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Large and small sets with respect to homomorphisms and products of groups

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Abstract. We study the behaviour of large, small and medium subsets with respect to homomorphisms and products of groups. Then we introduce the definition of a P-small set in abelian groups and we investigate the relations between this kind of smallness and the previous one, giving some examples that distinguish them.

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1. Introduction

The notion of a large subset of a group, even if with different names, has been deeply investigated in the recent past (see for example [F54], [CR54], [EK72], [DPS89]). It was then natural to consider a dual notion of smallness, and different definitions has been proposed for this purpose. In the first part of this paper we adopt the definition given in [BM99] and we study the behaviour of large, small and medium subsets with respect to homomorphisms and products of groups. In the second part we consider another definition of smallness introduced by I. Prodanov in [DPS89] and we investigate the relations between the two kind of smallness.

1.1. **Preliminaries.** The basic definitions and propositions from which we start can be found in [BM99]; we recall here them for clearness.

Definition 1.1. A subset X of a group G is said to be:

i) *large* if there exists a finite subset $F \subseteq G$ such that $FX = G = XF$;
ii) *small* if for every finite subset $F \subseteq G$ the subset $G \setminus FXF$ is large;
iii) *medium* if it is neither large nor small.

To avoid trivialities, in this paper all groups are infinite. First of all we want to establish an equivalence between the previous definition of smallness and a slightly different one, that allows us to simplify some proofs.

Definition 1.2. We call a subset S of a group G *asymmetrically small* if for every finite subset $K \subseteq G$ both $G \setminus KS$ and $G \setminus SK$ are large.

Proposition 1.3. *A subset S of a group G is small if and only if it is asymmetrically small.*

Proof. If S is small then taking $K' = K \cup \{e\}$, where with e we denote the identity element of G, we have that $G \setminus K'SK' = G \setminus (KSK \cup KS \cup SK \cup S)$ is small, and so *a fortiori* $G \setminus KS$ and $G \setminus SK$ are small.

Conversely we observe that from the proof of [BM99, Theorem 1.4] we immediately have that the difference between a large subset and an asymmetrically small subset is large. So if S is asymmetrically small then $G \setminus S$ is extralarge (see [BM00, Definition 3]). Moreover it is immediate to prove that if S is asymmetrically small then gSg' is still asymmetrically small for every $g, g' \in G$. Then if S is small for every finite subset $K \subseteq S$ we have that $G \setminus KSK = G \setminus (\bigcup_{i,j=1}^{n} k_i S k_j) = \bigcap_{i,j=1}^{n} G \setminus k_i S k_j$ is an intersection of extralarge subsets and so it is large. □

2. Homomorphisms

We need, to simplify the proofs, this simple but useful lemma:

Lemma 2.1. *Let $\varphi : G \to G'$ an homomorphism, $X \subseteq G, Y \subseteq G'$. Then the following hold:*

(i) $\varphi^{-1}(Y) = (\ker \varphi)\varphi^{-1}(Y) = \varphi^{-1}(Y)(\ker \varphi)$.
(ii) $\varphi^{-1}(\varphi(X)) = (\ker \varphi)X = X(\ker \varphi)$.

With respect to epimorphisms we have two similar, and in certain sense dual, results for large and small subsets.

Proposition 2.2. *Let $\varphi : G \to G'$ an epimorphism. Then the following hold:*

(i) *If X is large in G then $\varphi(X)$ is large in G'*
(ii) *If Y is large in G' then $\varphi^{-1}(Y)$ is large in G.*
(iii) *Y is large in G' if and only if $\varphi^{-1}(Y)$ is large in G.*

Proof. (i) Let $F \subseteq G$ be finite such that $FX = G$; then $\varphi(F)\varphi(X) = \varphi(FX) = \varphi(G) = G'$ and since $\varphi(F)$ is finite we have finished.
(ii) Let $F' \subseteq G'$ be finite such that $F'Y = G'$ and $F \subseteq G$ finite that $\varphi(F) = F'$. Then we have $G = \varphi^{-1}(G') = \varphi^{-1}(F'Y) = \varphi^{-1}(\varphi(F)Y) = \varphi^{-1}(\varphi(F))(\varphi^{-1}(Y) = F(\ker \varphi)\varphi^{-1}(Y) = F\varphi^{-1}(Y)$ for Lemma 2.1.
(iii) follows from i) e ii).

