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# The chainable continua are the spaces approximated by finite COTS

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Abstract. We show that the chainable continua (also called *snake-like* or *arc-like*) continua (see [7]), are precisely the Hausdorff reflections of inverse limits of sequences of finite COTS under maps which are continuous and are separating: whenever $C, D \subseteq Y$ are closed and disjoint, then $f^{-1}[C]$ and $f^{-1}[D]$ are contained in disjoint open sets. The finite connected ordered topological spaces (COTS) defined in [4] have a very simple structure and are used as finite approximations to intervals of the real line, in digital topology (for computer image processing) (see [8]).
We also obtain similar characterizations of circle-like continua and generalized Knaster continua.

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## 1. Introduction

It is known that every compact Hausdorff space is the Hausdorff reflection of an inverse limit of finite $T_0$-spaces (see [2], or see [5] for a treatment in our notation). The finite spaces used for reconstructing $X$ in this way are quotients of $X$ and may indeed be rather complicated spaces. On the other hand, the connected ordered topological spaces (COTS) defined in [4] have a very simple structure and are used in digital topology as approximations to intervals of the real line, (see [8]).

Part of the motivation for the work in [5] is the recently-developed theory of *skew compact* spaces; these are $T_0$ spaces $(X, \tau)$ in which there is a second

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topology $\tau^*$ such that $\tau \vee \tau^*$ is compact, and for each $x, y \in X$:

$$x \in \mathrm{cl}_\tau\{y\} \Leftrightarrow y \in \mathrm{cl}_{\tau^*}\{x\},$$

and

if $x \notin \mathrm{cl}_\tau\{y\}$ then there are disjoint $T, U$ so that $x \in T \in \tau$ and $y \in U \in \tau^*$.

Skew compact spaces behave very much like compact $T_2$ spaces; in particular, limits of inverse systems of skew compact spaces, in which each map is continuous, not only between the original topologies, but also between the second topologies, are skew compact. These second topologies are uniquely determined by the first: the $\tau^*$-closed sets are those $\tau$-compact sets $C$ which are *$\tau$-saturated*, that is, if $\mathrm{cl}_\tau\{y\} \cap C \neq \varnothing$ then $y \in C$. All finite $T_0$ spaces are skew compact, with the $\tau^*$-closed sets precisely the $\tau$-open sets. As a result, each map which is continuous between the original topologies of finite $T_0$ spaces, is also continuous between these second topologies, so limits of inverse systems of finite $T_0$ spaces and continuous maps are skew compact.

There are many equivalent definitions of a chainable (previously called a *snake-like* or *arc-like*) continuum (see [7]), but the one which interests us here is the characterization of chainable continua as being precisely the inverse limits of sequences of unit intervals and continuous maps. Since unit intervals can be approximated (in the sense of [8]) by COTS, it is natural to ask whether chainable continua can also be approximated in this way. The aim of this paper is to answer this question in the affirmative: chainable continua are precisely the Hausdorff reflections of inverse limits of sequences of finite COTS under maps which are continuous and separating: whenever $C, D \subseteq Y$ are closed and disjoint, then $f^{-1}[C]$ and $f^{-1}[D]$ are contained in disjoint open sets. We also obtain similar characterizations of circle-like continua and generalized Knaster continua.

These results lead to alternate digital representations of continua in these classes. Current methods of computation involving these continua are modifications of linear methods, and often inadequate because of nonlinearity inherently involved with graphics attempting to illustrate complicated continua. It is hoped that better algorithms will result from a consideration of this work, but none are proposed below.

## 2. Approximating chainable continua with COTS

Our aim in this section is to show that every chainable continuum is the Hausdorff reflection of an inverse limit of finite connected ordered topological spaces. Below, we let $\mathbb{I}$ denote the unit interval. Since a chainable continuum $X$ is the inverse limit of a sequence of unit intervals $X = \varprojlim(\mathbb{I}_n, f_n)$, we begin by showing that $X$ is homeomorphic to $Y = \varprojlim(\mathbb{I}_n, g_n)$, where each of the maps $g_n$ is continuous and piecewise linear.

The following lemma is a classical theorem of calculus:

**Lemma 2.1.** *Let $f : \mathbb{I} \to \mathbb{I}$ be a continuous function. Then given $\epsilon > 0$, there is a continuous piecewise-linear function $g : \mathbb{I} \to \mathbb{I}$ such that $|f(x) - g(x)| < \epsilon$ for all $x \in \mathbb{I}$.*

Now suppose that $f$ is a piecewise linear continuous function and let $N = \{0 = a_0, a_1, \ldots, a_{n-1}, a_n = 1\}$ be the points at which $f$ is not differentiable. Let $\epsilon > 0$ and let $\{r_0, r_1, \ldots, r_{n-1}, r_n\}$ be distinct real numbers such that $|f(a_i) - r_i| < \epsilon$ for each $i \in \{0, 1, \ldots, n\}$. By defining $g(a_i) = r_i$ for each $i$ and extending linearly on each interval $(a_i, a_{i+1})$ we obtain a continuous piecewise linear, nowhere locally constant function $g : \mathbb{I} \to \mathbb{I}$ such that $|f(x) - g(x)| < \epsilon$ for each $x \in \mathbb{I}$. Thus each continuous function $f : \mathbb{I} \to \mathbb{I}$ can be uniformly approximated by continuous piecewise-linear nowhere locally constant functions. The following is now an immediate consequence of Theorem 3 of [1]:

**Theorem 2.2.** *If $X = \varprojlim(\mathbb{I}_n, f_n)$ is the inverse limit of a sequence of unit intervals with continuous maps, then there exists a sequence $\{g_n : n \in \omega\}$ of continuous piecewise-linear nowhere locally constant self-maps of $\mathbb{I}$ such that $X = \varprojlim(\mathbb{I}_n, g_n)$.*

We now turn to the problem of representing each chainable continuum as the Hausdorff reflection of an inverse limit of finite COTS.

