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Topological normal forms of high degree for the simplest bifurcations

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Abstract. This paper is devoted to study the topological normal forms of families of maps on $\mathbb{R}$ which, under nondegeneracy conditions of high degree, also present the simplest bifurcations.

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1. Introduction

Let us consider uniparametric families of maps

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad f(x, \mu) = f_\mu(x)$$

and

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad g(y, \nu) = g_\nu(y),$$

where $\mu, \nu \in \mathbb{R}$ are the parameters and $x, y \in \mathbb{R}$ the variables in the state space under consideration.

f and g are called *locally topologically equivalent* near the origin, if there exists a map

$$(x, \mu) \rightsquigarrow (h_\mu(x), p(\mu)),$$

defined in a small neighborhood of $(x, \mu) = (0, 0)$ in the direct product $\mathbb{R} \times \mathbb{R}$ and such that

(i) $p : \mathbb{R} \longrightarrow \mathbb{R}$ is a homeomorphism defined in a small neighborhood of $\mu = 0$, with $p(0) = 0$;
(ii) $h_\mu : \mathbb{R} \longrightarrow \mathbb{R}$ is a parameter-dependent homeomorphism defined in a small neighborhood U_μ of $x = 0$, with $h_0(0) = 0$ and mapping orbits of the first system in U_μ onto orbits of the second one in $h_\mu(U_\mu)$, preserving the direction of time.

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Roughly speaking, if there is a qualitative change in the behavior of the maps of f when $\mu_0 \in \mathbb{R}$ is crossed, one says that μ_0 is a *bifurcation value* or that a *bifurcation* occurs at μ_0. More rigourously, it can be said that the appearance of a topologically non-conjugate system under variation of the parameter is called a *bifurcation.*

A *bifurcation diagram* of the family is a stratification of its parameter space induced by the topological conjugacy, together with representative phase portraits for each system.

Sometimes, for bifurcations of a family f, near a fixed point of a map belonging to the family, it is possible to construct a simple polynomial family

$$g(y, \nu), \quad y \in \mathbb{R}, \nu \in \mathbb{R} \tag{1.1}$$

which presents at the map corresponding to the parameter value $\nu = 0$ the fixed point $y = 0$, satisfying the same *bifurcation conditions.* With same bifurcation conditions we mean to present the same eigenvalue (of unit modulus) at the fixed point and some other conditions that will have the form of inequalities (equalities)

$$D_i(g) \neq 0 \qquad (D_i(g) = 0),$$

where D_i are some algebraic functions of partial derivatives of g, evaluated on $(y, \nu) = (0, 0)$. The inequalities (equalities) D_i which only involve partial derivatives with respect to the state variable y are called *nondegeneracy conditions* (*degeneracy conditions*), while those involving parameters are known as *transversality conditions.*

A family of the form (1.1) is said to be a *topological normal form* for a bifurcation if any family f with the same bifurcation conditions is locally topologically equivalent to it, near the corresponding fixed point.

In [12] or [13], it is shown that under some nondegeneracy conditions on f and f^2 up to the third order, the standard bifurcations for families of one dimensional maps (saddle node, transcritical, pitchfork, flip) appear.

In [4], we can see that similar results are obtained if those nonzero conditions are fulfilled by partial derivatives of order greater than three. However, in spite of it proves that the same number of fixed points (or period-2 points in the flip case) appears and with identical type of stability, the task of finding a homeomorphism providing the topological equivalence between any family verifying certain bifurcation conditions and the corresponding simpler polinomial family is very complicated. Even in the simplest cases that appear in [13], a complete proof remains unpublished (see [1], [8]). The main difficulty is to encounter the analytic expression of the homeomorphism which provides the equivalence, instead of the presence of higher order terms in the Taylor expansion of the original map.

The theory of normal forms, which began with Poincaré and was developed by Arnold, is a mathematical technique which allows to reduce the expression of the equation that defines a dynamical system (e.g., see [6], [2]). The interest of such (polynomial) topological normal forms is that their bifurcation diagrams, which can be often easily obtained, have a universal meaning.

Here, our goal is to show how one can reduce the expression defining a family of discrete dynamical systems, which verifies certain bifurcation conditions of higher degree, to a topologically equivalent simpler form.

