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Some results on best proximity pair theorems

P. S. Srinivasan and P. Veeramani

Abstract. Best proximity pair theorems are considered to expound the sufficient conditions that ensure the existence of an element $x_\circ \in A$, such that

$$d(x_\circ, Tx_\circ) = d(A, B)$$

where $T\colon A \to 2^B$ is a multifunction defined on suitable subsets A and B of a normed linear space E. The purpose of this paper is to obtain best proximity pair theorems directly without using any multivalued fixed point theorem. In fact, the well known Kakutani's fixed point theorem is obtained as a corollary to the main result of this paper.

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1. Introduction

Let T be a multifunction from A to B where A and B are non-empty subsets of a normed linear space. Best proximity pair theorem analyzes the conditions under which the problem of minimizing the real valued function $x \to d(x, Tx)$ has a solution. It is evident that

$$d(x, Tx) \geq d(A, B) \quad \text{for all } x \in A.$$

Therefore, a nice solution to the above optimization problem will be one for which the value $d(A, B)$ is attained.

Consider the fixed point equation $Tx = x$ where T is a non-self operator. If this equation does not have a solution then the next attempt is to find an element x in a suitable space such that x is close to Tx in some sense. In fact, the "Best approximation pair theorems" and "Best proximity pair theorems" are pertinent to be explored in this direction. In the setting of a normed linear space E if T is a mapping with domain A, then a best approximation theorem provides sufficient conditions that ascertain the existence of an element $x_\circ$, known as best approximant, such that

$$d(x_\circ, Tx_\circ) = d(Tx_\circ, A)$$

where $d(X,Y) := \mathrm{Inf}\{\|x-y\| : x \in X \text{ and } y \in Y\}$ for any non-empty subsets X and Y of the space E. A classical best approximation theorem, due to Ky Fan [3], states that if K is a non-empty compact convex subset of a Hausdorff locally convex topological vector space E with a continuous seminorm p and $T\colon K \longrightarrow E$ is a single valued continuous map, then there exists an element $x_\circ \in K$ such that

$$p(x_\circ - Tx_\circ) = d(Tx_\circ, K)$$

Later, Reich [7] has weakened the compactness condition and thereby obtained the generalization of the above theorem. This result has been further extended by Sehgal and Singh [11], to the one for continuous multifunctions. Further, they have also proved a generalization of a best approximation theorem due to Prolla [6]. The authors Vetrivel, Veeramani and Bhattacharyya [15] have established the existence of a best approximant for continuous Kakutani factorizable multifunctions which unifies and generalizes the known results on best approximations.

On the other hand, even though a best approximation theorem guarantees the existence of an approximate solution, it is contemplated to find an approximate solution which is optimal. The best proximity pair theorem sheds light in this direction. Indeed, a best proximity pair theorem due to Sadiq Basha and Veeramani [8] provides sufficient conditions that ensure the existence of an element $x_\circ \in A$, such that

$$d(x_\circ, Tx_\circ) = d(A,B)$$

where $T\colon A \longrightarrow 2^B$ is a Kakutani factorizable multifunction defined on suitable subsets A and B of a locally convex topological vector space E. The pair $(x_\circ, Tx_\circ)$ is called a *best proximity pair* of T. Because of the fact that

$$d(x,\ Tx) \geq d(Tx,\ A) \geq d(A,\ B), \quad \text{for all } x \in A,$$

an element x_o satisfying the conclusion of a best proximity pair theorem is a best approximant but the refinement of the closeness between x_o and its image Tx_o is demanded.

Apart from the purpose of seeking an approximate solution which is optimal, these best proximity pair theorems also blend the results on best approximations and proximal points of a pair of sets considered by the authors Beer and Pai [1], Sahney and Singh [10], and Xu [16], of providing sufficient conditions for the non-emptiness of the set

$$\mathrm{Prox}(A,B) := \{(a,b) \in A \times B : d(a,b) = d(A,B)\}.$$

Best proximity pair theorems obtained by Sadiq Basha and Veeramani [8], [9] hinges on Lassonde's multivalued fixed point theorem. The purpose of this paper is to elicit best proximity pair theorems without using Lassonde's theorem. Indeed, the main best proximity pair theorem obtained in this paper does not employ any multivalued fixed point theorems in the proof.

2. Preliminaries

Let X and Y be non-empty sets. A *multivalued map* or *multifunction* T from X to Y denoted by $T\colon X \to 2^Y$, is defined to be a function which assigns to each element of $x \in X$, a non-empty subset Tx of Y. Fixed points of $T\colon X \to 2^X$ will be the points $x \in X$ such that $x \in Tx$.

For further discussion, let X and Y be any two normed linear spaces.