By a *calculus partition* (or simply *partition*) of $\mathbb{I}$ we mean a finite subset $S$ of $\mathbb{I}$ and without loss of generality, we will assume that each partition contains 0 and 1. Given a partition, $S = \{0 = s_0, \ldots, s_m = 1\}$, we define a COTS $C_S$ by:

$$C_S = S \cup \{(s_{i-1}, s_i) : 1 \le i \le m\}$$

topologized in such a way that the elements of $C_S \setminus S$ are open and the minimal neighbourhood of $s_i \in S$ is of the form $\{(s_{i-1}, s_i), s_i, (s_i, s_{i+1})\}$. Then the space $C_S$ will be called the *COTS associated with the partition $S$* of $\mathbb{I}$; also, $p_S : \mathbb{I} \to C_S$ will denote the natural projection.

**Lemma 2.3.** *Suppose $f : \mathbb{I} \to \mathbb{I}$ is a piecewise-linear continuous nowhere locally constant function, and that $D, R$ are partitions of $\mathbb{I}$ such that $f^{-1}[R] \subseteq D$, and $D$ also contains $0, 1$ and all points at which $f$ is not differentiable. Then there is a unique continuous map $g : C_D \to C_R$ (which we call the $(D$-$R)$-approximation to $f$) such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{f} & \mathbb{I} \\ \downarrow{\scriptstyle p_D} & & \downarrow{\scriptstyle p_R} \\ C_D & \xrightarrow[g]{} & C_R \end{array}$$

*Proof.* Define $g : C_D \to C_R$ as follows:

(1) if $d_i \in D$, then $g(d_i)$ is defined to be the unique element (either a singleton or an open interval) of $C_R$ containing $f(d_i)$, and
(2) $g[(d_{i-1}, d_i)]$ is the unique element of $C_R$ containing $f[(d_{i-1}, d_i)]$.

To see that $g$ is well defined, we need only show that the image under $f$ of each interval $(d_{i-1}, d_i)$ is contained in a unique element of $C_R$. But since $f$ is continuous, $f[(d_{i-1}, d_i)]$ is an interval and since $D \supseteq f^{-1}[R]$, it can contain no element of $R$ and so is contained in a unique element of $C_R$. Of course, for each singleton or interval $p_D(x) \in C_D$, this element has been selected so that $g(p_D(x)) = p_R(f(x))$, in other words, to make the above diagram commute.

Both $C_R$ and $C_D$ are Alexandroff spaces and hence to show that $g$ is continuous at $x \in C_D$, it suffices to show that if $V$ is the minimal open neighbourhood of $g(x) \in C_R$ and $U$ is the minimal open neighbourhood of $x \in C_D$, then $g[U] \subseteq V$. If $\{x\}$ is open in $C_D$, then $U = \{x\}$ and there is nothing to prove. If on the other hand $\{x\}$ is closed, then there are two cases to consider:

(1) $V = \{(r_{i-1}, r_i)\}$ for some $i \leq k$; then $x = d_j$ for some $j \leq m$. But then by the continuity of $f$,

$$f[(d_{j-1}, d_j)] \cap V \neq \varnothing \neq f[(d_j, d_{j+1})] \cap V$$

and hence $f[(d_{j-1}, d_j)] \cup f[(d_j, d_{j+1})] \subseteq V$ so $g[U] \subseteq V$ where $U = \{(d_{j-1}, d_j), d_j, (d_j, d_{j+1})\}$ is the minimal open neighbourhood of $d_j$.

(2) $V = \{(r_{i-1}, r_i), r_i, (r_i, r_{i+1})\}$ is the minimal open neighbourhood of $r_i$ for some $i \leq k$; again, $x = d_j$ for some $j \leq m$ and an argument similar to that of (1) now applies.

□

Suppose next that we are given an inverse sequence $\{f_n : n \in \omega\}$ of continuous piecewise linear nowhere locally constant maps of the unit interval ($\mathbb{I}_n = \mathbb{I}$ for each $n \in \omega$).

$$\ldots \xrightarrow{f_3} \mathbb{I}_3 \xrightarrow{f_2} \mathbb{I}_2 \xrightarrow{f_1} \mathbb{I}_1 \xrightarrow{f_0} \mathbb{I}_0$$

We use Lemma 2.3 to define an inverse sequence of COTS and continuous maps:

$$\ldots \xrightarrow{g_3} C_3 \xrightarrow{g_2} C_2 \xrightarrow{g_1} C_1 \xrightarrow{g_0} C_0$$

First let $Q_k$ denote the finite set of dyadic rationals in $\mathbb{I}$ which are expressible in the form $j/2^k$ for some $j \in \mathbb{N}$ and let $N_k = \{0 = d_0 < \cdots < d_{m_k} = 1\}$ denote the finite set of points at which $f_k$ is not differentiable. Since $f_k$ is one-to-one on each interval $(d_{i-1}, d_i)$, it is a finite-to-one map.