A fixed point $p \in \mathbb{R}$ of $f \in \mathrm{Diff}(\mathbb{R}^m)$ is *hyperbolic* if the linearization $Df(p)$ has no eigenvalues of unit modulus. From Hartman-Grobman's theorem it follows (see [6] or [9]) that to study local bifurcations of fixed points in parametric families $f_\mu(x)$, it suffices to consider those parameters μ_0 for which the corresponding map has a non-hyperbolic fixed point p_0. In one dimensional families, an eigenvalue equals 1 or -1 are the only two possibilities. We deal with each one in Sections 2 and 3, respectively.

2. An eigenvalue equal to 1

Fold Bifurcation. A uniparametric family

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

undergoes a fold bifurcation if the family possesses a unique curve of fixed points in the $x - \mu$ plane passing through the bifurcation point that locally lies on one side of $\mu = 0$.

In [4], one can see nondegeneracy conditions of higher order to the appearance of this bifurcation. Concretely, the following result is given.

Theorem 2.1. *Suppose that a one-parameter family*

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

of C^{2n} maps has at $\mu_0 = 0$ the fixed point $x_0 = 0$ and let $f_x(0,0) = 1$.

Assume that the following conditions are satisfied:

(F$_1$)
- $f_{xx}(0,0) = f_{xxx}(0,0) = \cdots = f_{x^{2n-1}}(0,0) = 0$,
- $f_{x^{2n}}(0,0) \neq 0$

(F$_2$) $f_\mu(0,0) \neq 0$

Then the family undergoes a fold bifurcation.

Remark 2.2. The case $n = 2$ can be found in the majority of the literature about bifurcation theory (e.g., see [6], [12], [2] or [7])

For each $n \in \mathbb{N}\backslash\{0\}$, one observes that

$$x + s_1\mu + s_2 x^{2n}, \tag{2.2}$$

where $s_1, s_2 \in \{-1, 1\}$, are the simplest families verifying the statement of theorem 2.1. As we have said in the introduction, the task of proving the (topological) equivalence between (2.2) and any family satisfying the hypothesis of theorem 2.1, giving the pertinent parameter-dependent homeomorphism, is unexpectedly complicated. We approach to it in the following sense. The Taylor expansion of a smooth f with the above properties is

$$f(x,\mu) = x + f_\mu(0,0)\mu + \frac{1}{(2n)!} f_{x^{2n}}(0,0)x^{2n} + \mathcal{O}(\mu^2 + |x||\mu| + x|^{2n+1}).$$

Making the rescaling

$$x = \frac{\eta}{\left|\frac{1}{(2n)!} f_{x^{2n}}(0,0)\right|^{\frac{1}{2n-1}}}$$

and introducing the new parameter

$$\nu = \left|\frac{1}{(2n)!} f_{x^{2n}}(0,0)\right|^{\frac{1}{2n-1}} |f_\mu(0,0)|\mu$$

we obtain the form

$$\eta \rightsquigarrow \eta + s_1\nu + s_2\eta^{2n} + \mathcal{O}(\nu^2 + |\eta||\nu| + \eta^{2n+1}),$$

where

$$s_1 = \operatorname{sign}(f_{x\mu}(0,0)) \text{ and } s_2 = \operatorname{sign}(f_{x^{2n}}(0,0)).$$

In fact, it is the family (2.2), but with some higher order terms.

Transcritical Bifurcation. A uniparametric family

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

undergoes a transcritical bifurcation if the family possesses two curves of fixed points in the $x - \mu$ plane passing through the origin and existing on both sides of $\mu = 0$, changing the stability of the fixed points.

As in the preceding case, in [4], we have the following result about nondegeneracy conditions of higher order.

Theorem 2.3. *Suppose that a one-parameter family*

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

of C^{2n+1} maps has at the map corresponding to the parameter value $\mu_0 = 0$ the fixed point $x_0 = 0$ and let $f_x(0,0) = 1$, $f_\mu(0,0) = 0$.

Assume that the following conditions are satisfied:

(T_1)
- $f_{xx}(0,0) = f_{xxx}(0,0) = \cdots = f_{x^{2n-1}}(0,0) = 0,$
- $f_{x^{2n}}(0,0) \neq 0$

(T_2) $f_{x\mu}(0,0) \neq 0$

Then the family undergoes a transcritical bifurcation.

Remark 2.4. The case $n = 2$ can be seen in [6], [12] or [2].