Let $T\colon X \to 2^Y$ be a multivalued map. T is said to be *upper semi-continuous* (resp. *lower semi-continuous*) if $T^{-1}(A) = \{x \in X : T(x) \cap A \neq \varnothing\}$ is closed (resp. open) in X whenever A is a closed (resp. open) subset of Y. Also, T is said to be continuous if it is both lower semi-continuous and upper semi-continuous.

A multifunction $T\colon X \to 2^Y$ is said to have compact and convex values if for each $x \in X$, $T(x)$ is a compact and convex subset of Y. Also, T is said to be a *compact multifunction* if $T(X)$ is a compact subset of Y.

A multifunction $T\colon X \to 2^Y$ is said to be closed if $Gr(T) := \{(x, y) : x \in X \text{ and } y \in Tx\}$ is a closed subset of $X \times Y$.

It is known that if T is an upper semi-continuous multifunction with compact values, then $T(K)$ is compact whenever K is a compact subset of X. It is a noteworthy fact that the composition of two convex valued maps need not be convex valued.

A multivalued map $T\colon X \longrightarrow 2^Y$ is said to be a *Kakutani multifunction*[5] if the following conditions are satisfied

(i) T is upper semi-continuous and
(ii) Either T is singleton or for each $x \in X$ Tx is non-empty, compact and convex (in which case Y is assumed to be a convex set).

The collection of all Kakutani multifunctions from X to Y is denoted by $\mathcal{K}(X, Y)$.

A multifunction is said to be a *Kakutani factorizable multifunction* [5] if the multifunction can be expressed as a composition of finitely many Kakutani multifunctions.

The class of all Kakutani factorizable multifunctions from X to Y is denoted by $\mathcal{K}_C(X, Y)$.

A non-empty set A of X is said to be *approximately compact* if for each y in X and each sequence $\{x_n\}$ in A satisfying the condition that $\|x_n - y\| \longrightarrow d(y, A)$, there is a subsequence of $\{x_n\}$ converging to an element of A.

Let A be any non-empty subset of X. Then $P_A\colon X \longrightarrow 2^A$ defined by

$$P_A(x) = \{a \in A : \|a - x\| = d(x, A)\}$$

is the set of all best approximations in A to any element $x \in X$.

It is known that if A is an approximately compact convex subset of X, $P_A(x)$ is a non-empty compact convex subset of A and the multivalued map P_A is upper semi-continuous on X.

Let A and B be any two non-empty subsets of a normed linear space. The following notions are also recalled.

$$\begin{aligned} d(A,B) &:= \operatorname{Inf}\{d(a,b) : a \in A \text{ and } b \in B\} \\ A_\circ &:= \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\} \\ B_\circ &:= \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\} \end{aligned}$$

If $A = \{x\}$, then $d(A,B)$ is written as $d(x,B)$. Also, if $A = \{x\}$ and $B = \{y\}$, then $d(x,y)$ denotes $d(A,B)$ which is precisely $\|x - y\|$.

3. Main results

This section is devoted to the principal results on best proximity pair theorems.

The proof of the main theorem requires the following lemma:

Lemma 3.1. *Let A, B, C and D be non-empty closed subsets of a normed linear space E. Let $T\colon A \to 2^B$ be an upper semi-continuous compact multifunction with closed values. Further, let $F\colon C \to 2^D$ be a compact and closed multifunction. Then the set*

$$K = \{(x,y) \in A \times C : d(Tx, Fy) = d(B,D)\}$$

is closed in $A \times C$

Proof. Let $(x_n, y_n) \in K$ such that $x_n \to x$ and $y_n \to y$. This implies that

$$d(Tx_n, Fy_n) = d(B,D).$$

As the multifunctions T and F are compact valued, it is possible to choose for every n, $a_n \in Tx_n$ and $b_n \in Fy_n$ such that

$$d(a_n, b_n) = d(Tx_n, Fy_n).$$

Hence

$$d(a_n, b_n) = d(B,D).$$

Since the multifunctions T and F are compact, without loss of generality, it may be assumed that $a_n \to a$ and $b_n \to b$. From the above facts, it is easy to see that

$$d(a,b) = d(B,D).$$

It follows from the upper semi-continuity of T, that $a \in Tx$. Also, as F is a closed multifunction, $b \in Fy$. Now,

$$\begin{aligned} d(Tx, Fy) &\leq d(a,b) \\ &= d(B,D). \end{aligned}$$

Therefore, $(x,y) \in K$ and hence K is closed. □

The following theorem is the first in the sequel of obtaining best proximity pair theorems. The proof does not subsume any multivalued fixed point theorem. The crux in the proof is the technique of simplicial approximation carried out in the lines of [5].