Let $R_0 = Q_0 = \{0, 1\}$ and let $C_0 = C_{R_0}$, the COTS associated with the partition $R_0$. Then $R_1 = N_0 \cup {f_0}^{-1}[R_0] \cup Q_1$ is a finite set (since $f^{-1}[R_0]$ is finite by the previous paragraph), so let $C_1 = C_{R_1}$ and let $g_0$ be the ($R_1$-$R_0$)-approximation of $f_0$ as constructed in Lemma 2.3. Having defined maps $g_0, g_1, \ldots, g_{k-2}$ with COTS domains $C_0, C_1, \ldots, C_{k-1}$ approximating the maps $f_0, f_1, \ldots, f_{k-2}$, we let $R_k = N_{k-1} \cup Q_k \cup {f_k}^{-1}[R_{k-1}]$, $C_k$ be the COTS associated with the partition $R_k$, and $g_{k-1}$ be the ($R_k$-$R_{k-1}$)-approximation to $f_{k-1}$.

In what follows, for simplicity of notation, $p_n$ will denote the natural projection $p_{C_n} : \mathbb{I}_n \to C_n$.

**Theorem 2.4.** *The following diagram commutes:*

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{f_3} & \mathbb{I}_3 & \xrightarrow{f_2} & \mathbb{I}_2 & \xrightarrow{f_1} & \mathbb{I}_1 & \xrightarrow{f_0} & \mathbb{I}_0 \\
 & & \downarrow{\scriptstyle p_3} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} & & \downarrow{\scriptstyle p_0} \\
\cdots & \xrightarrow[g_3]{} & C_3 & \xrightarrow[g_2]{} & C_2 & \xrightarrow[g_1]{} & C_1 & \xrightarrow[p_0]{} & C_0
\end{array}$$

*Proof.* This follows directly from the definition of the $g_k$ and Lemma 2.3. □

Our aim now is to establish the relationship between $X = \varprojlim(\mathbb{I}_n, f_n)$ and $X^* = \varprojlim(C_n, g_n)$.

Throughout the rest of the paper, if $(Y_n, h_n)$ is an inverse sequence, then we let $h_{mn}$ denote the map $h_n \circ h_{n-1} \circ \cdots \circ h_{m-1} : Y_m \to Y_n$ if $m > n$, and denote the identity map on $Y_n$ if $m = n$.

**Theorem 2.5.** *$X$ is the Hausdorff reflection of $X^*$.*

*Proof.* In the proof of Theorem 2.4, [5], the following is shown:Let $(X, \tau)$ be a compact $T_2$-space and let $\mathcal{F}$ a collection of finite sets of open subsets of $X$ which is directed by $\subseteq$, and whose union, $\bigcup \mathcal{F}$, is a base for $\tau$. Then $(X, \tau)$ is the Hausdorff reflection of the inverse limit of the system $\{(X_F, \pi_{GF}) \mid F, G \in \mathcal{F},\ F \subseteq G\}$, where $X_F$ is the partition of $X$ into the nonempty Boolean combinations of elements of $F$, equipped with the quotient topology induced by the natural quotient, and for $F \subseteq G$, $\pi_{GF}$ is the natural quotient from $X_G$ to $X_F$. We use this in the following proof, and in that of Theorem 3.2 below.

Let us denote by $\pi_n$ the projection from $X$ to $\mathbb{I}_n$. By the above, it suffices to show that if we define:

$$\mathcal{F}_k = \{(p_k \circ \pi_k)^{-1}[V_k] :\ V_k \text{ is open in } C_k\}$$

then

$$\mathcal{F} = \{\mathcal{F}_k : k \in \omega\}$$

is directed by $\subseteq$ and $\bigcup \mathcal{F}$ is a base for $X$.

That $\mathcal{F}$ is directed follows immediately from the definition of $\mathcal{F}$ and the fact that if $U_n \in \mathcal{F}_n$, then $g_{mn}^{-1}[U_n] \in \mathcal{F}_m$. So now suppose that $U$ is open in $X$ and $x \in U$. Then there is some $k \in \omega$ and some open set $U_k \subseteq \mathbb{I}_k$ such that $x \in \pi_k^{-1}[U_k] \subseteq U$. Now, since for each $m \in \omega$, the partition $R_m$ used in the construction of the COTS $C_m$ is such that $R_m \supseteq Q_m$, it follows that we can find $m \geq k$ and an open set $V_m \subseteq C_m$ such that $\pi_m(x) \in p_m^{-1}[V_m] \subseteq f_{km}^{-1}[U_k]$ and we are done. □

## 3. Hausdorff reflections of inverse limits of COTS

A (continuous) map $f : X \to Y$ is called *separating* if whenever $C, D \subseteq Y$ are closed and disjoint, then $f^{-1}[C]$ and $f^{-1}[D]$ are contained in disjoint open sets. Of course, every continuous map from a normal space is separating. An inverse sequence of spaces $\{Y_m : m \in \omega\}$ and maps $\{h_n : Y_{n+1} \to Y_n : n \in \omega\}$, is *separating* if for each $n$ there is an $m \geq n$ such that $h_{mn}$ is a separating map. In this section we show that the Hausdorff reflection of the inverse limit of a sequence of finite COTS and separating maps is a chainable continuum.

