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On some properties of T_0-ordered reflection

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Abstract

In [12], the authors give an explicit construction of the T_0-ordered reflection of an ordered topological space $(X, \tau, \leq)$. All ordered topological spaces such that whose T_0-ordered reflections are T_1-ordered spaces are characterized. In this paper, some properties of the T_0-ordered reflection of a given ordered topological space $(X, \tau, \leq)$ are studies. The class of morphisms in **ORDTOP** *orthogonal to all T_0-ordered topological space is characterized.*

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1. Introduction

Among the oldest separation axioms in topology, there are three famous ones T_0, T_1 and T_2.

The T_0-, T_1- and T_2-reflections of a topological space have long been of interest to categorical topologist. The construction of these reflections in the category **TOP** of all topological spaces are given in [10].

In [2], the authors introduced some new separation axioms using the T_i-reflections $T_i(X)$ $i \in \{0, 1, 2\}$ as follow:

Definition 1.1. Let i, j be two integers such that $0 \leq i < j \leq 2$. A topological space X is said to be a $T_{(i,j)} -$ *space* if $T_i(X)$ is a T_j-space.

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Precisely, there are two new types of separation axioms namely $T_{(0,1)}-$ and $T_{(0,2)}-$spaces.

The characterization of these spaces are completely stated in [2, Theorem 3.5.] and [2, Theorem 3.12.].

After this, in [12], H. Künzi and T. A. Richmond are interested in the corresponding concepts of T_i-ordered reflections in the category **ORDTOP** with ordered topological spaces $(X, \tau, \leq)$ as objects and continuous increasing maps as morphisms (or arrows).

The authors show that, given an ordered topological space $(X, \tau, \leq)$, the T_i-ordered reflection, for $i \in \{0, 1, 2\}$ of $(X, \tau, \leq)$ is obtained as a quotient of X (for more information see [12, Theorem 2.5.]).

Let $(X, \tau, \leq)$ be an ordered topological space. For $A \subseteq X$, the increasing hull of A is $i\,(A) = \{y \in X\ :\ \exists\, x \in A\; x \leq y\}$. A subset A of X is an increasing set if $i\,(A) = A$ and we denote by $I\,(A)$ the closed increasing hull of A, that is, the smallest closed increasing set containing A. Decreasing set, decreasing hull $d(A)$ and the closed decreasing hull $D(A)$ are defined dually. A subset A which satisfy $A = I(A) \cap D(A)$ is called a $c-set$. We denote by $C(A)$ the smallest $c-set$ containing A.

An ordered topological space $(X, \tau, \leq)$ is said to be T_0-ordered if the following equivalent conditions hold.

(1) $[I(x) = I(y)\ and\ D(x) = D(y)] \Longrightarrow x = y.$
(2) $C(x) = C(y) \Longrightarrow x = y.$
(3) If $x \neq y$, there exist a monotone open neighborhood of one of the points which does not contain the other point.

Let $(X, \tau, \leq)$ be an ordered topological space. For $x, y \in X$, define an equivalence relation on X by $x \approx y$ if and only if $[I\,(x) = I\,(y)$ and $D\,(x) = D\,(y)]$, which is equivalent to $C\,(x) = C\,(y)$.

Order the set ${}^{X}\!/\!\approx$ by the finite step order defined by :

$$\overline{z_0} \leq^0 \overline{z_n} \quad \Longleftrightarrow \quad \begin{array}{c} \exists\, z_1, ..., z_{n-1}\ and\ \exists\, z_i^{'}, z_i^* \in \overline{z_i}\ (i = 0, 1, ..., n) \\ \text{with } z_i^{'} \leq z_{i+1}^* \quad \forall\, i = 0, 1, ..., n-1 \end{array}.$$

T. A. Richmond and H-P. A. Künzi show that $\left({}^{X}\!/\!\approx, {}^{\tau}\!/\!\approx, \leq^0\right)$ is the T_0-ordered reflection of X.

This paper consists of some investigations into the T_0-ordered reflection of an ordered topological space $(X, \tau, \leq)$.

In the first section we give the characterization of an ordered topological space $(X, \tau, \leq)$ such that its T_0-ordered reflection is T_1^K-ordered and we characterize ordered topological spaces whose T_0-ordered reflections are T_2-ordered. [2, Theorem 3.5] and [2, Theorem 3.12] are recovered.