□

Proposition 2.3. *Let $\varphi : G \to G'$ an epimorphism. Then the following hold:*

(i) *If $\varphi(X)$ is small in G' then X is small in G.*
(ii) *If $\varphi^{-1}(Y)$ is small in G then Y is small in G'.*
(iii) *Y is small in G' if and only if $\varphi^{-1}(Y)$ is small in G.*

Proof. (i) Let $F \subseteq G$ be finite and consider $G \setminus FX$; we have then $\varphi^{-1}(G' \setminus \varphi(F)\varphi(X)) \subseteq G \setminus FX$ and since $\varphi(F)$ is finite then $G' \setminus \varphi(F)\varphi(X)$ is large in G', and so for Proposition 2.2 $\varphi^{-1}(G' \setminus \varphi(F)\varphi(X))$, and *a fortiori* $G \setminus FX$, are large in G.

(ii) Let $F' \subseteq G'$ finite and $F \subseteq G$ such that $\varphi(F) = F'$. Then $G \setminus F\varphi^{-1}(Y)$ is large in G, and so $\varphi(G \setminus F\varphi^{-1}(Y))$ is large in G' for Proposition 2.2. But $\varphi(G \setminus F\varphi^{-1}(Y)) \subseteq G' \setminus \varphi(F\varphi^{-1}(Y)) = G' \setminus F'Y$, from which we immediately obtain the conclusion: let be $g' = \varphi(g)$ with $g \notin F\varphi^{-1}(Y)$ and suppose that $g' \in \varphi(F\varphi^{-1}(Y)$, that is it exists $h \in F\varphi^{-1}(Y)$ such that $g' = \varphi(h)$. Then we obtain $g \in h \ker\varphi \subseteq F\varphi^{-1}(Y)\ker\varphi = F\varphi^{-1}(Y)$ for Lemma 2.1, a contradiction.

(iii) Follows from (i) and (ii).

□

We notice here that from Proposition 2.3 we immediately obtain as a corollary Proposition 1.7 of [BM99]: if $H \trianglelefteq G$ has infinite index then taking $\varphi : G \to G/H$ the quotient map we have that $\varphi(H) = \{0_{G/H}\}$ is obviously small in G/H and so H is small in G.

Studying the behaviour of this notions with respect to monomorphisms we discovered this interesting result:

Proposition 2.4. *Let G be a group, $H \leq G$ a subgroup. If $S \subseteq H$ is small in H then it is small in G.*

Proof. We prove that for every $F \subseteq G$ finite $G \setminus FS$ is large in G; the proof for $G \setminus SF$ is symmetric. First we suppose that H has finite index in G. Let T be a set of representatives of left cosets of H with $1 \notin H$, such that $G = H \cup TH$ and the union is disjoint. Let now $F \subseteq G$ be finite, and we write $F = F' \cup F''$, where $F' = F \cap H$ and $F'' = F \cap TH$; so $G \setminus FS = G \setminus (F' \cup F'')S = G \setminus F'S \cap G \setminus F''S$. Since $F'S \subseteq H$, we have $G \setminus F'S = (H \setminus F'S) \cup (TH)$; moreover $G \setminus F''S \supseteq G \setminus F''H \supseteq G \setminus TH = H$. Consequently $(G \setminus F'S) \cap (G \setminus F''S) \supseteq ((H \setminus F'S) \cup TH) \cap H = H \setminus F'S$ that is large in G since H has finite index (see [BM99, Proposition 1.9]).

Now we suppose that H has infinite index in G, and we keep the same notations. We observe that $F'' = \{t_1h_1, \dots, t_nh_n\}$ for suitable $t_i \in T$ and $h_i \in H$, and so $F''S = \bigcup_{i=1}^n t_ih_iS$. Then $G \setminus F''S = H \cup T'H \cup (\bigcup_{i=1}^n t_i(H \setminus h_iS))$ where $T' = T \setminus \{t_1, \dots, t_n\}$. So we can write $(G \setminus F'S) \cap (G \setminus F''S) = (H \setminus F'S) \cup T'H \cup (\bigcup_{i=1}^n t_i(H \setminus h_iS))$. Let then $K, K_1, \dots, K_n \subseteq H$ be finite such that $K(H \setminus F'S) = H = (H \setminus F'S)K$ and $K_i(H \setminus h_iS) = H = (H \setminus h_iS)K_i$ for $i = 1, \dots, n$. At this point we notice that for $i = 1, \dots, n$ we have $t_iK_it_i^{-1}(t_i(H \setminus$