If $f : X \to Y$ and $g : Y \to Z$ are continuous and either is separating, then their composition is easily seen to be separating; thus in a separating inverse sequence for each $n$ there is an $m \geq n$ such that if $p \geq m$ then $h_{pn}$ is separating.

In an inverse sequence $Y_n$ of finite $T_0$ spaces, it suffices to check that whenever $x, y \in Y_n$ are closed, distinct points, then for some $m \geq n$, $h_{mn}^{-1}[x]$ and $h_{mn}^{-1}[y]$ are contained in disjoint open sets: For any subset $E$ of a topological space, the *saturation* of $E$ is $\mathrm{n}(E) = \bigcap\{T \text{ open} |\ E \subseteq T\}$; in a finite space, this is open, and if $E$ is closed, then $\mathrm{n}(E) = \bigcup\{\, \mathrm{n}(x) \mid x \in E, \{x\} \text{ closed}\}$, so if $F$ is another closed set, and $\mathrm{n}(x) \cap \mathrm{n}(y) = \varnothing$ whenever $x \in E, y \in F$, then $\mathrm{n}(E) \cap \mathrm{n}(F) = \varnothing$.

Further, there are only a finite number of pairs of distinct closed sets, so for each $n \in \omega$, there exists $m \geq n$ such that for each pair of distinct closed sets $C, D \subseteq Y_n$, $h_{mn}^{-1}[C]$ and $h_{mn}^{-1}[D]$ are contained in disjoint open sets. Now, by passing to a subsequence of the inverse spectrum (which will have the same inverse limit as the original sequence), we can assume that each map $h_n$ is separating. In the sequel, we assume that we have an inverse sequence of COTS $Y_n$ and separating maps $f_n : Y_{n+1} \to Y_n$.

**Theorem 3.1.** *The Hausdorff reflection of the limit of a separating inverse sequence of finite $T_0$ spaces and continuous maps is its subspace of closed points. This subspace is a retract of the limit.*

*Proof.* Denote by $F$ the set of closed points of $\varprojlim (X_n, g_n)$, let $\rho_m : \varprojlim X_n \to X_m$ be the projection maps, and let $x, y \in F$ be distinct. Then

$$\begin{aligned}\varnothing &= \mathrm{cl}(x) \cap \mathrm{cl}(y)\\ &= \big(\textstyle\bigcap_{m\in\omega} \rho_m^{-1}[\mathrm{cl}(\rho_m(x))]\big) \cap \big(\textstyle\bigcap_{m\in\omega} \rho_m^{-1}[\mathrm{cl}(\rho_m(y))]\big)\\ &= \textstyle\bigcap_{m\in\omega} \rho_m^{-1}[\mathrm{cl}(\rho_m(x)) \cap \mathrm{cl}(\rho_m(y))],\end{aligned}$$

and so, since $\varprojlim X_n$ is compact and $\omega$ is directed,

$$\mathrm{cl}(\rho_n(x)) \cap \mathrm{cl}(\rho_n(y)) = \varnothing$$

for some $n \in \omega$. By definition of separating inverse sequence, for some $m \geq n$,

$$g_{mn}^{-1}[\mathrm{cl}(\rho_n(x))] \text{ and } g_{mn}^{-1}[\mathrm{cl}(\rho_n(y))]$$

lie in disjoint open sets, $T, U$, so $\rho_m^{-1}[T], \rho_m^{-1}[U]$ are disjoint open sets containing $x, y$ respectively. This shows that the subspace $F$ is Hausdorff.

Each element of $\varprojlim X_n$ has in its closure a closed point, since this space is $T_0$ and (skew)compact (the intersection of a maximal chain of closures of points is a minimal closed set, thus a closed singleton). But if two closed points were in the closure of the same point, then this point would be in any open set containing either, contradicting the existence of disjoint open sets containing them, established in the last paragraph; therefore, each element of $\varprojlim X_n$ has in its closure a *unique* closed point.

Thus a map $\pi : \varprojlim X_n \to F$ is defined by letting $\pi(x)$ be the unique element of $\mathrm{cl}(x) \cap F$. It is continuous: In fact, each closed $C$ is compact in the skew compact $X$ and it is then easy to check that

$$\pi^{-1}[C] = \{x \mid \mathrm{cl}(x) \cap C \neq \varnothing\} = \mathrm{n}(C),$$

the saturation of $C$ (not usually open in this infinite space), is compact. Also here, if $y \in \pi^{-1}[C]$ then $\mathrm{cl}(y) \subseteq \pi^{-1}[C]$, so $\pi^{-1}[C]$ is closed by [6], theorem 3.1.

To complete the proof, it will suffice to show that if $f : \varprojlim X_n \to H$, $H$ Hausdorff, then $f$ factors through $\pi$. But for such a map, $f|F : F \to H$, and for each $x \in \varprojlim X_n$, if $y \in \mathrm{cl}(x) \cap F$ then

$$(f|F) \circ \pi(x) = f(y) \in \mathrm{cl}\{f(x)\} = \{f(x)\},$$

so $(f|F) \circ \pi = f$. $\square$

With slight changes, the definitions and proof can be adapted to show the result for arbitrary (nonsequential) inverse limits of separating spectra.