The second investigation deals with some categorical properties of the category $\mathbf{ORDTOP_0}$, of T_0-ordered topological spaces. More precisely, a characterization of the class of morphisms in **ORDTOP** rendered invertible, by the T_0-ordered reflection functor, is given. [2, Theorem 2.4] is seen to be a particular case of our result.

2. Separation axioms

Given an ordered topological space $(X, \tau, \leq)$, the construction of its T_0−ordered reflection denoted by $\left({}^{X}/\approx, {}^{\tau}/\approx, \leq^0\right)$ satisfies some categorical properties:

For each ordered topological space $(Y, \gamma, \sqsubseteq)$ and each continuous increasing map f from $(X, \tau, \leq)$ to $(Y, \gamma, \sqsubseteq)$, there exists a unique continuous increasing map $\overset{\approx}{f} : \left({}^{X}/\approx, {}^{\tau}/\approx, \leq^0\right) \longrightarrow \left({}^{Y}/\approx, {}^{\gamma}/\approx, \sqsubseteq^0\right)$ such that the following diagram commutes:

$$\begin{array}{ccc} (X, \tau, \preceq) & \xrightarrow{\quad f \quad} & (Y, \gamma, \sqsubseteq) \\ {\scriptstyle q_X}\downarrow & \circlearrowleft & \downarrow{\scriptstyle q_Y} \\ \left({}^{X}/\approx, {}^{\tau}/\approx, \leq^0\right) & \xrightarrow[\overset{\approx}{f}]{} & \left({}^{Y}/\approx, {}^{\gamma}/\approx, \sqsubseteq^0\right) \end{array}$$

where q_X is the canonical surjection map.

From the above properties, it is clear that we have a covariant functor from the category of ordered topological spaces **ORDTOP** into the full subcategory **ORDTOP$_\mathbf{0}$** of **ORDTOP** whose objects are T_0−ordered topological spaces.

In [12], the authors characterize those ordered topological spaces whose T_0−ordered reflections are T_1−ordered as follows:

Theorem 2.1 ([12, Theorem 3.2])**.** *The following statements are equivalent:*

(1) *The T_0 − ordered reflection ${}^{X}/\approx$ of X is T_1 − ordered.*

(2) *$\overline{x} \nleq^0 \overline{y}$ in ${}^{X}/\approx$ implies there exists an open increasing neighborhood of x not containing y and there exists an open decreasing neighborhood of y not containing x.*

(3)
$$i\left(\overline{x}\right) = \bigcap \{N : N \, is \, an \, open \, increasing \, neigborhood \, of \, x\}$$
$$and$$
$$d\left(\overline{x}\right) = \bigcap \{N : N \, is \, an \, open \, decreasing \, neigborhood \, of \, x\}$$
$$\forall x \in X.$$

On the other hand, recall that an ordered topological space $(X, \tau, \leq)$ is said to be a T_1^K−ordered space if, for any point x in X, we have $C(x) = \{x\}$ (for more information see [13]). The following theorem characterizes ordered topological spaces whose T_0−ordered reflections are T_1^K−ordered.

Theorem 2.2. *Let $(X, \tau, \leq)$ be an ordered topological space. The following statements are equivalent:*

(i) *The $T_0 - ordered\, reflection$ ${}^{X}/\approx$ of X is T_1^K − ordered;*

(ii) *For each $x \in X$ and each monotone closed subset F of X such that $F \cap C(x) \neq \varnothing$, we have $x \in F$;*

(iii) *For each monotone open subset O of X containing x, we have $C(x) \subseteq O$.*

Proof.

- $(i) \Rightarrow (ii)$

  Suppose that ${}^{X}/_{\approx}$ is $T_1^K - ordered$.

  Let F be a closed monotone subset of X such that $F \cap C(x) \neq \varnothing$ and $y \in F \cap C(x)$.

  Clearly, $q_X(y) \in q_X(F) \cap C(q_X(x))$. Thus, we can see that $q_X(x) = q_X(y)$. Now, since F is monotone and consequently a saturated subset of X, $q_X(x) = q_X(y) \subseteq F$. Therefore, $x \in F$.