$h_iS)) = t_iH$ and that $(t_i(H \setminus h_iS))K_i = t_iH$. So we have:

$$\Big(K\cup\{1\}\cup(\bigcup_{i=1}^{n} t_iK_it_i^{-1})\Big)\Big((G\setminus F'S)\cap(G\setminus F''S)\Big)$$

$$= \Big(K\cup\{1\}\cup(\bigcup_{i=1}^{n} t_iK_it_i^{-1})\Big)\Big((H\setminus F'S)\cup T'H\cup(\bigcup_{i=1}^{n} t_i(H\setminus h_iS))\Big)$$

$$\supseteq \big(H\cup T'H\cup(\bigcup_{i=1}^{n} t_iH)\big) = H\cup TH = G$$

and so $G \setminus FS$ is large from the left. Analogously we obtain:

$$\Big((G\setminus F'S)\cap(G\setminus F''S)\Big)\Big(K\cup\{1\}\cup(\bigcup_{i=1}^{n} K_i)\Big)$$

$$= \Big((H\setminus F'S)\cup T'H\cup(\bigcup_{i=1}^{n} t_i(H\setminus h_iS))\Big)\Big(K\cup\{1\}\cup(\bigcup_{i=1}^{n} K_i)\Big)$$

$$\supseteq \big(H\cup T'H\cup(\bigcup_{i=1}^{n} t_iH)\big) = H\cup TH = G$$

and so $G \setminus FS$ is large from the right.

□

As a consequence we obtain this proposition for monomorphisms:

Proposition 2.5. *Let $\varphi : G \to G'$ be a monomorphism. Then the following hold:*

(i) *If $\varphi(X)$ is large in G' then X is large in G.*
(ii) *If X is small in G then $\varphi(X)$ is small in G'.*

There are simple counterexamples that show that these are the best results we can get for epimorphisms and monomorphisms: for instance if we consider the immersion $\iota : (\mathbb{Z}, +) \hookrightarrow (\mathbb{R}, +)$ clearly $\mathbb{Z}$ is large in itself but it is small in $(\mathbb{R}, +)$. Clearly from the previous results we obtain that these notions, as expected, are invariant for isomorphisms:

Corollary 2.6. *Let $\varphi : G \to G'$be an isomorphism. Then the following hold:*

(i) *X is large in G if and only if $\varphi(X)$ is large in G'*
(ii) *X is small in G if and only if $\varphi(X)$ is small in G'*
(iii) *X is medium in G if and only if $\varphi(X)$ is medium in G'.*

3. Products

Here we consider finite and infinite direct products of groups.

From Proposition 2.2 we straightaway obtain the following:

Proposition 3.1. *Let $(G_i)_{i\in I}$ be a family of groups and $L \subseteq \prod_{i\in I} G_i$ large. If $\pi_i : \prod_{i\in I} G_i \to G_i$ denotes the i-th canonical projection, then $\pi_i(L)$ is large in G_i for every $i \in I$.*

But we can obtain more. In fact we have this nice characterization of largeness in arbitrary products of groups.

Theorem 3.2. *Let $(G_i)_{i\in I}$ a family of groups and $L_i \subseteq G_i$ for every $i \in I$. Then $\prod_{i\in I} L_i$ is large in $\prod_{i\in I} G_i$ if and only if L_i is large in G_i for every $i \in I$ and $L_i = G_i$ except for a finite subset of indexes.*

Proof. $\Rightarrow$) Clearly L_i is large in G_i for every $i \in I$ by Prop. 3.1 By contradiction suppose now that exists $I' \subseteq I$ infinite such that $L_i \neq G_i$ if $i \in I'$ and let's show that then $\prod_{i\in I} L_i$ is not large in $\prod_{i\in I} G_i$. In fact let $F \subseteq \prod_{i\in I} G_i$ be finite, $F = \{f_1, \ldots, f_n\}$; as I' is infinite there are $i_1, \ldots, i_n \in I'$ all distinct, and so we have $L_{i_1} \neq G_{i_1}, \ldots, L_{i_n} \neq G_{i_n}$. Then for every $h = 1, \ldots, n$ let be $g_h \in G_{i_h} \setminus L_{i_h}$, and define $f \in \prod_{i\in I} G_i$ by putting $f(i_h) = f_h(i_h)g_h$ and $f(i) = e_i$, where e_i is the identity element of G_i, if $i \notin \{i_1, \ldots, i_n\}$. Now if $f \in F(\prod_{i\in I} L_i)$ then there exist $\bar{h} \in \{1, \ldots, n\}$ and $l \in \prod_{i\in I} L_i$ such that $f = f_{\bar{h}}\, l$, and so $f(i_{\bar{h}}) = f_{\bar{h}}(i_{\bar{h}})l(i_{\bar{h}})$, that is $f_{\bar{h}}(i_{\bar{h}})g_{\bar{h}} = f_{\bar{h}}(i_{\bar{h}})l(i_{\bar{h}})$, from which $l(i_{\bar{h}}) = g_{\bar{h}} \notin L_{i_{\bar{h}}}$, absurd. Having found for every finite $F \subseteq \prod_{i\in I} G_i$ an element $f \notin F(\prod_{i\in I} L_i)$ we can conclude that $\prod_{i\in I} L_i$ is not large.
$\Leftarrow$) Let $\{i_1, \ldots, i_n\}$ be the subset of indexes for which $L_{i_j} \subsetneq G_{i_j}$. For every $j = 1, \ldots, n$ there exists $F_j \subseteq G_{i_j}$ finite such that $F_j L_{i_j} = G_{i_j} = L_{i_j} F_j$. Thus define $F = \{f \in \prod_{i\in I} G_i : f(i_j) \in F_j \text{ and } f(i) = e_i \text{ if } i \notin \{i_1, \ldots, i_n\}\}$; clearly $|F| = |F_1| \cdots |F_n| < +\infty$. Let's show that $F(\prod_{i\in I} L_i) = \prod_{i\in I} G_i = (\prod_{i\in I} L_i)F$: let $h \in \prod_{i\in I} G_i$ be arbitrarily chosen; for every $j = 1, \ldots, n$ $\;h(i_j) \in G_{i_j}$ and so there exist $f_j \in F_j$ and $l_j \in L_{i_j}$ such that $h(i_j) = f_j l_j$. We then define $f \in \prod_{i\in I} G_i$ by putting $f(i_j) = f_j$ and $f(i) = e_i$ if $i \notin \{i_1, \ldots, i_n\}$, so that $f \in F$, and l by putting $l(i_j) = l_j$ and $l(i) = h(i)$ if $i \notin \{i_1, \ldots, i_n\}$, so that $l \in \prod_{i\in I} L_i$ as for $i \notin \{i_1, \ldots, i_n\}$ $L_i = G_i$. Obviously, $h = fl$ by their definition, and so we have proved $F(\prod_{i\in I} L_i) = \prod_{i\in I} G_i$; symmetrically you obtain $\prod_{i\in I} G_i = (\prod_{i\in I} L_i)F$. □

