib(G_\alpha) \leq \omega$. Therefore $|G_\alpha| \leq 2^{ib(G_\alpha)\cdot\psi(G_\alpha)} \leq 2^\omega$ for every $\alpha \in \kappa$ (see Theorem 4.6 in [11]). If $\kappa \leq 2^\omega$, then $|G| \leq \kappa \cdot 2^\omega \leq 2^\omega$.

If, on the other hand, $\kappa > 2^\omega$, consider an arbitrary $\alpha \in \kappa$. Then, by a result of Comfort and Sacks (see [2]), there exists a countably compact group H_α such that $G_\alpha \leqslant H_\alpha \leqslant G$ and $|H_\alpha| = |G_\alpha| \leq 2^\omega$. Now, since κ is regular and $\kappa > 2^\omega$, there is $\alpha' \in \kappa$ such that $H_\alpha \subseteq G_{\alpha'}$ and therefore $\psi(H_\alpha) \leq \psi(G_{\alpha'}) \leq \omega$. We have shown that for every $\alpha < \kappa$ there exists a countably compact group H_α such that $G_\alpha \leqslant H_\alpha \leqslant G_{\alpha'}$ for some $\alpha' > \alpha$. We shall now define a function $\nu : \kappa \longrightarrow \kappa$ in such a way that for every $\alpha < \kappa$, $G_\alpha \leqslant H_{\nu(\alpha)}$ and $H_{\nu(\alpha)} \leqslant H_{\nu(\beta)}$ if $\alpha \leq \beta$. Let $\nu(0) = 0$. Pick any $\alpha < \kappa$ and suppose that $\nu(\beta)$ has been defined for every $\beta < \alpha$. Then, for each $\beta < \alpha$, there exists $\xi_\beta < \kappa$ such that $H_{\nu(\beta)} \leqslant G_{\xi_\beta}$. The family $\mathcal{A} = \{G_{\xi(\beta)} : \beta < \alpha\} \cup \{G_\alpha\}$ has cardinality strictly less than κ, which is regular. We can therefore choose $\nu(\alpha) < \kappa$ such that $\xi_\beta \leq \nu(\alpha)$ if $\beta < \alpha$ and $\alpha \leq \nu(\alpha)$. Clearly $G_\alpha \leqslant G_{\nu(\alpha)} \leqslant H_{\nu(\alpha)}$ and $H_{\nu(\alpha)} \leqslant H_{\nu(\alpha)}$ if $\beta < \alpha$. Hence $G = \bigcup\{H_{\nu(\alpha)} : \alpha < \kappa\}$.

So we have the countably compact group G expressed as the union of an increasing chain of subgroups $\{H_{\nu(\alpha)} : \alpha \in \kappa\}$, each countably compact and with countable pseudocharacter. But these two last facts imply that each $H_{\nu(\alpha)}$ has countable character as well. Applying Proposition 4.1 in [11] we get

$$w(H_{\nu(\alpha)}) = ib(H_{\nu(\alpha)}) \cdot \chi(H_{\nu(\alpha)}) \leq \omega$$

for every $\alpha \in \kappa$. Given that any $X \subseteq G$ with cardinality at most ω_1 is contained in some H_α and has, therefore, countable weight, we conclude, applying Theorem 1 in [9], that $w(G) \leq \omega$. Thus, $|G| \leq 2^\omega$. □

Note that the above result is not valid in general for countably compact spaces. In order to see this, consider any cardinal $\kappa > 2^\omega$ and let X be the set of all ordinals in κ with countable cofinality. Then X with the order topology is a countably compact space with countable pseudocharacter, but $|X| > 2^\omega$. It is not valid for pseudocompact groups either, in the next example we construct a pseudocompact topological group G that can be expressed as the union of an increasing chain of subgroups (which will be totally bounded, being subgroups of a totally bounded group), each of countable pseudocharacter, and such that $|G| = \mathfrak{c}^+$, where $\mathfrak{c}$ is the cardinality of the continuum. Recall that a group is called *boolean* if all its elements have order 2.