In this situation the question is if any family satisfying the hypothesis of this last theorem is (locally) topologically equivalent to

$$x + s_1\mu x + s_2 x^{2n}, \tag{2.3}$$

where $s_1, s_2 \in \{-1, 1\}$.

On one hand, the Taylor expansion of a smooth f with the above properties is

$$x + f_{x\mu}(0,0)x\mu + \frac{1}{(2n)!} f_{x^{2n}}(0,0)x^{2n} + \mathcal{O}(|x|\mu^2 + x^2|\mu| + |x|^{2n+1} + |\mu|^2).$$

Making the rescaling

$$x = \frac{\eta}{\left|\frac{1}{(2n)!} f_{x^{2n}}(0,0)\right|^{\frac{1}{2n-1}}}$$

and introducing the new parameter

$$\nu = |f_{x\mu}(0,0)|\mu$$

we obtain the form

$$\eta \rightsquigarrow \eta + s_1\nu\eta + s_2\eta^{2n} + \mathcal{O}(|\eta|\nu^2 + \eta^2|\nu| + |\eta|^{2n+1} + \nu^2),$$

where

$$s_1 = \operatorname{sign}(f_{x\mu}(0,0)) \text{ and } s_2 = \operatorname{sign}(f_{x^{2n}}(0,0)).$$

On the other hand, in spite of it, it would be nicer to prove that, after the truncation of the higher order terms in the last equation, a topologically equivalent family remains.

Pitchfork Bifurcation. A uniparametric family

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

undergoes a pitchfork bifurcation if the family possesses two curves of fixed points in the $x - \mu$ plane passing through the bifurcation point; one curve exists on both sides of $\mu = 0$ and the other lies locally on one side of $\mu = 0$.

For this bifurcation, we have:

Theorem 2.5. *Suppose that a one-parameter family*

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

of C^{2n+2} maps has at the map corresponding to the parameter value $\mu_0 = 0$ the fixed point $x_0 = 0$ and let $f_x(0,0) = 1$, $f_\mu(0,0) = 0$.

Assume that the following conditions are satisfied:

(P$_1$) • $f_{xx}(0,0) = f_{xxx}(0,0) = \cdots = f_{x^{2n}}(0,0) = 0$,
• $f_{x^{2n+1}}(0,0) \neq 0$

(P$_2$) $f_{x\mu}(0,0) \neq 0$

Then the family undergoes a pitchfork bifurcation.

Proof. See [4]. □

Remark 2.6. As in the two preceding cases, the case $n = 3$ can be encountered in the literature (e.g., see [12]).

It is evident that the simplest families which verify the conditions of theorem 2.5 are

$$x + s_1\mu x + s_2 x^{2n+1}, \tag{2.4}$$

where $s_1, s_2 \in \{-1, 1\}$.

In particular, the Taylor expansion of a smooth f with the above properties is

$$x + f_{x\mu}(0,0)x\mu + \frac{1}{(2n+1)!} f_{x^{2n+1}}(0,0)x^{2n+1} + \mathcal{O}(|x|\mu^2 + x^2|\mu| + |x|^{2n+2} + \mu^2).$$

Making the rescaling

$$x = \frac{\eta}{\left|\frac{1}{(2n+1)!} f_{x^{2n+1}}(0,0)\right|^{\frac{1}{2n}}}$$

and introducing the new parameter

$$\nu = |f_{x\mu}(0,0)|\mu$$

we obtain the form

$$\eta \rightsquigarrow \eta + s_1\nu\eta + s_2\eta^{2n+1} + \mathcal{O}(|\eta|\nu^2 + \eta^2|\nu| + \eta^{2n+2} + \nu^2),$$

where

$$s_1 = \operatorname{sign}(f_{x\mu}(0,0)) \text{ and } s_2 = \operatorname{sign}(f_{x^{2n+1}}(0,0)).$$

Topological equivalence with the simplest form (2.4), however, is not proved yet.

3. An Eigenvalue equal to -1

Flip or Period Doubling Bifurcation. A uniparametric family

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

undergoes a flip bifurcation if the family possesses two curves in the $x - \mu$ plane passing through the bifurcation point; one curve of fixed points exists on both sides of $\mu = 0$ and the other one, of periodic points of period two, lies locally on one side of $\mu = 0$.