For the next theorem we will require some new notation. For a nonempty interval $x \subseteq \mathbb{I}$, we let $M(x)$ denote its midpoint. Also recall that if $x \in C$, $C$ a finite COTS, then $x^+$ denotes the immediate successor of $x$ and $x^-$ the immediate predecessor of $x$ if such exist; thus if $x$ is closed, $x^{++}$ and $x^{--}$ are (if they exist) the closed points nearest to $x$.

Finally, if $k$ is a positive integer and $C$ is a COTS with $k$ open points, then $C$ is clearly homeomorphic to one of the following spaces:

$$F(k) = \{\tfrac{i}{k} \mid i = 0, \dots, k\} \cup \{(\tfrac{i}{k}, \tfrac{i+1}{k}) \mid {\scriptstyle i=0,\dots,k-1}\},$$
$$L(k) = \{\tfrac{i}{k} \mid i = 0, \dots, k-1\} \cup \{(\tfrac{i}{k}, \tfrac{i+1}{k}) \mid i = 0, \dots, k-2\} \cup \{(\tfrac{k-1}{k}, 1]\},$$
$$R(k) = \{\tfrac{i}{k} \mid i = 1, \dots, k\} \cup \{(\tfrac{i}{k}, \tfrac{i+1}{k}) \mid i = 1, \dots, k-1\} \cup \{[0, \tfrac{i}{k})\},$$
$$O(k) = \{\tfrac{i}{k} \mid i = 1, \dots, k-1\} \cup \{(\tfrac{i}{k}, \tfrac{i+1}{k}) \mid i = 1, \dots, k-2\} \cup \{[0, \tfrac{i}{k}), (\tfrac{k-1}{k}, 1]\},$$

equipped with the quotient topology from $\mathbb{I}$.

Note that the inverse image under the quotient map $p : \mathbb{I} \to C$, of a closed point $x \in C$ is a single point in $\mathbb{I}$ and to simplify the notation slightly we identify this point with $x$. Also adjacent points are identified with points in $\mathbb{I}$ which are $\frac{1}{k}$ apart from each other.

**Theorem 3.2.** *A space is a chainable continuum if and only if it is the Hausdorff reflection of the limit of an inverse sequence of finite COTS and separating maps.*

*Proof.* The construction discussed in and before Theorem 2.5 and in Theorem 2.4 of [5], in fact yields a separating inverse spectrum. To see this using the notation there, let $A, B$ be disjoint closed sets in $C_n$. Then $\pi_n^{-1}[A]$, $\pi_n^{-1}[B]$ are disjoint closed sets in the normal space $X$, so they are contained in disjoint open sets, $V, W$ respectively. For the base $\bigcup \mathcal{F}$, there are basic covers of $\pi_n^{-1}[A]$ by subsets of $V$, and of $\pi_n^{-1}[B]$ by subsets of $W$. By compactness of $X$, there are finite such covers, $T_1, \dots, T_j$ and $U_1, \dots, U_i$ and since $\mathcal{F}$ is directed, there is a single $\mathcal{F}_m \supseteq \mathcal{F}_n$ containing each element of both these finite covers. But then $\bigcup_{i=1}^{j} \pi_m[T_i]$, $\bigcup_{i=1}^{k} \pi_m[U_i]$ are disjoint open sets in $C_m$ containing $g_{mn}^{-1}[A]$, $g_{mn}^{-1}[B]$, respectively.

To show the converse, that the Hausdorff reflection of the limit of an inverse sequence of finite COTS and separating maps, is always a chainable continuum, we first assume that the maps are onto. Then there are two possibilities: either for some $n$, $C_n$ has at least two closed points, or there is no such $n$. In the second case, for each $n$, $g_n^{-1}(c)$, is a nonempty closed set, thus contains $c$, where by abuse of notation, $c$ denotes the unique closed point in each $C_n$ so $g_n(c) = c$; and since each $g_n$ is onto, once $C_n$ achieves its maximal cardinality, (which is at most 3), it is one-to-one on $C_n \setminus \{c\}$, and thus a homeomorphism on $C_n$. So the Hausdorff reflection of the inverse limit is a singleton. Below, we assume to the contrary that for some $n \in \omega$, $C_n$ has at least two closed points.

For each $m \in \omega$, we denote the canonical quotient map from $\mathbb{I}$ onto $C_m$ by $p_m$ and suppose that $C_m$ has $k_m$ open points; then for each $m$ we can assume that $C_m$ is one of the COTS $F(k_m), L(k_m), O(k_m)$ or $R(k_m)$. Now fix $n \in \omega$ and let $A = \{x \in C_{n+1} \mid x \text{ and } g_n(x) \text{ are both closed}\}$; then $A \neq \varnothing$, in fact for each closed point $c \in C_n$, $g_n^{-1}(c)$ contains a closed point $d$, for it is a nonempty (since $g_n$ is onto) closed set in a finite $T_0$ space, and necessarily $d \in A$. If $J \subseteq C_{n+1} \setminus A$ is an interval in the COTS $C_{n+1}$, then by continuity and the fact that for each closed point $z \in J$, $g_n(z)$ is open, $g_n(z^-) = g_n(z) = g_n(z^+)$ and so for each point $\omega \in J$, $g_n(\omega)$ is open, implying that $g_n$ is constant on the connected set $J$.