Example 3.2. Let D be the group $\{0,1\}$ with the discrete topology and put $H=(D^\omega)^{\mathfrak{c}^+}$. We will define a dense pseudocompact subgroup $G \leqslant H$. Let Σ be any boolean group of cardinality $\mathfrak{c}^+$ and let us write Σ as follows:

$$\Sigma = \{x_\alpha : \alpha < \mathfrak{c}^+\}.$$

For each set $S \subseteq \mathfrak{c}^+$, let $H_S = (D^\omega)^S$ and put $\mathcal{A} = \bigcup\{H_S : S \subseteq \mathfrak{c}^+, |S| \leq \omega\}$. Note that if S is a countable subset of $\mathfrak{c}^+$, then $|H_S| \leq (2^\omega)^\omega = \mathfrak{c}$, so $|\mathcal{A}| \leq \mathfrak{c}\cdot(\mathfrak{c}^+)^\omega = \mathfrak{c}^+$. We can therefore write $\mathcal{A}$ in the following way:

$$\mathcal{A} = \{q_\alpha : \alpha < \mathfrak{c}^+\}.$$

We are going to define recursively a family $\{\widetilde{f}_\xi : \xi < \mathfrak{c}^+\}$ of group homomorphisms from Σ to D^ω. In order to do this, we will first define, for each $\alpha < \mathfrak{c}^+$, a subset $P_\alpha \subseteq \mathfrak{c}^+$ and a family of homomorphisms $\{f_{\xi,\alpha} : X_{\xi,\alpha} \longrightarrow D^\omega\}_{\xi\in P_\alpha}$ in such a way that the following conditions are satisfied:

(i) P_α is a non-empty subset of $\mathfrak{c}^+$ and $|P_\alpha| \leq \mathfrak{c}$;
(ii) $X_{\xi,\alpha} < \Sigma$ and $|X_{\xi,\alpha}| \leq \mathfrak{c}$ for every $\xi \in P_\alpha$;
(iii) $P_\beta \subseteq P_\alpha$ if $\beta \leq \alpha$;
(iv) for every $\xi \in P_\beta$, $X_{\xi,\beta} \subseteq X_{\xi,\alpha}$ if $\beta \leq \alpha$;
(v) for every $\xi \in P_\beta$, $f_{\xi,\alpha} \restriction X_{\xi,\beta} = f_{\xi,\beta}$ if $\beta \leq \alpha$.

Fix $\alpha < \mathfrak{c}^+$ and suppose P_β and $\{f_{\xi,\beta} : X_{\xi,\beta} \longrightarrow D^\omega\}_{\xi\in P_\beta}$ defined for every $\beta < \alpha$. If α is a limit ordinal, put $P_\alpha = \bigcup\{P_\beta : \beta < \alpha\}$ and, for every $\xi \in P_\alpha$,

$$X_{\xi,\alpha} = \bigcup\{X_{\xi,\beta} : \beta < \alpha,\ \xi \in P_\beta\}$$

and

$$f_{\xi,\alpha} = \bigcup\{f_{\xi,\beta} : \beta < \alpha,\ \xi \in P_\beta\}.$$

Conditions (i)–(v) are clearly satisfied. Let us assume now that α is a successor ordinal, say $\alpha = \beta + 1$. The construction consists of several stages. In the first stage we make sure that $x_\beta \in X_{\xi,\alpha}$, so that every element of Σ will be in the domain of some $f_{\xi,\alpha}$. Stages 2 and 3 ensure that the resulting subgroup of H is pseudocompact. Finally, with Stage 4 we guarantee that the resulting group G can be expressed as the union of an increasing chain of subgroups of countable pseudocharacter. The known fact that if X is a subgroup of a boolean group Y, then every homomorphism from X to D can be extended to Y will be used extensively.