So we have a strengthening of Prop. 3.1:

Corollary 3.3. *If $L \subseteq \prod_{i\in I} G_i$ is large then $\pi_i(L)$ is large for every $i \in I$ and $\pi_i(L) = G_i$ except for a finite subset of indexes.*

We will see in Prop. 3.7 that even if $\pi_i(L) = G_i$ for every $i \in I$ it is possible that L is small.

For small subsets the situation is again, in a certain sense, dual. A first result is the following:

Proposition 3.4. *Let G_1, G_2 be groups and $S \subseteq G_1 \times G_2$ be such that either $\pi_1(S)$ or $\pi_2(S)$ are small. Then S is small in $G_1 \times G_2$.*

Proof. Suppose for example that $\pi_1(S) = S_1$ is small, and let's show that $S_1 \times G_2$ is small, from which, since $S \subseteq S_1 \times S_2 \subseteq S_1 \times G_2$, we have clearly the proof. Let $K \subseteq G_1 \times G_2$ be finite, and let's consider $(G_1 \times G_2) \setminus K(S_1 \times G_2)K$. Let be $K_1 = \pi_1(K)$; then we have $K(S_1 \times G_2)K = K_1 S_1 K_1 \times G_2$.. Then $(G_1 \times G_2) \setminus K(S_1 \times G_2)K = (G_1 \times G_2) \setminus (K_1 S_1 K_1 \times G_2) = (G_1 \setminus K_1 S_1 K_1) \times$

G_2; as S_1 is small in G_1 then $G_1 \setminus K_1S_1K_1$ is large in G_1, and consequently $(G_1 \setminus K_1S_1K_1) \times G_2$ is large in $G_1 \times G_2$ by Prop. 3.2. □

Corollary 3.5. *Let $(G_i)_{i\in I}$ be a family of groups and $S \subseteq \prod_{i\in I} G_i$ be such that there exists $j \in I$ for which $\pi_j(S)$ is small in G_j; then S is small in $\prod_{i\in I} G_i$. Consequently a product of small subsets is small.*

In order to go on, and to prove some results for medium subsets too, we need to introduce, as in [BM99, Theorem 1.2], the following notion: we call a subset of G of the form gK (respectively Kg) *right circle* (respectively *left circle*) *of center g and radius K*. It is easy then to prove the next lemma:

Lemma 3.6. *Let G be a group, $A \subseteq G$. then:*

(i) *$G \setminus A$ is not large in G if and only if either A contains right circles for any finite radius or A contains left circles for any finite radius.*

(ii) *$G \setminus A$ is large in G if and only if there is $F \subseteq G$ finite such that A does not contain neither left nor right circle of radius F.*

At this point we can prove the following proposition, that gives a general example of a small subsets with full projection for every index:

Proposition 3.7. *Let $(G_i)_{i\in I}$ a family of groups, and for every $i \in I$ let $E_i \subseteq \prod_{i\in I} G_i$ be defined by $E_i = \prod_{j\in I} E_{ij}$, where $E_{ii} = G_i$ and $E_{ij} = \{e_j\}$ for $j \neq i$. Then $S = \bigcup_{i\in I} E_i$ is small in $\prod_{i\in I} G_i$.*