If $J$ is a maximal non-empty (topologically) open interval contained in $C_{n+1} \setminus A$, then necessarily one of the following occurs:

1) $J = (x, y)$ with $x, y \in A$ and $g_n(x) \neq g_n(y)$, or
2) $J = (x, y)$ with $x, y \in A$ and $g_n(x) = g_n(y)$, or
3) $J$ is an initial open interval $(\leftarrow, x)$ or a final open interval $(y \rightarrow)$ of $C_{n+1}$, where again, $x, y \in A$.

Let $f_n : \mathbb{I}_{n+1} \to \mathbb{I}_n$ be the piecewise linear continuous map defined as follows:

If case 1) occurs, then (the graph of) $f_n \mid p_{n+1}^{-1}[x, y]$ is the segment joining $(x, g_n(x))$ to $(y, g_n(y))$. (Recall that since $x$ and $y$ are closed, they are identified with the points $p_{n+1}^{-1}[x]$ and $p_{n+1}^{-1}[y]$ respectively in $\mathbb{I}_{n+1}$.)

If case 2) occurs, then let $z_J$ be the midpoint of the open interval $p_{n+1}^{-1}[J] \subseteq \mathbb{I}_{n+1}$; note that in this case, since $J$ always contains an odd number of points, $z_J$ is actually an element of $C_{n+1}$. We define (the graph of) $f_n \mid p_{n+1}^{-1}[x, y]$ to be the union of the segments joining $(x, g_n(x))$ to $(M(p_{n+1}^{-1}[z_J]), w)$ and $(M(p_{n+1}^{-1}[z_J]), w)$ to $(y, g_n(y))$ where $w = \frac{1}{2}(g_n(x) + M(p_n^{-1}[g_n(z_J)]))$.

If case 3) occurs and $J$ is an initial open interval $(\leftarrow, x)$ in $C_{n+1}$, then we define (the graph of) $f_n$ on the initial interval $p_{n+1}^{-1}(\leftarrow, x]$ of $\mathbb{I}_{n+1}$ to be the segment joining $(0, u)$ to $(x, g_n(x))$, where $u = M(p_n^{-1}[(r_J)])$ and $r_J$ is the constant value of $g_n$ on $J$. Similarly, if $J$ is a final interval $(y, \rightarrow)$ of $C_{n+1}$, then we define (the graph of) $f_n$ on the final interval $p_{n+1}^{-1}[y, \rightarrow)$ of $\mathbb{I}_{n+1}$ to be the segment joining $(y, g_n(y))$ to $(1, v)$, where $v = M(p_n^{-1}[(r_J)])$. Then:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{f_3} & \mathbb{I}_3 & \xrightarrow{f_2} & \mathbb{I}_2 & \xrightarrow{f_1} & \mathbb{I}_1 & \xrightarrow{f_0} & \mathbb{I}_0 \\
 & & \downarrow{\scriptstyle p_3} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} & & \downarrow{\scriptstyle p_0} \\
\cdots & \xrightarrow[g_3]{} & C_3 & \xrightarrow[g_2]{} & C_2 & \xrightarrow[g_1]{} & C_1 & \xrightarrow[p_0]{} & C_0
\end{array}$$

is certainly a commutative diagram.

Let $\phi_m : \varprojlim(\mathbb{I}_k, f_k) \to \mathbb{I}_m$ and $\pi_m : \varprojlim(C_k, g_k) \to C_m$ denote the projections. We complete the proof in this case by showing that $\psi : \varprojlim(\mathbb{I}_n, f_n) \to \varprojlim(C_n, g_n)$, defined by $\pi_n \circ \psi_n = p_n \circ \phi_n$ is a homeomorphism onto its subspace $F$ of closed points and thus by Theorem 3.1 onto its Hausdorff reflection. The first of these spaces is a chainable continuum, so the second one is as well.

Since $g_n$ is separating and continuous, it is clear that if case 1) occurs, then $y \geq x^{++++}$, so the absolute value of the slope of $f_n$ on the interval $[x, y]$ is

$$\mid g_n(y) - g_n(x) \mid / (y - x) \leq (1/k_n)/2(1/k_{n+1}) = k_{n+1}/2k_n.$$

If cases 2) or 3) occur, then it is straightforward to check that the absolute value of the slope of any segment defining the graph of $f_n$ is again at most $k_{n+1}/2k_n$.

Observe that in a COTS, if two distinct points are in the closure of any point, then the latter point is unique and must be open. We use this to define a measure of distance (which need not be a metric), by:

$$d_n(v, w) = \begin{cases} k_n|\phi_n(v) - \phi_n(w)| & \text{if } x \text{ open and } p_n \circ \phi_n(v), p_n \circ \phi_n(w) \in \mathrm{cl}(x) \\ 2 & \text{if no such } x \text{ exists.} \end{cases}$$

Thus if $d_{n+1}(v, w) \leq 1$ or equivalently, if for some open $x$,

$$p_{n+1} \circ \phi_{n+1}(v), p_{n+1} \circ \phi_{n+1}(w) \in \mathrm{cl}(x)$$

then $\mid \phi_{n+1}(v) - \phi_{n+1}(w) \mid \leq 1/k_{n+1}$. In this case we have that

$$\begin{aligned} d_n(v, w) &= k_n \; |\phi_n(v) - \phi_n(w)| \\ &= k_n \; |f_n(\phi_{n+1}(v)) - f_n(\phi_{n+1}(w))| \\ &\leq (k_{n+1}/2k_n) k_n |\phi_{n+1}(v) - \phi_{n+1}(w)| \\ &= \tfrac{1}{2} d_{n+1}(v, w), \end{aligned}$$

so in particular:

$$d_n(v, w) \leq \tfrac{1}{2} d_{n+1}(v, w) \tag{3.1}$$

If for each $j$ there is some open $x_j$ in $C_j$ such that $p_j(\phi_j(v)), p_j(\phi_j(w)) \in \mathrm{cl}(x_j)$ then applying induction to the inequality (3.1), we obtain for each $n, m \in \omega$, $d_n(v, w) \leq d_{n+m}(v, w)/2^m \leq 1/2^m$, so each $\phi_n(v) = \phi_n(w)$ and hence $v = w$.