Stage 1. Consider an arbitrary $\xi \in P_\beta$. Put $Y_{\xi,\alpha} = \langle X_{\xi,\beta} \cup \{x_\beta\}\rangle < \Sigma$. By inductive hypothesis, the homomorphism $f_{\xi,\beta} : X_{\xi,\beta} \longrightarrow D^\omega$ has already been defined, so, for every $n \in \omega$, $\pi_n \circ f_{\xi,\beta} : X_{\xi,\beta} \longrightarrow D$ is a homomorphism, where $\pi_n : D^\omega \longrightarrow D$ is the natural projection. For each $n \in \omega$, let $h_n : Y_{\xi,\alpha} \longrightarrow D$ be a homomorphic extension of $\pi_n \circ f_{\xi,\beta}$. Then the diagonal product

$$g_{\xi,\alpha} = \triangle_{n\in\omega} h_n : Y_{\xi,\alpha} \longrightarrow D^\omega$$

is an extension of $f_{\xi,\beta}$ to $Y_{\xi,\alpha}$, that is, $g_{\xi,\alpha} \restriction X_{\xi,\beta} = f_{\xi,\beta}$. We have defined, for every $\xi \in P_\beta$, a subgroup $Y_{\xi,\alpha} < \Sigma$ and a homomorphism $g_{\xi,\alpha}$ in such a way that $X_{\xi,\beta} \leqslant Y_{\xi,\alpha} < \Sigma$; $|Y_{\xi,\alpha}| \leq \mathfrak{c}$; $x_\beta \in Y_{\xi,\alpha}$; $g_{\xi,\alpha} : Y_{\xi,\alpha} \longrightarrow D^\omega$ and $g_{\xi,\alpha} \restriction X_{\xi,\beta} = f_{\xi,\beta}$.

Stage 2. Now let us consider $q_\beta \in \mathcal{A}$. Then, by definition of $\mathcal{A}$, there exists a countable subset S of $\mathfrak{c}^+$ such that $q_\beta \in H_S$, that is, $q_\beta \in (D^\omega)^S$. Let K be the subgroup of Σ generated by $\bigcup\{Y_{\xi,\alpha} : \xi \in P_\beta \cap S\}$. Then, as $|P_\beta| \leq \mathfrak{c}$ and for each $\xi \in P_\beta$, $|Y_{\xi,\alpha}| \leq \mathfrak{c}$, $|K| \leq \mathfrak{c}$ and we can therefore find $x \in \Sigma \setminus K$. For each $\xi \in P_\beta \cap S$, put

$$Z_{\xi,\alpha} = \langle Y_{\xi,\alpha} \cup \{x\}\rangle = Y_{\xi,\alpha} + \{0, x\} < \Sigma.$$

Since $x \notin K$, we can define, for each n, a homomorphism $j_n : Z_{\xi,\alpha} \longrightarrow D$ such that $j_n(x) = q_\beta(\xi)(n)$ and $j_n \restriction Y_{\xi,\alpha} = \pi_n \circ g_{\xi,\alpha}$. Then, defining $h_{\xi,\alpha}$ as the diagonal product of the family $\{j_n : n \in \omega\}$ we get, for every $\xi \in P_\beta \cap S$, a subgroup $Z_{\xi,\alpha}$ of Σ and a homomorphism $h_{\xi,\alpha}$ that satisfy the following conditions: $Y_{\xi,\alpha} \lneqq Z_{\xi,\alpha} < \Sigma$; $|Z_{\xi,\alpha}| \leq \mathfrak{c}$; $h_{\xi,\alpha} : Z_{\xi,\alpha} \longrightarrow D^\omega$; $h_{\xi,\alpha} \restriction Y_{\xi,\alpha} = g_{\xi,\alpha}$ and

$$h_{\xi,\alpha}(x) = q_\beta(\xi). \tag{3.1}$$

Stage 3. If, on the other hand, $\xi \in S \setminus P_\beta$, let $V_{\xi,\alpha}$ be the subgroup of Σ generated by x and define $j(x) = q_\beta(\xi)$. Then j can be extended to a homomorphism $l_{\xi,\alpha} : V_{\xi,\alpha} \longrightarrow D^\omega$, so we have, in this case: $V_{\xi,\alpha} < \Sigma$; $|V_{\xi,\alpha}| \leq \mathfrak{c}$; $l_{\xi,\alpha} : V_{\xi,\alpha} \longrightarrow D^\omega$ and