Proof. Let $F \subseteq G = \prod_{i\in I}$ be finite, and let's show that $G \setminus FS$ is large from the right; as you will see the proof for the remaining cases is analogous. We in fact will show that there is $K \subseteq G$ finite such that $FS \not\supseteq gK$ for any $g \in G$, so that we obtain what desired for Lemma 3.6. If $F = \{f_1, \dots, f_n\}$ we choose, for each $i \in I$, $n+1$ distinct elements $x_{i,1}, \dots, x_{i,n+1} \in G_i$; let then be $x_1 = (x_{i,1})_{i\in I}, \dots, x_{n+1} = (x_{i,n+1})_{i\in I}$, and $K = \{x_1, \dots, x_{n+1}\}$. If there exist $g \in G$ such that $gK \subseteq FS = \bigcup_{h=1}^{n} f_hS$, we would find $j_1, j_2 \in \{1, \dots, n+1\}$ distinct and $f_{\bar h} \in F$ such that $gx_{j_1}, gx_{j_2} \in f_{\bar h}S$. So there would be $i_1, i_2 \in I$ such that

$$\begin{cases} gx_{j_1} \in f_{\bar h}E_{i_1} \\ gx_{j_2} \in f_{\bar h}E_{i_2}\ . \end{cases}$$

Hence taking $i \neq i_1, i_2$, we would have $\pi_i(gx_{j_1}) = \pi_i(f_{\bar h}) = \pi_i(gx_{j_2})$, and so $x_{i,j_1} = x_{i,j_2}$, a contradiction. □

If we restrict our attention to abelian groups we obtain a result for medium subsets, and we can learn more about products of small subsets.

Proposition 3.8. *Let G_1, G_2 be abelian groups, $M_1 \subseteq G_1, M_2 \subseteq G_2$ medium respectively in G_1 and G_2. Then $M_1 \times M_2$ is medium in $G_1 \times G_2$.*

Proof. First of all $M_1 \times M_2$ is not large by Prop. 3.1. Let then $K_1 \subseteq G_1, K_2 \subseteq G_2$ be finite such that $G_1 \setminus (M_1 + K_1)$ and $G_2 \setminus (M_2 + K_2)$ are not large. Let be $H \subseteq G_1 \times G_2$ finite, and call H_1, H_2 its projections. Then by Lemma 3.6 there exist $g_1 \in G_1$ and $g_2 \in G_2$ such that $M_1 + K_1 \supseteq g_1 + H_1$ and

$M_2 + K_2 \supseteq g_2 + H_2$. Consequently $(M_1 \times M_2) + (K_1 \times K_2) = (M_1 + K_1) \times (M_2 + K_2) \supseteq (g_1 + H_1) \times (g_2 + H_2) = (g_1, g_2) + (H_1 \times H_2) \supseteq (g_1, g_2) + H$. Therefore $(M_1 \times M_2) + (K_1 \times K_2)$ contains circles for every finite radius, and so $(G_1 \times G_2) \setminus ((M_1 \times M_2) + (K_1 \times K_2))$ is not large, and then $M_1 \times M_2$ is not small. □

Proposition 3.9. *Let G, H be two groups, $M \subseteq G$ medium and $L \subseteq H$ large. Then $M \times L$ is medium in $G \times H$.*

Proof. Clearly $M \times L$ is not large. Let then K be a finite subset of G such that $G \setminus KMK$ is not large, and K' a finite subset of H such that $K'L = H = LK'$. Then $(G \times H) \setminus (K \times K')(M \times L)(K \times K') = (G \setminus KMK') \times H$ that is not large by Prop. 3.1, and so $M \times L$ is not small. □

Corollary 3.10. *A finite product of subsets is small if and only if at least one of the factors is small.*

4. P-small and small subsets

In [DPS89] I. R. Prodanov introduced, for the case of abelian groups, the following definition of small subset:

Definition 4.1. Let G be an abelian group, and $S \subseteq G$. Then we call S *small in the sense of Prodanov (brief. P-small)* if there exists $X = \{x_n\}_{n \in \mathbb{N}} \subseteq G$ countable such that $(S + x_i) \cap (S + x_j) = \varnothing$ if $i \neq j$.