In the sequel, we denote $\varprojlim(\mathbb{I}_j, f_j)$ by $X$ and $\varprojlim(C_j, g_j)$ by $Y$. Now suppose that for some $v, w \in X$, $\psi(v) = \psi(w)$; then $p_j(\phi_j(v)) = p_j(\phi_j(w))$ for each $j \in \omega$ and hence, for each $j$, there is some open point $x_j$ in $C_j$ such that

$p_j(\phi_j(v)), p_j(\phi_j(w)) \in \mathrm{cl}(x_j)$. Hence by the previous paragraph, we must have that $v = w$, so $\psi$ is one-to-one.

Next, we show that for each $v \in X = \varprojlim(\mathbb{I}_j, f_j)$, $\psi(v)$ is closed. For suppose to the contrary that there is some $t \in \mathrm{cl}(\psi(v)) \setminus \{\psi(v)\}$, then for each $j$, $t_j \in \mathrm{cl}(p_j\phi_j(v))$ and for some $n_0$, $t_j \neq p_j\phi_j(v)$, for each $j \geq n_0$. As a result, for each $j \geq n_0$, $t_j \in \mathrm{cl}(p_j\phi_j(v))$ and $p_j\phi_j(v)$ is open; also for $j < n_0$, $t_j, p_j\phi_j(v) \in \mathrm{cl}(x_j)$, where $x_j$ is an open point among ${t_j}^-, t_j, {t_j}^+$. Now let $w_n = p_n^{-1}(t_n)$; then $w = (w_n)_{n\in\omega} \in X$ and $\psi(w) = t$, so for each $j \in \omega$, there is an open $x_j$ such that $p_j\phi_j(w) = t_j, p_j\phi_j(v) \in \mathrm{cl}(x_j)$, so $v = w$, hence $\psi(v) = \psi(w) = t$, contradicting our hypothesis.

We now show that $\psi$ maps onto the closed points of $Y = \varprojlim(C_n, g_n)$. If $y \in Y$ is closed, it follows that for all $z \in Y$ distinct from $y$, there is some $n \in \omega$ such that $\pi_n(z) \notin \mathrm{cl}[\pi_n(y)]$. We consider the set

$$A = \cap\{(p_n \circ \phi_n)^{-1}[\mathrm{cl}(\pi_n(y))] : n \in \omega\} \subseteq X.$$

Note first that $A$ is the intersection of a nested sequence of closed subsets of the compact $T_2$-space $X$ and hence $A \neq \varnothing$. We now claim that if $a \in A$, then $\psi(a) = y$. To prove our claim, note that $a \in (p_n \circ \phi_n)^{-1}[\mathrm{cl}(\pi_n(y))]$ for each $n \in \omega$ and hence $(p_n \circ \phi_n)(a) = (\pi_n \circ \psi)(a) \in \mathrm{cl}[\pi_n(y)]$ for each $n$. Thus $\psi(a) = y$ and we are done.

Notice also that since $p_n$ and $\phi_n$ are continuous, the composition of $\psi$ with each projection $\pi_n$ is continuous. Thus $\psi$ is a continuous one-to-one and onto map between two compact Hausdorff spaces and hence is a homeomorphism. This ends proof of the special case in which the bonding maps are onto.

Consider now an inverse sequence $(C_n, g_n)$ in which the bonding maps $g_n$ are not assumed to be surjective. For each $n \in \omega$, let $D_n = \bigcap_{m\geq n} g_{mn}[C_m]$, with the subspace topology, and let $h_n = g_n|D_{n+1}$. Clearly, the bonding maps $h_n$ of the sequence $(D_n, h_n)$ are continuous. Furthermore, since $g_{mn}[C_m]$ is a connected subset of $C_n$, it is an interval in $C_n$ and hence $D_n$ is a COTS. We will show that the bonding maps $h_n$ are surjective. To this end, suppose that $x \in D_n \setminus h_n[D_{n+1}]$, then $x \notin g_n[D_{n+1}]$ and so for some $m \geq n+1$,

$$x \notin \bigcap g_n g_{m(n+1)}[C_m] = g_{mn}[C_m] \supseteq D_n,$$

a contradiction. Certainly $\varprojlim(D_n, h_n) \subseteq \varprojlim(C_n, g_n)$, but if $x \in \varprojlim(C_n, g_n)$, then for each $m \geq n$, $x_n = g_{mn}(x_m)$ so $x_n \in \bigcap_{m>n} g_{mn}[C_m]$; in a similar way we have that $x_m \in \bigcap_{p>m} g_{pm}[C_p]$ and so $x_n = g_{mn}|D_m(x_m) = h_m(x_m)$, proving that $x \in \varprojlim(D_n, h_n)$. Thus $\varprojlim(C_n, g_n) = \varprojlim(D_n, h_n)$, and the latter is a chainable continuum if the sequence is separating. □

## 4. Circle-like and other continua

Another well-known family of metric continua defined by means of inverse sequences is mentioned in the title of this section. Recall from [3] that a *circle-like* (or *circularly chainable continuum*) is an inverse limit of circles (copies of

$S^1$). The methods of the previous two sections extend (*mutatis mutandis*) to this type of continua.