$$l_{\xi,\alpha}(x) = q_\beta(\xi). \tag{3.2}$$

Stage 4. Choose now $\xi_\alpha \in \mathfrak{c}^+ \setminus (P_\beta \cup S)$, this is possible because $|P_\beta| \leq \mathfrak{c}$ and $|S| \leq \aleph_0$. Consider

$$X_\alpha = \langle\{x_\nu : \nu < \alpha\}\rangle < \Sigma.$$

If $\alpha \geq \mathfrak{c}$, put $W_{\xi_\alpha,\alpha} = X_\alpha$, otherwise put $W_{\xi_\alpha,\alpha} = X_\mathfrak{c} = \langle\{x_\nu : \nu < \mathfrak{c}\}\rangle$. Then $W_{\xi_\alpha,\alpha}$ is a boolean group of cardinality $\mathfrak{c}$ and can therefore be represented as $\bigoplus_{\alpha\in\mathfrak{c}} D_\alpha$, where each $D_\alpha = D$. So $W_{\xi_\alpha,\alpha}$ is algebraically isomorphic to D^ω. Let $k_{\xi_\alpha,\alpha} : W_{\xi_\alpha,\alpha} \longrightarrow D^\omega$ be an isomorphism. Then we have the following situation: $W_{\xi_\alpha,\alpha} < \Sigma$; $|W_{\xi_\alpha,\alpha}| = \mathfrak{c}$; $X_\alpha \subseteq W_{\xi_\alpha,\alpha}$ and

$$k_{\xi_\alpha,\alpha} : W_{\xi_\alpha,\alpha} \longrightarrow D^\omega \text{ is an isomorphism.} \tag{3.3}$$

Finally we are in a position to define P_α and $f_{\xi,\alpha}$ for every $\xi \in P_\alpha$. Put $P_\alpha = P_\beta \cup S \cup \{\xi_\alpha\}$ and, for every $\xi \in P_\alpha$, define $X_{\xi,\alpha}$ and $f_{\xi,\alpha}$ as follows:

(a) If $\xi \in P_\beta \setminus S$, then $X_{\xi,\alpha} = Y_{\xi,\alpha}$ and $f_{\xi,\alpha} = g_{\xi,\alpha}$ (see Stage 1);
(b) if $\xi \in P_\beta \cap S$, then $X_{\xi,\alpha} = Z_{\xi,\alpha}$ and $f_{\xi,\alpha} = h_{\xi,\alpha}$ (see Stage 2);
(c) if $\xi \in S \setminus P_\beta$, then $X_{\xi,\alpha} = V_{\xi,\alpha}$ and $f_{\xi,\alpha} = l_{\xi,\alpha}$ (see Stage 3);
(d) if $\xi = \xi_\alpha$, then $X_{\xi,\alpha} = W_{\xi_\alpha,\alpha}$ and $f_{\xi,\alpha} = k_{\xi_\alpha,\alpha}$ (see Stage 4).

It is easy to see that conditions (i)–(v) are satisfied. Note also that, for every countable $S \subseteq \mathfrak{c}^+$, every element $q_\beta \in H_S$ was considered in Stages 2 and 3 of the construction, therefore $\bigcup\{P_\alpha : \alpha < \mathfrak{c}^+\} = \mathfrak{c}^+$. For each $\xi < \mathfrak{c}^+$ put $X^\xi = \bigcup\{X_{\xi,\,\alpha} : \alpha < \mathfrak{c}^+, \xi \in P_\alpha\}$. Then $X^\xi < \Sigma$ and, since $X_{\xi,\beta} \subseteq X_{\xi,\alpha}$ and $f_{\xi,\beta} \subseteq f_{\xi,\alpha}$ for $\beta \leq \alpha$, $f_\xi = \bigcup\{f_{\xi,\alpha} : \alpha < \mathfrak{c}^+\} : X^\xi \longrightarrow D^\omega$ is a homomorphism.