Theorem 3.2. *Let X be a simplex and let Y and Z be non-empty closed convex sets in a finite dimensional space E. Let $\Lambda\colon X \to 2^Z$ be a compact and closed multifunction. Then the following statements are equivalent.*

(i) *For every $f \in \mathcal{C}(X,Y)$, there exists $x \in X$ such that $d(fx, \Lambda x) = d(Y,Z)$.*

(ii) *For every $F \in \mathcal{K}(X,Y)$, there exists $x \in X$ such that $d(Fx, \Lambda x) = d(Y,Z)$.*

Proof. Suppose that (i) holds. It is proved that (ii) holds using the same construction as that of Lassonde [5].

For $p = 1, 2, \ldots$, let Σ^p be a simplicial subdivision of X of mesh size lower than $1/p$. Let $a_o^p, \cdots, a_{m_p}^p$ be the vertices of Σ^p. Also, let $\lambda_o^p, \cdots, \lambda_{m_p}^p$ be the co-ordinate maps associated with those vertices. It follows that, each point $x \in X$ can be uniquely written as $x = \sum_{i=0}^{m_p} \lambda_i^p(x) a_i^p$. For each vertex a_i^p of Σ^p a point b_i^p in Fa_i^p is chosen. With this choice, let a single valued function $f^p\colon X \to Y$ be defined as $f^p(x) = \sum_{i=0}^{m_p} \lambda_i^p(x) b_i^p$. Evidently, f^p is continuous. Now, by property (i), it follows that

$$\text{(3.1)} \qquad \text{for every } p, \text{ there exists } x^p \in X \text{ such that } d(f^p x^p, \Lambda x^p) = d(Y,Z).$$

Let the dimension of the simplex X be n. Let B denote the unit ball of the Euclidean space spanned by X. Let $a_{i_o}^p, \cdots, a_{i_n}^p$ be the vertices of any n-simplex of Σ^p containing x^p. Then, $a_{i_k}^p \in x^p + (1/p)B$ for each $k = 0, \cdots, n$, $\lambda_i^p(x^p) = 0$ for all $i \notin \{i_o, \cdots, i_n\}$ and $\lambda^p(x^p) := \left(\lambda_{i_o}^p(x^p), \cdots, \lambda_{i_n}^p(x^p)\right) \in \Lambda_n$, where

$$\Lambda_n = \left\{ (\lambda_o, \cdots, \lambda_n) \in \mathbf{R}^{n+1} : \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

Let the multifunction $T\colon \Lambda_n \times X^{n+1} \to 2^Y$ be defined as

$$T(\lambda, a_o, \cdots, a_n) = \sum_{i=0}^{n} \lambda_i F a_i.$$

The following fact is immediate

$$\text{(3.2)} \qquad f^p(x^p) = \sum_{k=0}^{n} \lambda_{i_k}^p(x^p) b_{i_k}^p \in T\left(\lambda^p(x^p), a_{i_o}^p, \cdots, a_{i_n}^p\right).$$

Now, from 3.1 for every p, there exist x^p in X, $\lambda^p \in \Lambda_n$ and $(a_o^p, \cdots, a_n^p) \in X^{n+1}$ such that $a_i^p \in x^p + (1/p)B$ for each $i = 0, \cdots, n$ and

$$\begin{aligned} d(T\left(\lambda^p, a_o^p, \cdots, a_n^p\right), \Lambda x^p) &\leq d(f^p x^p, \Lambda x^p) && \text{(by 3.2)} \\ &= d(Y,Z). && \text{(by 3.1)} \end{aligned}$$

As $T\left(\lambda^p, a_o^p, \cdots, a_n^p\right) \subseteq Y$ and $\Lambda x^p \subseteq Z$,

$$d(Y,Z) \leq d(T\left(\lambda^p, a_o^p, \cdots, a_n^p\right), \Lambda x^p).$$

Therefore, for every p, there exists x^p in X, $\lambda^p \in \Lambda_n$ and $(a_o^p, \cdots, a_n^p) \in X^{n+1}$ such that $a_i^p \in x^p + (1/p)B$ for each $i = 0, \cdots, n$ and

$$d\left(T\left(\lambda^p, a_o^p, \cdots, a_n^p\right), \Lambda x^p\right) = d(Y, Z). \tag{3.3}$$

Now, T is compact-valued upper semi-continuous, since it can be written as gG, where the multifunction $G\colon \Lambda_n \times X^{n+1} \to 2^{\Lambda_n \times Y^{n+1}}$ defined as

$$G(\lambda, a_o, \cdots, a_n) = \{\lambda\} \times Fa_o \times \cdots \times Fa_n$$

is compact and upper semi-continuous with closed values and the single valued function $g\colon \Lambda_n \times Y^{n+1} \to Y$ given by

$$g(\lambda, b_o, \cdots, b_n) = \sum_{i=o}^{n} \lambda_i b_i$$

is continuous since Y is convex.