- $(ii) \Rightarrow (i)$

  Let $y \in X$ be such that $q_X(y) \in C(q_X(x))$. Clearly $C(y) \subseteq C(x)$. Conversely, since $I(y) \cap C(x)$ is non empty then by (ii) $x \in I(y)$ and by the same way we say that $x \in D(y)$. Therefore $x \in C(y)$ and $C(x) \subseteq C(y)$.

  Finally we can see that $C(x) = C(y)$, so we have $q_X(x) = q_X(y)$.

- $(ii) \Rightarrow (iii)$

  Let $x \in X$ and O be a monotone open subset of X such that $x \in O$. If $C(x) \nsubseteq O$ then by (ii) $x \notin O$ which is false.

- $(iii) \Rightarrow (ii)$

  Let F be a closed monotone subset of X such that $F \cap C(x) \neq \varnothing$. If $x \notin F$ then $x \in F^C$, since F^C is a monotone open subset of X then by (iii) we obtain $C(x) \subseteq F^C$ which is false.

□

As an immediate consequence of Theorem 2.2, we have the following corollary.

Corollary 2.3 ([2, Theorem 3.5])**.** *Let (X, τ) be a topological space. Then the following statements are equivalent:*

(i) *X is a $T_{(0,1)}$-space;*

(ii) *For each open subset U of X and each $x \in U$, we have $\overline{\{x\}} \subseteq U$;*

(iii) *For each $x \in X$ and each closed subset C of X such that $\overline{\{x\}} \cap C \neq \varnothing$, we have $x \in C$.*

Proof. Let (X, τ) be a topological space. It is enough to consider the ordered topological space $(X, \tau, =)$ in Theorem 2.2.

□

Now, let us introduce the following notation and definition:

Notation 2.4. Let $(X, \tau, \leq)$ be an ordered topological space and z in X. We denote by:

$$T(z) := \{x \in X \, : \, C(x) = C(z)\}.$$

Definition 2.5. Let $(X, \tau, \leq)$ be an ordered topological space. Defines on X *the finite step preorder* $\preceq_{(X,\leq)}$ *related to* $\leq$, by $x \preceq_{(X,\leq)} y$ if there exists $z_0, ..., z_n$ *and* $\exists \, z_i^{'}, z_i^{*} \in T(z_i)$ $(i = 0, 1, ..., n)$ such that $z_0 = x$, $z_n = y$ and $z_i^{'} \leq z_{i+1}^{*} \quad \forall \, i = 0, 1, ..., n-1$.

for short, we denote $\preceq_{(X,\leq)}$ also by $\preceq_{\leq}$.

Remarks 2.6.

- It is clear that $G(\leq) \subseteq G(\preceq_{\leq})$.
- If X is a T_0-ordered space, we have $\leq = \preceq_{\leq}$.
- For each $x, y \in X$, we have equivalence between $x \preceq_{\leq} y$ and $q_X(x) \leq^0 q_X(y)$.

Recall that an ordered topological space $(X, \tau, \leq)$ is said to be T_2-ordered if there is an increasing neighborhood of x disjoint form some decreasing neighborhood of y whenever $x \nleq y$, which is equivalent to the order $\leq$ being closed in $(X, \tau) \times (X, \tau)$.

Now, we are in position to give the characterization of ordered topological spaces whose T_0-ordered reflections are T_2-ordered.

Theorem 2.7. *Let $(X, \tau, \leq)$ be an ordered topological space. Then the following statements are equivalent:*

(i) *The $T_0 - ordered\, reflection$ $X/\approx$ of X is $T_2 - ordered$;*
(ii) *If $x \npreceq_{(X,\leq)} y$ there exists an increasing neighborhood of x disjoint from some decreasing neighborhood of y;*
(iii) *The graph $G(\preceq_{\leq})$ of $\preceq_{\leq}$ is closed in $X \times X$.*

Proof.