It is then natural to ask what are the relations between this definitions and the previous one. A first result is the following theorem:

Theorem 4.2. *Let G be an abelian group, $S \subseteq G$. If S is P-small then it is small.*

Proof. Let $X = \{x_n\}_{n \in \mathbb{N}}$ be such that the subsets $S + x_i$ are pairwise disjoint. By contradiction suppose now that there exists $F \subseteq G$ finite such that $G \setminus (S + F)$ it is not large. Then by Lemma 3.6 $S + F$ contains circles for any finite radius. If $|F| = n$ we consider the radius $\{-x_1, -x_2, \dots, -x_{n+1}\}$; then there exists $g \in G$ such that $S + F \supseteq g + \{-x_1, -x_2, \dots, -x_{n+1}\}$. So we have that for every $i \in \{1, \dots, n+1\}$ there exist $s_i \in S, f_i \in F$ such that $s_i + f_i = g - x_i$. Since $|F| = n$ we can find i, j such that $f_i = f_j = f$ and so we obtain:

$$\begin{cases} s_i + f = g - x_i \\ s_j + f = g - x_j \end{cases}$$

it follows then $s_i + x_i = g - f = s_j + x_j$, a contradiction. □

The converse is in general not true, as we will show soon. Before we give the natural generalization of the previous definition for non abelian case:

Definition 4.3. Let G be a group, $S \subseteq G$;

(a) S is *left P-small* if there exists $X = \{x_n\}_{n \in \mathbb{N}} \subseteq G$ such that $x_i S \cap x_j S = \varnothing$ if $i \neq j$.

(b) S is *right P-small* if there exists $X = \{x_n\}_{n\in\mathbb{N}} \subseteq G$ such that $Sx_i \cap Sx_j = \varnothing$ if $i \neq j$.

(c) S is *P-small* if it is left and right P-small.

We notice immediately that, unlike the abelian case, P-smallness does not imply smallness:

Example 4.4. Let $F_2 = \langle a, b\rangle$ be the free group of two generators a, b; every $w \in F_2$ can be written in the reduced form $w = y_1 \cdot y_2 \cdots y_k$ where $k \geq 1$ and y_i is a non-trivial power of a or b such that if y_i is a power of a then y_{i-1} and y_{i+1} are powers of b and vice versa. We consider then the following subsets:
$S_1 = \{1\} \cup \{w \in F_2 : y_1, y_k \text{ are powers of } a\}$
$S_2 = \{w \in F_2 : y_1, y_k \text{ are powers of } b\}$
$S_3 = \{w \in F_2 : y_1 \text{ is a power of } a,\ y_k \text{ of } b\}$
$S_4 = \{w \in F_2 : y_1 \text{ is a power of } b,\ y_k \text{ of } a\}$
It is clear that each of them is P-small both on the left and on the right: for example if we consider S_1, the two family of subsets $\{b^n S_1\}_{n\geq 1}$ and $\{S_1 b^n\}_{n\geq 1}$ have pairwise disjoint members. Because $\bigcup_{i=1}^{4} S_i = F_2$ at least one among the S_i must be not small, otherwise we will obtain that F_2 is small by [BM99, Theorem 1.2], and so that subset is P-small but not small.

Remark 4.5. It is not difficult however to prove that every S_i then must be not small: see [DMM01, Example 2.16].

So the idea is to introduce another kind of "strongly" P-smallness that ensures that a subset of a non abelian group that has that property it is also small; you can refer to [DMM01] for such a definition and further developments in this direction.

Now we want to prove what we have announced before:

Proposition 4.6. *Let G be a group, $H \trianglelefteq G$ an infinite normal subgroup of infinite index and $T \subseteq G$ a complete set of representatives of cosets of H . Then $H \cup T$ is small but it is neither left nor right P-small.*

Proof. According to [DMM01, Corollary 2.15] $H \cup T$ is small. We show here that $H \cup T$ is not right P-small (the proof for the left is symmetric). We prove something more: however you choose $x_1, x_2 \in G$ you obtain $(H \cup T)x_1 \cap (H \cup T)x_2 \neq \varnothing$. In fact we can write $x_1 = h_1t_1, x_2 = h_2t_2$ for suitable $h_1, h_2 \in H$, and $t_1, t_2 \in T$. Then if $t_1 = t_2 = t$ we have $(H \cup T)h_1t_1 \cap (H \cup T)h_2t_2 \supseteq Hh_1t_1 \cap Hh_2t_2 = Ht_1 \cap Ht_2 = Ht \neq \varnothing$.