We define a *digital circle* $S_C$ to be the quotient space of a finite COTS $C$ with an odd number of elements in which the first and last element in the ordering are identified. The graph of a function $f$ between digital circles $S_C$ and $S_D$ can be represented as a graph in the subset $C \times D$ of the digital plane (see [4]) in which the left and right edges as well as the top and the bottom are identified. Similarly, the graph of a continuous function $f : S^1 \to S^1$ can be represented as a graph in the unit square $\mathbb{I}^2$ in which again, the left and right edges and the top and bottom are identified. Somewhat laboriously, it can now be verified that the results of Sections 1 and 2 hold in this new setting and hence we have proved:

**Theorem 4.1.** *A topological space $X$ is a circle-like continuum if and only if it is the Hausdorff reflection of an inverse sequence of digital circles and separating bonding maps.*

The term *Knaster continuum* has been used in a number of papers (see for example [9]) to denote a continuum which is an inverse limit of unit intervals under certain types of tent maps (which are necessarily open). We define a *generalized Knaster continuum* to be an inverse limit of unit intervals with open bonding maps.

In order to deal with generalized Knaster continua, note that a (piecewise linear) continuous map from $\mathbb{I}$ to $\mathbb{I}$ is open if and only if:

1) Whenever $x$ is a local maximum, then $f(x) = 1$ and if $x$ is a local minimum, then $f(x) = 0$ and since the end-points (that is to say, the non-cutpoints) are local maxima or minima, $f$ maps end-points to end-points, $f[\{0,1\}] \subseteq \{0,1\}$.
2) $f$ is strictly monotone between successive local maxima and minima.

In order to extend Theorem 3.2 to generalized Knaster continua, we need to characterize those functions $g$ between COTS which give rise (in the way they are defined in Theorem 3.2) to a function $f : \mathbb{I} \to \mathbb{I}$ satisfying conditions 1) and 2) and inversely, which arise from open maps $f$ in the construction described in the proof of Theorem 2.5. Note first that if $f : \mathbb{I} \to \mathbb{I}$ is open, then the map $g$ constructed in Theorem 2.5 will satisfy the following conditions:

i) If $g : C \to D$ has a local maximum at $x$, then $g(x)$ is the largest element of $D$ and if $g$ has a local minimum at $x$, then $g(x)$ is the smallest element of $D$ and since again, the end-points are local maxima or minima, $g$ maps the end-points of $C$ to the end-points of $D$.
ii) $g$ is monotone (but not necessarily strictly monotone) on each interval which contains no local maximum or minimum.

Such maps do not have to be open and it is easy to see that open maps between COTS need not satisfy *i*) and *ii*) (for example, a constant map whose range is open). We call a continuous function $g$ between finite COTS $C$ and $D$ which satisfies conditions *i*) and *ii*) a *digital Knaster map*.

Now suppose that $g : C \to D$ is a digital Knaster map and consider the piecewise linear map $f$ constructed from $g$ as in Theorem 3.2. It is clear that $f$ will be monotone (but not necessarily strictly monotone) on each subinterval of $[0, 1]$ which contains no local extremum of $f$. Furthermore, if the greatest element $1 \in D$ is closed, then each interval $J \subseteq g^{-1}[1]$ contains some closed point $x$ and so $f(x) = 1$. A similar argument applies if the least element $0 \in D$ is closed. If on the other hand the greatest element $r$ of $D$ is open, say $r = (1 - 1/n, 1]$, then each maximal subinterval contained in $g^{-1}[r]$ has open endpoints and $f$ will assume the value $1 - 1/2n$ at the midpoint of the corresponding (Euclidean) interval. Again, a similar argument holds in case the least element of $D$ is open. Thus in all cases, the map $f$ constructed from $g$ is monotone between successive maxima and minima and the value of $f$ at each local minimum is either 0 (if the least element of $D$ is closed) or $1/2n$ (if the least element of $D$ is open) and correspondingly, at each maximum is either 1 or $1 - 1/2n$. Let $T$ be the range of $f$; then $f : \mathbb{I} \to T$ satisfies the following two conditions:

a) If $T = [l, r]$, then whenever $x$ is a local maximum, $f(x) = r$ and if $x$ is a local minimum, then $f(x) = l$ and $f[\{0, 1\}] \subseteq \{l, r\}$.
b) $f$ is monotone between successive local maxima and minima.

Since such maps can clearly be uniformly approximated by maps satisfying condition a) and

b′) $f$ is strictly monotone between successive local maxima and minima,

it follows from Theorem 3 of [1], that the inverse sequence constructed in Theorem 3.2 has as its limit a generalized Knaster continuum. Thus we have proved:

**Theorem 4.2.** *A topological space $X$ is a generalized Knaster continuum if and only if it is the Hausdorff reflection of an inverse sequence of COTS whose bonding functions are separating digital Knaster maps.*

We do not know whether Theorems 3.2–4.2 hold if the sequences are not required to be separating.

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