- $(i) \Longrightarrow (ii)$
  Let x, y be two points in X such that $x \npreceq_{\leq} y$. Then $q_X(x) \nleq^0 q_X(y)$. Since $X/\approx$ is T_2-ordered, there exists an increasing neighborhood U of $q_X(x)$ disjoint from some decreasing neighborhood V of $q_X(y)$. Now, we can see that $q_X^{-1}(U)$ is an increasing neighborhood of x disjoint from $q_X^{-1}(V)$, which is a decreasing neighborhood of y.
- $(ii) \Longrightarrow (iii)$
  Let $x, y \in X$ such that $(x, y) \notin G(\preceq_{\leq})$ which means that $x \npreceq_{\leq} y$. Then, there exists an increasing neighborhood U of x disjoint from some decreasing neighborhood V of y. Clearly, we can see that $U \times V$ is a neighborhood of (x, y) and we have $(U \times V) \cap G(\preceq_{\leq}) = \varnothing$. Therefore, $G(\preceq_{\leq})$ is closed in $X \times X$.
- $(iii) \Longrightarrow (i)$
  For this implication we can see that $G(\preceq_{\leq}) = q_X^{-1} \times q_X^{-1}\left(G\left(\leq^0\right)\right)$. Then , $G\left(\leq^0\right)$ is closed and thus $X/\approx$ is T_2-ordered.

□

By the same way as in Corollary 2.3, the following result holds immediately.

Corollary 2.8 ([2, Theorem 3.12])**.** *Let (X, τ) be a topological space. Then the following statements are equivalent:*

(i) *X is a $T_{(0,2)}$-space;*
(ii) *For each $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, there are two disjoints open sets U and V in X with $x \in U$ and $y \in V$.*

3. The class of morphisms in ORDTOP orthogonal to all T_0-ordered spaces

It is worth noting that reflective subcategories arise throughout mathematics, via several examples such as the free group and free ring functors in algebra, various compactification functors in topology, and completion functors in analysis: cf. [14, p. 90]. Recall from [14, p. 89] that a subcategory D of a category C is called reflective (in C) if the inclusion functor $I : D \longrightarrow C$ has a left adjoint functor $F : C \longrightarrow D$; i.e., if, for each object A of C, there exist an object $F(A)$ of D and a morphism $\mu_A : A \longrightarrow F(A)$ in C such that, for each object X in D and each morphism $f : A \longrightarrow X$ in C, there exists a unique morphism $\tilde{f} : F(A) \longrightarrow X$ in D such that $\tilde{f} \circ \mu_A = f$.

The concept of reflections in categories has been investigated by several authors (see for example [3], [4], [5], [6],[9], [11], [15]). This concept serves the purpose of unifying various constructions in mathematics.

Historically, the concept of reflections in categories seems to have its origin in the universal extension property of the Stone-Čech compactification of a Tychonoff space.

A morphism $f : A \longrightarrow B$ and an object X in a category C are called orthogonal [7], if the mapping $hom_C(f;X) : hom_C(B;X) \longrightarrow hom_C(A;X)$ that takes g to gf is bijective. For a class of morphisms Σ (resp., a class of objects D), we denote by $\Sigma^{\perp}$ the class of objects orthogonal to every f in Σ (resp., by $D^{\perp}$ the class of morphisms orthogonal to all X in D) [7].

The orthogonality class of morphisms $D^{\perp}$ associated with a reflective subcategory D of a category C satisfies the following identity $D^{\perp\perp} = D$ [1, Proposition 2.6]. Thus, it is of interest to give explicitly the class $D^{\perp}$. Note also that, if $I : D \longrightarrow C$ is the inclusion functor and $F : C \longrightarrow D$ is a left adjoint functor of I, then the class $D^{\perp}$ is the collection of all morphisms of C rendered invertible by the functor F (i.e. $D^{\perp} = \{f \in hom_C : F(f) \text{ is an isomorphism of } D\}$)[1, Proposition 2.3].

This section is devoted to the study of the orthogonal class $\mathbf{ORDTOP_0}^{\perp}$; hence we will give a characterization of morphisms rendered invertible by the functor of the T_0-ordered reflection.

Recall that a continuous map $q : Y \to Z$ is said to be a *quasihomeomorphism* if $U \to q^{-1}(U)$ defines a bijection $\mathcal{O}(Z) \to \mathcal{O}(Y)$ [8], where $\mathcal{O}(Y)$ is the set of all open subsets of the space Y.

Then the following definition is more natural.

Definition 3.1. Let $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ be an increasing continuous map between two ordered topological spaces. f is said to be an *ordered − quasihomeomorphism* if $U \longmapsto f^{-1}(U)$ defines a bijection between the set of saturated open (resp. closed) sets of Y and the set of saturated open (resp. closed) sets of X.

Examples 3.2.

(1) $q_X : X \longrightarrow X/\approx$ is an ordered-quasihomeomorphism.