It follows that for $n \in \omega$, the homomorphism $\pi_n \circ f_\xi : X^\xi \longrightarrow D$ can be homomorphically extended to $\widetilde{f_{\xi,n}} : \Sigma \longrightarrow D$. By putting

$$\widetilde{f}_\xi = \triangle_{n\in\omega}\widetilde{f_{\xi,n}} : \Sigma \longrightarrow D^\omega$$

we have, for every $\xi \in \mathfrak{c}^+$, a homomorphism $\widetilde{f}_\xi$ from Σ to D^ω with $\widetilde{f}_\xi \upharpoonright X^\xi = f_\xi$. Let ϕ be the diagonal product of the $\widetilde{f}_\xi$'s:

$$\phi = \triangle_{\xi\in\mathfrak{c}^+}\widetilde{f}_\xi : \Sigma \longrightarrow (D^\omega)^{\mathfrak{c}^+}$$

and define $G = \phi(\Sigma) < H$. We now show that G is a dense pseudocompact subgroup of H of cardinality $\mathfrak{c}^+$ which can be written as the union of an increasing chain of subgroups, each with countable pseudocharacter. H being compact, all its subgroups are totally bounded. To see that G is a dense pseudocompact subgroup of H it suffices to show that for every countable subset $S \subseteq \mathfrak{c}^+$, $\pi_S(G) = (D^\omega)^S$, where $\pi_S : (D^\omega)^{\omega_1} \longrightarrow (D^\omega)^S$ is the projection. So, let S be any countable subset of $\mathfrak{c}^+$ and take $y \in (D^\omega)^S$, then $y \in \mathcal{A}$ and so $y = q_\beta$ for some $\beta < \mathfrak{c}^+$. At step $\beta + 1$ of our construction we defined $x \in X_{\xi,\beta+1}$ in such a way that $f_{\xi,\beta+1}(x) = q_\beta(\xi) = y(\xi)$ for all $\xi \in S$ (see (3.1), (3.2), (b) and (c)). So $x \in X^\xi$ and $\widetilde{f}_\xi(x) = f_\xi(x) = f_{\xi,\beta+1}(x) = y(\xi)$ for every $\xi \in S$. We conclude that $\pi_S(\phi(x)) = y$ and therefore $\pi_S(G) = (D^\omega)^S$. To see that $|G| = \mathfrak{c}^+$ it is enough to observe that given any two different elements of Σ, say x_α and x_β, with $\alpha > \beta$, at step $\alpha + 1$ of our construction we defined an ordinal $\xi_{\alpha+1}$ such that $f_{\xi\alpha+1,\alpha+1}(x_\beta) \neq f_{\xi\alpha+1,\alpha+1}(x_\alpha)$ (see (3.3) and (d)), and therefore $f_{\xi\alpha+1}(x_\beta) \neq f_{\xi\alpha+1}(x_\alpha)$, whence ϕ is injective. Finally, for every $\alpha < \mathfrak{c}^+$ let $X_\alpha = \langle\{x_\nu : \nu < \alpha\}\rangle$ and $G_\alpha = \phi(X_\alpha)$. Then $\Sigma = \bigcup\{X_\alpha : \alpha < \mathfrak{c}^+\}$ and $G = \bigcup\{G_{\alpha+1} : \alpha < \mathfrak{c}^+\}$. It remains to see that all the G_α's have countable pseudocharacter. Pick any element $a \in G_{\alpha+1}$, so there exists $x \in X_{\alpha+1}$ such that $\phi(x) = a$. Put $\beta = \alpha + 1$. Then, at step β of our construction we defined an ordinal ξ_β such that $f_{\xi_\beta,\beta} : X_\beta \longrightarrow D^\omega$ is an injective homomorphism (see (3.3) and (d)). So, by definition of f_{ξ_β}, $f_{\xi_\beta} \upharpoonright X_\beta$ is injective. Let $\pi_{\xi_\beta} : (D^\omega)^{\mathfrak{c}^+} \longrightarrow D^\omega$ be the projection. Then, since $\pi_{\xi_\beta}(a) = f_{\xi_\beta}(x)$, the restriction $\pi_{\xi_\beta} \upharpoonright G_\beta : G_\beta \longrightarrow D^\omega$ is injective. Therefore $\psi(G_\beta) \leq \psi(D^\omega) = \omega$.

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