If $t_1 \neq t_2$ there exist $t \in T, t \neq 1$ and $h \in H$ such that $t_1t_2^{-1} = ht$; we can then write $t_1 = htt_2 = hth_2^{-1}h_2t_2$; since H is normal there is $h' \in H$ such that $th_2^{-1} = h't$, so we obtain $t_1 = hh'th_2t_2 = hh'tx_2$. Then $(hh')^{-1}t_1 = tx_2$ and therefore $Ht_1 \cap Tx_2 \neq \varnothing$; since $Ht_1 = Hh_1t_1 = Hx_1$ we again obtain $(H \cup T)x_1 \cap (H \cup T)x_2 \supseteq Hx_1 \cap Tx_2 \neq \varnothing$. □

Corollary 4.7. *Let G be an abelian group, $H \leq G$ an infinite subgroup of infinite index, and $T \subseteq G$ a transversal of H. Then $H \cup T$ is small but not P-small.*

We also have a strengthening of [BM99, Proposition 1.8] in the abelian case:

Proposition 4.8. *Let G be an abelian group, $S \subseteq G$ such that $|S| < |G|$. Then S is P-small in G.*

Proof. Since $|\langle S\rangle| < |G|$, any transversal T of $\langle S\rangle$ must be infinite. So taking a countable subset $T' = \{t_1, t_2, \dots, t_n, \dots\} \subseteq T$ you obtain $\langle S\rangle + t_i \cap \langle S\rangle + t_j = \varnothing$ if $i \neq j$, and so *a fortiori* S is P-small. □

We want to finish this section giving some "concrete" examples distinguishing smallness and P-smallness in the group of integers. The definition of small and large in the semigroup of natural numbers $\mathbb{N}$ are given in [AMM01]. We recall them here:

Definition 4.9. Let be $X \subseteq \mathbb{N}$; then:

(i) X is *large in* $\mathbb{N}$ if there exists $F \subseteq \mathbb{N}$ finite such that $X \pm F \equiv (X + F) \cup (X - F) \supseteq \mathbb{N}$;
(ii) X is *small in* $\mathbb{N}$ if for every $F \subseteq \mathbb{N}$ finite $\mathbb{N} \setminus (X \pm F)$ is large in $\mathbb{N}$.
(iii) X is *medium in* $\mathbb{N}$ if it is neither large nor small in $\mathbb{N}$

We need now the following Lemma:

Lemma 4.10. *Let $X \subseteq \mathbb{Z}$ be symmetric, that is $X = -X$. Then the following hold:*

(i) X is large in $\mathbb{Z}$ if and only if $X \cap \mathbb{N}$ is large in $\mathbb{N}$.
(ii) X is small in $\mathbb{Z}$ if and only if $X \cap \mathbb{N}$ is small in $\mathbb{N}$
(iii) X is medium in $\mathbb{Z}$ if and only if $X \cap \mathbb{N}$ is medium in $\mathbb{N}$

Proof. For simplicity we use the notation $X_+ = X \cap \mathbb{N}$ and $X_- = X \cap -\mathbb{N}$; notice that if $X \subseteq \mathbb{Z}$ is symmetric, then $-X_+ = X_-$.

(i) If $X \subseteq \mathbb{Z}$ symmetric is large there exists $F \subseteq \mathbb{Z}$ finite and symmetric such that $X + F = \mathbb{Z}$. If by contradiction X_+ is not large in $\mathbb{N}$ then $X_+ \pm F_+ \not\supseteq \mathbb{N}$. Then also $(X_+ \pm F_+) \cup -(X_+ \pm F_+) \neq \mathbb{Z}$. But for the symmetry of the subsets we have $(X_+ \pm F_+) \cup -(X_+ \pm F_+) = X + F$, and we get a contradiction.

Conversely if X_+ is large in $\mathbb{N}$ than there exists $F \subseteq \mathbb{N}$ finite such that $X_+ \pm F \supseteq \mathbb{N}$; then also $-(X_+ \pm F) \supseteq -\mathbb{N}$, and so we obtain $\mathbb{Z} = (X_+ \pm F) \cup -(X_+ \pm F) = X + (F \cup -F)$, therefore X is large in $\mathbb{Z}$.

(ii) Suppose that X is small in $\mathbb{Z}$. Then taken $F \subseteq \mathbb{N}$ we want to prove that $\mathbb{N} \setminus (X_+ \pm F)$ is large in $\mathbb{N}$. Consider $F' = F \cup -F$; then $\mathbb{Z} \setminus (X + F')$ is large in $\mathbb{Z}$ for hypothesis; moreover, since $X + F'$ is symmetric, $\mathbb{Z} \setminus (X + F')$ is symmetric too and so $(\mathbb{Z} \setminus (X + F')) \cap \mathbb{N}$ is large in $\mathbb{N}$ for (i); but $(\mathbb{Z} \setminus (X + F')) \cap \mathbb{N} \subseteq \mathbb{N} \setminus (X_+ \pm F)$, so we have finished.

Conversely suppose that X_+ is small in $\mathbb{N}$ and let's prove that X is small in $\mathbb{Z}$. If this was false we could find $F \subseteq \mathbb{Z}$ finite such that $\mathbb{Z} \setminus X + F$ is not large; then also $\mathbb{Z} \setminus X + F'$, where $F' = F \cup -F$ is not large. Since $\mathbb{Z} \setminus (X + F')$ is symmetric, for (i) $(\mathbb{Z} \setminus (X + F')) \cap \mathbb{N}$ is not large in N, and since $(\mathbb{Z} \setminus (X + F')) \cap \mathbb{N} \supseteq \mathbb{N} \setminus (X_+ \pm F'_+)$ we obtain that $\mathbb{N} \setminus (X_+ \pm F'_+)$ is not large in $\mathbb{N}$, against the hypothesis.