(2) Let $q : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ be an increasing continuous map between two ordered topological spaces.

If $\begin{array}{cccc} \tilde{q}: & (X,\tau) & \longrightarrow & (Y,\gamma) \\ & x & \longmapsto & q(x) \end{array}$ is a quasihomeomorphism then q is an ordered-quasihomeomorphism.

The converse does not hold as shown in the following example:

(3) Let $X = [0, 3]$ with the topology induced by the usual topology of $\mathbb{R}$. Define on X the order $\preceq$ by $G(\preceq) = \{(a, b) : a, b \in \mathbb{Q} \cap X \, and \, a \leq b\} \cup \left\{(\sqrt{5}, x) : x \in (\mathbb{Q} \cap X) \cup \left\{\sqrt{2}, \sqrt{5}\right\}\right\} \cup \left\{(x, \sqrt{5}) : x \in (\mathbb{Q} \cap X) \cup \left\{\sqrt{2}, \sqrt{5}\right\}\right\} \cup \triangle X$.

q_X is an ordered-quasihomeomorphism which is not a quasihomeomorphism: $]0, 2[$ is an open set, since it is not saturate then there is no an open subset V of $X/\approx$ such that $]0, 2[= q_X^{-1}(V)$.

Let us give an important property of ordered-quasihomeomorphisms.

Proposition 3.3. *If $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ are continuous increasing maps between ordered topological spaces and two of the three maps f, g, $g \circ f$ are $ordered-quasihomeomorphisms$, then so is the third one.*

Proof.

- Suppose that f and g are two ordered-quasihomeomorphisms. For any saturated closed subset U of X, let V be the unique saturated closed subset of Y such that $U = f^{-1}(V)$ and let W the unique saturated closed subset in Z such that $V = g^{-1}(W)$. It is clear that W is the unique saturated closed subset of Z such that $U = (g \circ f)^{-1}(W)$. We conclude that $g \circ f$ is an ordered-quasihomeomorphism.
- Suppose that g and $g \circ f$ are ordered-quasihomeomorphisms. Let U be a saturated closed subset in X. Since $g \circ f$ is an ordered-quasihomeomorphism, there exists a unique saturated closed subset W in Z such that $U = (g \circ f)^{-1}(W) = f^{-1}\left(g^{-1}(W)\right)$. Now, $V = g^{-1}(W)$ is a saturated closed subset of Y satisfying $U = f^{-1}(V)$. Let us show that V is the unique saturated closed subset of Y such that $U = f^{-1}(V)$. For this, let V' be a saturated closed subset in Y such that $U = f^{-1}(V')$. There exists a unique saturated closed subset W' in Z such that $V' = g^{-1}(W')$. So
$$(g \circ f)^{-1}(W) = U = f^{-1}(V') = f^{-1}\left(g^{-1}(W')\right) = (g \circ f)^{-1}(W').$$
Finally $W = W'$ and consequently $V = g^{-1}(W) = g^{-1}(W') = V'$.
- Suppose that f and $g \circ f$ are ordered-quasihomeomorphisms. If V is a saturated closed set in Y, $f^{-1}(V)$ is a saturated closed set in X. Then there exits a unique saturated closed set W in Z such that $(g \circ f)^{-1}(W) = f^{-1}(V)$. It is easy to show that W is the unique

saturated closed set in Z such that $V = g^{-1}(W)$. We conclude that f is an ordered-quasihomeomorphism.

□

Now, let'us introduce the following definition:

Definition 3.4. Let $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ be an increasing continuous map between two ordered topological spaces. We say that f is $strongly-increasing$ (for short $s-increasing$) if it satisfies : $x \preceq_{\leq} y$ if and only if $f(x) \preceq_{(Y,\sqsubseteq)} f(y)$ for all $x, y \in X$.

Examples 3.5.

(1) Let $(X, \tau, \leq)$ be an ordered topological space. Then q_X is a s-increasing map.

(2) An increasing map need not to be s-increasing map. Indeed, take $(X, \tau, \leq)$ of the example in 3.2 (3) and f the following map.

$$\begin{array}{cccc} f: & [0,3] & \longrightarrow & [0,3] \\ & x & \longmapsto & 0 \end{array}$$

Clearly for any $\alpha \in [0,3] \setminus \left(\mathbb{Q} \cup \left\{\sqrt{2}, \sqrt{5}\right\}\right)$ we have $f(\alpha) \preceq_{\leq} f(0)$ but $\alpha \not\preceq_{\leq} 0$.