Consider the following set

$$H := \{(\lambda, a_o, \cdots, a_n, x) \in \Lambda_n \times X^{n+1} \times X : d\left(T(\lambda, a_o, \cdots, a_n), \Lambda x\right) = d(Y, Z)\}.$$

By Lemma 3.1, H is closed in the compact space $\Lambda_n \times X^{n+1} \times X$.

Without loss of generality, assume that the sequence $\{x^p\}$ converges to a point $\hat{x} \in X$ and the sequence $\{\lambda^p\}$ to a point $\hat{\lambda} \in \Lambda_n$ as p tends to infinity. From 3.3, it follows that the sequence $\{a_i^p\}$ converges to $\hat{x}$ for each $i = 0, \cdots, n$. Since for every p, $(\lambda^p, a_o^p, \cdots, a_n^p, x^p) \in H$, and H is closed, $(\hat{\lambda}, \hat{x}, \cdots, \hat{x}) \in H$. Further, this implies that

$$d(T(\hat{\lambda}, \hat{x}, \cdots, \hat{x}), \Lambda\hat{x}) = d(Y, Z).$$

But,

$$\begin{aligned} T(\hat{\lambda}, \hat{x}, \cdots, \hat{x}) &= \sum_{i=0}^{n} \lambda_i F\hat{x} \\ &\subseteq F\hat{x}, \quad \text{since } F\hat{x} \text{ is convex.} \end{aligned}$$

Therefore,

$$\begin{aligned} d(F\hat{x}, \Lambda\hat{x}) &\leq d(T(\hat{\lambda}, \hat{x}, \cdots, \hat{x}), \Lambda\hat{x}) \\ &= d(Y, Z). \end{aligned}$$

The condition (ii) implying (i) is immediate as $\mathcal{C}(X, Y) \subseteq \mathcal{K}(X, Y)$. □

By choosing $Z = X$ and Λ to be the identity function in Theorem 3.2, the following corollary is immediate.

Corollary 3.3. *Let X be a simplex and Y be a non-empty compact convex set in a finite dimensional space E.*

Then the following statements are equivalent.

(i) *For every $f \in \mathcal{C}(X, Y)$, there exists $x \in X$ such that $d(x, fx) = d(X, Y)$.*
(ii) *For every $F \in \mathcal{K}(X, Y)$, there exists $x \in X$ such that $d(x, Fx) = d(X, Y)$.*

Remark 3.4. The above corollary contains the multivalued fixed point theorem due to Kakutani [4]. This follows by choosing $Y = X$ in the above corollary and noting the fact that condition (i) always holds by Brouwer's fixed point theorem [2].

The following corollary contains simplex analogue of a best proximity pair theorem due to Sadiq Basha and Veeramani [8].

Corollary 3.5. *Let A be a simplex and B be a non-empty compact convex set in a finite dimensional space E such that A_0 is also a simplex.*

Then the two following equivalent statements hold.

(i) *For every $f \in \mathcal{C}(A, B)$ with $f(A_0) \subseteq B_0$, there exists $x \in A$ such that $d(x, fx) = d(A, B)$.*

(ii) *For every $F \in \mathcal{K}(A, B)$ with $F(A_0) \subseteq B_0$, there exists $x \in A$ such that $d(x, Fx) = d(A, B)$.*

Proof. Choosing $X = A_0$ and $Y = B_0$ in the above corollary and noting the fact $d(A, B) = d(A_0, B_0)$, the proof of (i) being equivalent to (ii) follows.

It remains to show the validity of the statement (i). For this, consider the multifunction $P_A \circ f \colon A_0 \to 2^{A_0}$, where $f \in \mathcal{C}(A, B)$ with $f(A_0) \subseteq B_0$. As $P_A \circ f \in \mathcal{K}(A_0, A_0)$, by Remark 3.4 (or Kakutani's fixed point theorem) there exists $x \in A_0$ such that $x \in (P_A \circ f)x$. That is $d(x, fx) = d(A, fx)$. Since $f(A_0) \subseteq B_0$ it is clear that $d(A, fx) = d(A, B)$. Therefore $d(x, fx) = d(A, B)$. Therefore statement (i) holds.

Since (i) is equivalent to (ii), this completes the proof of the corollary. □

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P.S. Srinivasan and P. Veeramani
Department of Mathematics
Indian Institute of Technology
Madras, Chennai
600 036, India

E-mail address: pvmani@iitm.ac.in