(iii) Follows directly from (i) and (ii).

□

As a corollary we immediately obtain:

Corollary 4.11. *Let X be a subset of $\mathbb{Z}$. If $(X - X) \cap \mathbb{N}$ is not large in $\mathbb{N}$ then X is P-small in $\mathbb{Z}$.*

Proof. From [DMM01, Lemma 2.5] if $X - X$ is not large then it is P-small; so the conclusion follows from the previous lemma since $X - X$ is symmetric. □

We can now give a "concrete" example of a P-small subset:

Proposition 4.12. *For every $q \geq 2$ the set $X_q = \{q^k : k \in \mathbb{N}\}$ is P-small in $\mathbb{Z}$.*

Proof. We want to prove that $(X_q - X_q) \cap \mathbb{N}$ is not large in $\mathbb{N}$. First of all we notice that $(X_q - X_q) \cap \mathbb{N} = \{0\} \cup \{q^{k'} - q^k : k, k' \in \mathbb{N}, k' \geq k + 1\} = \{0\} \cup \{q^k(q^t - 1) : k \geq 0, t \geq 1\}$. Suppose now that $(X_q - X_q) \cap \mathbb{N}$ is large in $\mathbb{N}$, so that there exists $M > 0$ that is a superior bound to the differences between consecutives elements of $(X_q - X_q) \cap \mathbb{N}$ (see [AMM01, Proposition 1.1]). Let's take $\bar{k} \in \mathbb{N}$ such that $q^{\bar{k}} > M$ and consider $q^{\bar{k}}(q - 1) \in (X_q - X_q) \cap \mathbb{N}$. Its immediate successor will be $q^{\bar{k}+d}(q^{d'+1} - 1)$ where $d, d' \in \mathbb{N}$ and at least one of them is strictly positive. Therefore we get $q^{\bar{k}+d}(q^{d'+1} - 1) - q^{\bar{k}}(q - 1) = q^{\bar{k}} \cdot x$ with $x \geq 1$, and so $q^{\bar{k}+d}(q^{t+d'} - 1) - q^{\bar{k}}(q^t - 1) \geq q^{\bar{k}} > M$, a contradiction. □

As announced we conclude this section with another example distinguishing P-smallness from smallness. Before we need this two lemmas:

Lemma 4.13. *Let G be an abelian group, $S \subseteq G$. Then S is P-small in G if and only if there exists $X \subseteq G$ countable such that $(S - S) \cap (X - X) = \{0\}$.*

Proof. $\Rightarrow$) Let $X = \{x_i\}_{i \in \mathbb{N}}$ be such that $S + x_i \cap S + x_j = \varnothing$ if $i \neq j$. If $S - S \cap X - X \ni g \neq 0$ then we could find $i \neq j$ such that $x_i - x_j \in S - S$, from which follows that $S + x_i \cap S + x_j \neq \varnothing$, an absurd.

$\Leftarrow$) Taken X as in the hypothesis, if $S + x_i \cap S + x_j \neq \varnothing$ then $x_i - x_j \in S - S$ and so $x_i - x_j = 0$, that is $x_i = x_j$. □

Lemma 4.14. *Let S be a subset of $\mathbb{Z}$. Then if $\mathbb{N} \setminus (S - S) = F$, where F is a finite subset of $\mathbb{N}$, S is not P-small in $\mathbb{Z}$.*

Proof. Suppose that there exists $X \subseteq \mathbb{Z}$ infinite such that $(S - S) \cap (X - X) = \{0\}$; then it must be $(X - X) \cap \mathbb{N} \subseteq F \cup \{0\}$, absurd. □

At this point we can build the desired example:

Example 4.15. We define by induction the following subset $X = \{x_n\}_{n \in \mathbb{N}}$ of $\mathbb{N}$ taking $x_1 = 1$ and $x_{n+1} = x_n + n$. Therefore for every $n \geq 1$ $x_{n+1} - x_n = n$, from which $(X - X) \cap \mathbb{N} = \mathbb{N}$, and so for the previous corollary X is not P-small in $\mathbb{Z}$. Nevertheless $\lim_{n\to+\infty}\{x_{n+1}-x_n\} = \lim_{n\to+\infty} n = +\infty$, and so X is small in $\mathbb{N}$ for [AMM01, Proposition 1.2] and also in $\mathbb{Z}$, since $X \cap -\mathbb{N} = \{0\}$)

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