In order to give the main result of this section, we introduce the following definitions.

Definitions 3.6. Let $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ be an increasing continuous map.

(1) f is said to be $T-injective$ (or $T-one-to-one$) if, for each x, y in X : if there exists a monotone open subset of X which contains one of this points but not the other, then, the points $f(x), f(y)$ of Y, can be separated by a monotone open subset of Y.

(2) f is said to be $T-surjective$ (or $T-onto$) if, for each point $y \in Y$, there exists $x \in X$ such that we can not separate y and $f(x)$ by a monotone open subset of Y.

(3) f is said to be $T-bijective$ if it is both $T-injective$ and $T-surjective$.

Examples 3.7.

(1) Every onto continuous increasing map is T-onto.

(2) A T-onto map need not be onto as shown the following example : Let $X = \{0, 1, 2\}$ with the topology $\tau_X = \{\varnothing, X, \{0, 2\}, \{1\}\}$ and the order $\leq_X$ defined by his graph $G(\leq_X) = \{(0,0), (0,1), (0,2), (1,1), (1,2), (2,2)\}$. The map $f : (X, \tau_X, \leq_X) \longrightarrow (X, \tau_X, \leq_X)$ such that $f(X) = \{0\}$ is T-onto but not onto.

(3) A T-one-to-one map need not be one-to-one : $q_X : (X, \tau_X, \leq_X) \longrightarrow \left({}^{X}\!/\!\approx, {}^{\tau_X}\!/\!\approx, \leq_X^0\right)$ is T-one-to-one but not one-to-one.

(4) A one-to-one map need not be T-one-to-one : Let τ_d the discrete topology on X. Then the map $f : (X, \tau_d, \leq_X) \longrightarrow (X, \tau_X, \leq_X)$ defined by $f(x) = x$ for all $x \in X$ is a one-to-one map but not T-one-to-one.

Before giving the main result of this section we need a lemma:

Lemma 3.8. *Let $f : (X,\tau,\leq) \longrightarrow (Y,\gamma,\sqsubseteq)$ be an increasing continuous map. Then the following properties hold:*

(1) *f is T-injective if and only if $\overset{\approx}{f}$ is injective.*
(2) *f is T-surjective if and only if $\overset{\approx}{f}$ is surjective.*
(3) *f is T-bijective if and only if $\overset{\approx}{f}$ is bijective.*

Proof.

(1)
- Suppose that $\overset{\approx}{f}$ is injective : Let $x, y \in X$. If we can separate x and y by a monotone open subset of X then $q_X(x) \neq q_X(y)$. Since $\overset{\approx}{f}$ is injective then $\overset{\approx}{f}(q_X(x)) \neq \overset{\approx}{f}(q_X(y))$ which means $q_Y(f(x)) \neq q_Y(f(y))$. Therefore, we can separate $f(x)$ and $f(y)$ by a monotone open subset of Y.
- Conversely, suppose that f is T-injective : Let $x, y \in X$ be such that $q_X(x) \neq q_X(y)$ which means that we can separate x and y by one monotone open subset of X. Since f is T-injective we can separate $f(x)$ and $f(y)$ by one monotone open subset of Y which means that $q_Y(f(x)) \neq q_Y(f(y))$ and then $\overset{\approx}{f}(q_X(x)) \neq \overset{\approx}{f}(q_X(y))$.

(2)
- Suppose that $\overset{\approx}{f}$ is surjective : If $y \in Y$, since $\overset{\approx}{f}$ is a surjective map there exists $x \in X$ such that $\overset{\approx}{f}(q_X(x)) = q_Y(y)$. Thus, we have $q_Y(f(x)) = q_Y(y)$ and we can not separate $f(x)$ and y by a monotone open subset of Y.
- Conversely, suppose that f is T-onto. If we can't separate $f(x)$ and y ($x \in X$, $y \in Y$) then we have $\overset{\approx}{f}(q_X(x)) = q_Y(y)$, and we conclude that $\overset{\approx}{f}$ is an onto map.

(3) An immediate consequence of (1) and (2).

□

Now, we are in a position to give the main result of this section.