Once more, we have conditions of higher order required for a family to undergo a flip or period-doubling bifurcation.

Theorem 3.1. *Suppose that a one-parameter family*

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

of C^{2n+1} maps has at $\mu_0 = 0$ the fixed point $x_0 = 0$ and let $f_x(0,0) = -1$.

Assume that the following conditions are satisfied:

(PD$_1$)
- $(f^2)_{xxx}(0,0) = (f^2)_{x^4}(0,0) = \cdots = (f^2)_{x^{2n}}(0,0) = 0$,
- $(f^2)_{x^{2n+1}}(0,0) \neq 0$

(PD$_2$) $f_{x\mu}(0,0) \neq 0$

Then the family undergoes a flip or period-doubling bifurcation.

Proof. See [4]. □

Since $f_x(0,0) \neq 1$, the Implicit Function Theorem guaranties the existence and uniqueness of a curve, $x(\mu)$, for sufficiently small $|\mu|$, consisting on fixed points. Therefore, one can perform a coordinate shift, placing this fixed points at the origin of the state space and so, it can be assumed, without loss of generality, that $x = 0$ is a fixed point of each map f_μ of the family for sufficiently small $|\mu|$.

This being the case, it seems appropriated to say that the simplest family verifying the conditions of theorem 3.1 is one of the form

$$-(1 + s_1\nu)\eta + s_2\eta^{2n+1}, \tag{3.5}$$

where $s_1, s_2 \in \{-1, 1\}$.

From the conventional normal form theory, we know that the family can be written, after a suitable shift of coordinates, as follows (see [2] or [10])

$$\lambda(\mu)x + a_3(\mu)x^3 + \cdots + a_{2n+1}(\mu)x^{2n+1} + \mathcal{O}(x^{2n+3}), \tag{3.6}$$

where $\lambda(0) = -1$ and the first $a_i(0) \neq 0$ depends directly on the nondegeneracy conditions in the formulation of theorem 3.1. In fact, we have that

$$a_3(0) = \frac{1}{6}f_{xxx}(0,0) + \frac{1}{4}f_{xx}^2(0,0) = \frac{-1}{12}(f^2)_{xxx}(0,0),$$

and if $(f^2)_{xxx}(0,0) = 0$, what means that $a_3(0) = 0$, then

$$a_5(0) = \frac{1}{120}f_{x^5}(0,0) + \frac{1}{16}f_{x^4}(0,0)f_{xx}(0,0) - \frac{1}{8}f_{xx}^4(0,0 = \frac{-1}{240}(f^2)_{x^5}(0,0).$$

To the appearance of a (standard) flip bifurcation, $a_3(0) \neq 0$ is assumed. Further degeneracy can be introduced by taking

$$a_3(0) = a_5(0) = \cdots = a_{2l-1}(0) = 0, \quad \text{but} \quad a_{2l+1}(0) \neq 0, \tag{3.7}$$

which we will call *generalized flip singularity of type l*. A model that presents one of these generalized flip singularities can be seen in [3].

In this case, we have been able to go further, and some of the higher order terms in (3.6) have been eliminated, giving a simpler topologically equivalent form of the family.

First of all, we work on the particular value $l = 1$ and then we summarize the results for a general l. To begin with, let us consider the equation (3.6). As this only involves some odd terms, the transformation needs to involve these odd order terms only. In order to obtain the appropriate pattern of the nonlinear transformation, we start from the 3rd order terms. Making the nonlinear transformation

$$y = x + c_3x^3, \tag{3.8}$$

where $c_3 = c_3(\mu)$ is to be determined suitably, and supposing that upon applying this transformation the equation of (3.6) in the new coordinates can be expressed as

$$\lambda y + b_3y^3 + \cdots , \tag{3.9}$$

when we balance the 3rd order terms, we have

$$a_3 + \lambda^3 c_3 = b_3 + \lambda c_3.$$

So, evaluating this equation on $\mu = 0$, one has $b_3(0) = a_3(0) \neq 0$ for any choice of $c_3(0)$, what means that the 3rd order terms can not be eliminated.