Theorem 3.9. *Let $f : (X,\tau,\leq) \longrightarrow (Y,\gamma,\sqsubseteq)$ be an increasing continuous map between two ordered topological spaces. Then the following statements are equivalent:*

(1) *$\overset{\approx}{f}$ is an isomorphism;*
(2) *f satisfies the following properties.*
   (i) *f is s-increasing.*
   (ii) *f is T-onto.*
   (iii) *f is ordered-quasihomeomorphism.*

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$$\begin{array}{ccc} (X,\tau,\preceq) & \xrightarrow{\ f\ } & (Y,\gamma,\sqsubseteq) \\ {\scriptstyle q_X}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle q_Y} \\ \left({}^{X}\!/\!\approx, {}^{\tau}\!/\!\approx, \leq^0\right) & \xrightarrow[\overset{\approx}{f}]{} & \left({}^{Y}\!/\!\approx, {}^{\gamma}\!/\!\approx, \sqsubseteq^0\right) \end{array}$$

Proof.
(1) $\Rightarrow$ (2)

- By Lemma 3.8, f is T-onto .
- f is s-increasing.
  Since $\overset{\approx}{f}$ is an isomorphism, then $q_X(x) \leq^0 q_X(y)$ if and only if $\overset{\approx}{f}(q_X(x)) \sqsubseteq^0 \overset{\approx}{f}(q_X(y))$ which means that $f(q_Y(x)) \sqsubseteq^0 f(q_Y(y))$. Now, by Remarks 2.6, we can see that $x \preceq_{\leq} y$ if and only if $f(x) \preceq_{\sqsubseteq} f(y)$.
- By Proposition 3.3 and Example 3.2 it's clear that f is an ordered-quasihomeomorphism.

(2) $\Rightarrow$ (1)

- According to Lemma 3.8, $\overset{\approx}{f}$ is a surjective map.
- $\overset{\approx}{f}$ is injective.
  By Lemma 3.8, it is sufficient to show that f is T-one-to-one. To do this result, let $x, y \in X$ and U an open monotone neighborhood of x such that $y \notin U$. Since f is an ordered-quasihomemorphism, there exists a saturated open subset V of Y such that $f^{-1}(V) = U$. Let us show that V is monotone.
  Without loss of generality we can suppose U increasing.
  Let $a, b \in Y$ such that $a \in V$ and $a \sqsubseteq b$.
  Since f is T-onto, there exists $\alpha \in U$ and $\beta \in X$ such that $T(f(\alpha)) = T(a)$ and $T(f(\beta)) = T(b)$.
  Now, we can see that $f(\alpha) \preceq_{\sqsubseteq} f(\beta)$ and thus $\alpha \preceq_{\leq} \beta$. As U is increasing we have $\beta \in U$. Therefore $f(\beta) \in V$ and $b \in V$.
- $\overset{\approx}{f}^{-1}$ is increasing.
  Let $y, y' \in Y$ such that $q_Y(y) \sqsubseteq^0 q_Y(y')$. Since f is T-onto there exist $x, x' \in X$ such that $T(f(x)) = T(y)$ and $T(f(x')) = T(y')$. Then, we have $q_Y(f(x)) = q_Y(y)$, we have also $q_Y(y) = q_Y(f(x)) = \overset{\approx}{f}(q_X(x))$ so that $\overset{\approx}{f}^{-1}(q_Y(y)) = q_X(x)$. By the same way, we have $\overset{\approx}{f}^{-1}(q_Y(y')) = q_X(x')$. Since $q_Y(y) \sqsubseteq^0 q_Y(y')$ we have $q_Y(f(x)) \sqsubseteq^0 q_Y(f(x'))$ which means that $f(x) \preceq_{\sqsubseteq} f(x')$. Now, according the fact

that f is s-increasing, we have $x \preceq_{\leq} x'$ which is equivalent to $q_X(x) \leq^0 q_X(x')$ and finally $\overset{\approx}{f}^{-1}(q_Y(y)) \leq^0 \overset{\approx}{f}^{-1}(q_Y(y'))$.

- Now, let we show that $\overset{\approx}{f}$ is an open map.