Now, proceeding in the same manner with the 5th order terms, if we perform the change

$$y = x + c_3 x^3 + c_5 x^5 \tag{3.10}$$

we obtain, balancing the 5th order terms,

$$a_5 + 3\lambda^2 a_3 c_3 + \lambda^5 c_5 = b_5 + 3b_3 c_3 + \lambda c_5,$$

what proves that $b_5(0) = a_5(0)$ and the 5th order terms can not be eliminated for any choice of c_3 and c_5.

Working on the 7th order terms, if we make the nonlinear transformation

$$y = x + c_3 x^3 + c_5 x^5 + c_7 x^7, \tag{3.11}$$

for these terms we have

$$a_7 + 3\lambda^2 a_5 c_3 + 3\lambda a_3^2 c_3 + 5\lambda^4 c_5 a_3 + \lambda^7 c_7 = b_7 + 5c_3 b_5 + 3c_3^2 b_3 + 3c_5 b_3 + \lambda c_7. \tag{3.12}$$

Then, after evaluating on $\mu = 0$, we choose

$$c_3 = 0 \quad \text{and} \quad c_5(0) = \frac{-a_7(0)}{2a_3(0)}$$

what, applying the Implicit Function Theorem to the equation (3.12), allows us to make $b_7(\mu) = 0$ for sufficiently small $|\mu|$.

In general, if one performs the change

$$y = x + c_5 x^5 + c_7 x^7 + \cdots + c_{2k+1} x^{2k+1}, \tag{3.13}$$

balancing the $(2k+1)$th terms and evaluating in $\mu = 0$, one obtains by induction an equation of the form

$$(2k-4)a_3(0)c_{2k-1}(0) = b_{2k+1}(0) - a_{2k+1}(0) + P(\cdots), \quad k > 2,$$

where $P(\cdots)$ represents a summation of terms which are function of the known coefficients

$$c_3(0), c_5(0), \ldots, c_{2k-3}(0) \text{ and } a_3(0), a_5(0), \ldots, a_{2k-1}(0).$$

Obviously, this last equation allows us to make $b_{2k+1}(0) = 0$ with an appropriate choice of $c_{2k-1}(0)$, for all $k > 2$. Using again the Implicit Function Theorem, we have $b_{2k+1}(\mu) = 0$ for sufficiently small $|\mu|$.

In general, for a singularity of type l, i.e.,

$$a_3(0) = a_5(0) = \cdots = a_{2l-1}(0) = 0, \quad \text{but } a_{2l+1}(0) \neq 0, \tag{3.14}$$

the same procedure yields

$$b_3(0) = b_5(0) = \cdots = b_{2l-1}(0) = 0, \quad \text{but } b_{2l+1}(0) = a_{2l+1}(0) \neq 0 \tag{3.15}$$

for any choice of the coefficients $c_3, c_5, \ldots, c_{2l+1}$.

On the other hand, for $k > l$, making the nonlinear transformation

$$y = x + c_3x^3 + c_5x^5 + c_7x^7 + \cdots + c_{2k+1}x^{2k+1}, \quad k > l, \tag{3.16}$$

and balancing the $(2k+1)$th order terms, we have, after evaluating on $\mu = 0$,

$$(2k - 4l)a_{2l+1}(0)c_{2k+1-2l}(0) = b_{2k+1}(0) - a_{2k+1}(0) + P(\cdots), \tag{3.17}$$

what means that every term of order $2k+1$ greater than $2l+1$ can be eliminated, except in the case $k = 2l$. In such a case, we choose $c_{2l+1} = 0$ for simplicity.

Unfortunately, this procedure does not apply in the cases corresponding to an eigenvalue equals 1.

Thus, we have proved the following result.

Theorem 3.2. *Assume that the conventional normal form of an analytic family with the singularity of the presence of a fixed point with an eigenvalue equals -1 is given by (3.6).*

Suppose that the following degeneracy conditions are verified:

$$a_3(0) = a_5(0) = \cdots = a_{2l-1}(0) = 0, \;\; \textit{but } a_{2l+1}(0) \neq 0. \tag{3.18}$$

Then, its normal form can be written as

$$\lambda(\mu)x + a_3(\mu)x^3 + \cdots + a_{2l+1}(\mu)x^{2l+1} + a_{4l+1}(\mu)x^{4l+1}, \tag{3.19}$$

up to any order.

Remark 3.3. As in the other cases, the topological equivalence of any family satisfying the hypothesis of theorem 3.1 and the simplest family (3.5) is an open problem yet.

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