  Let U be an open set of ${}^{X}/{\approx}$. Since $q_X^{-1}(U)$ is an open saturated subset of X and f is an ordered-quasihomeomorphism then there exist a saturated open subset V of Y such that $f^{-1}(V) = q_X^{-1}(U) = f^{-1}\left(q_Y^{-1}\left(\overset{\approx}{f}(U)\right)\right)$. Let us show that $q_Y^{-1}\left(\overset{\approx}{f}(U)\right) = V$.

  Let $y \in V$, since f is a T-onto map there exists $x \in X$ such that $T(f(x)) = T(y)$. By saturation of V, $f(x) \in V$ and consequently $x \in f^{-1}(V) = q_X^{-1}(U) = q_X^{-1}\left(\overset{\approx}{f}^{-1}\left(\overset{\approx}{f}(U)\right)\right) = f^{-1}\left(q_Y^{-1}\left(\overset{\approx}{f}(U)\right)\right)$, and thus, $f(x) \in q_Y^{-1}\left(\overset{\approx}{f}(U)\right)$. Now the saturation of $q_Y^{-1}\left(\overset{\approx}{f}(U)\right)$ shows that $y \in q_Y^{-1}\left(\overset{\approx}{f}(U)\right)$. We conclude that $V \subseteq q_Y^{-1}\left(\overset{\approx}{f}(U)\right)$. The second inclusion is proved similarly.

  Therefore $\overset{\approx}{f}(U)$ is an open subset of ${}^{Y}/{\approx}$.

Finally, $\overset{\approx}{f}$ is a bijective open morphism with its inverse $\overset{\approx}{f}^{-1}$ is increasing; so that $\overset{\approx}{f}$ is an isomorphism. □

We close this paper by giving an immediate consequence of this theorem.

Corollary 3.10 ([2, Theorem 2.4])**.** *Let $q : (X, \tau) \longrightarrow (Y, \gamma)$ be a continuous map. Then the following statements are equivalent:*

(i) *q is a topologically onto quasihomeomorphism;*
(ii) *$T_0(q)$ is a homeomorphism.*

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References

[1] A. Ayache, O. Echi, *The envelope of a subcategory in Topology and group theory*, Int. J. Math. Math. Sci. **21** (2005), 3787–3404.

[2] K. Belaid, O. Echi and S. Lazaar, $T_{(\alpha,\beta)}$*-spaces and the Wallman compactification*, Int. J. Math. Math. Sc. **68** (2004), 3717–3735.

[3] C. Casacuberta, A. Frei and G. C. Tan, *Extending localization functors*, J. Pure Appl. Algebra **103** (1995), 149–165.

[4] A. Deleanu, A. Frei, and P. Hilton, *Generalized Adams completion*, Cah. Topologie Géom. Différ. Catég. **15** (1974), 61–82.

[5] R. El Bashir and J. Velebil, *Simultaneously reflective and coreflective subcategories of presheaves*, Theory Appl. Categ. **10** (2002), 410–423.

[6] A. Frei, *On completion and shape*, Bol. Soc. Brasil. Mat. **5** (1974), 147–159.

[7] P. J. Freyd and G. M. Kelly, *Categories of continuous functors (I)*, J. Pure Appl. Algebra, **2** (1972), 169–191.

[8] A. Grothendieck, J. Dieudonné, *Eléments de géométrie algébrique*, Springer-Verlag (1971).

[9] J. M. Harvey, *Reflective subcategories*, Ill. J. Math. **29** (1985), 365–369.

[10] H. Herrlich and G. Strecker, *Categorical topology–Its origins as exemplified by the unfolding of the theory of topological reflections and coreflections before 1971*, in Handbook of the History of General Topology, C.E. Aull and R. Lowen (eds.), Volume 1, Kluwer Academic Publishers, (1997), 255–341.

[11] H. Herrlich and G. Strecker, *H-closed spaces and reflective subcategories*, Math. Ann. **177** (1968), 302–309.

[12] H-P. A. Künzi and T. A. Richmond, T_i*-ordered reflections*, Appl. Gen. Topol. **6**, no. 2 (2005), 207–216.

[13] H-P. A. Künzi, A. E. Mccluskey and T. A. Richmond, *Ordered separation axioms and the Wallman ordered compactification*, Publ. Math. Debrecen **73/3–4** (2008), 361–377.

[14] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Math. vol. 5, Springer-Verlag, New York, (1971).

[15] W. Tholen, *Reflective subcategories*, Topology Appl. **27** (1987), 201–212.

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