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Bombay hypertopologies

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Dedicated by the first two authors to Professor S. Naimpally on the occasion of his 70th birthday.

Abstract. Recently it was shown that, in a metric space, the upper Wijsman convergence can be topologized with the introduction of a new far-miss topology. The resulting Wijsman topology is a mixture of the ball topology and the proximal ball topology. It leads easily to the generalized or g-Wijsman topology on the hyperspace of any topological space with a compatible LO-proximity and a cobase (i.e. a family of closed subsets which is closed under finite unions and which contains all singletons). Further generalization involving a topological space with two compatible LO-proximities and a cobase results in a new hypertopology which we call the *Bombay topology*. The *generalized locally finite Bombay topology* includes the known hypertopologies as special cases and moreover it gives birth to many new hypertopologies. We show how it facilitates comparison of any two hypertopologies by proving one simple result of which most of the existing results are easy consequences.

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1. Introduction.

The main purpose of this work is to give a unified treatment to the problem of comparing various hypertopologies with one another. We accomplish this by representing known hypertopologies as special cases of just *one* general hypertopology the *IL- locally finite Bombay hypertopology*. We prove one result, with a simple proof, giving necessary and sufficient conditions for one upper

Bombay topology to be coarser than another. We then show how this one result includes, as special cases, many known results scattered throughout the literature. Moreover, our approach gives simple transparent proofs in comparison with those in the original articles which involve intricate calculations.

Hypertopologies, which were born during the early part of the last century, have proven to be useful in Continua Theory, Topologies on Spaces of Functions, Optimization, Convex Analysis, Game Theory, Differential Equations, Image Analysis and Fractal Geometry, etc. Two early discoveries where:

(a) the Vietoris topology that can be defined in any topological space, and
(b) the Hausdorff metric topology which is available only in a metric space.

After the discovery of uniform spaces by Weil, the second one was generalized to Hausdorff-Bourbaki uniform topology. In a seminal paper, Michael [18] made a detailed study of above topologies in the middle of the last century.

Then in 1966 Wijsman [25], in studying Convex Analysis, introduced a *convergence* for nets of closed subsets of a metric space (X, d) viz:

$$A_n \to A \text{ iff for each } x \in X,\ d(x, A_n) \to d(x, A).$$

The lower part of this convergence is equivalent to convergence in the lower Vietoris topology. Attempts to topologize the upper part of the convergence were only partially successful until recently. The first attempt resulted in *the ball topology* ([1]) wherein a typical neighbourhood of a closed set consists of closed subsets that do not intersect a proper closed ball in the metric space. The next attempt led to *proximal ball topology* ([10]) wherein a typical neighbourhood of a closed set consists of closed sets that are far (w.r.t. the metric proximity) from a proper closed ball. This was more successful since it equalled the Wijsman topology in *nice* metric spaces, which included the normed linear spaces. But the topologization for all metric spaces defied attempts for the past 36 years.

A major part of the literature on hyperspaces consists of comparisons of various hypertopologies with one another. In this, the Wijsman topologies play an important role as the building blocks of many other topologies ([2]). But they are not easy to handle as the Wijsman topology depends strongly on the metric d. For example, even two uniformly equivalent metrics can give rise to unequal Wijsman topologies and non-uniformly equivalent metrics can induce equal Wijman topologies! Moreover, it is not easy to compare objects of different kinds and the absence of topologization of Wijsman topology was a major hurdle. So the literature contains long and complicated proofs involving epsilonetics.

Recently, one of us [20] discovered a simple way of topologizing the upper Wijsman convergence. In this approach, a typical neighbourhood of a closed set A in a metric space (X, d) consists of closed sets that do not intersect a proper closed ball B that is far from A in the metric proximity. With this new approach it is possible to give simple conceptual proofs of results involving comparisons of the Wijsman topologies among themselves and with others. This new way of looking at upper Wijsman topology interprets those results in a simple and direct way. Moreover, it leads to a generalization of the Wijsman topology to the hyperspace of any topological space satisfying some simple conditions.

When it was further scrutinized, it was found that the above generalization of the upper Wijsman topology can be further generalized to represent all known upper hypertopologies. In this way we arrived at the upper Bombay topology which can be defined in any topological space equipped with two compatible proximities and a cobase (i.e. a family of closed subsets which is closed under finite unions and which contains all singletons). Joining the upper Bombay topology to the lower one generated by a collection *IL* of locally finite families of open sets give rise to the *IL*-locally finite Bombay topology which includes all known hypertopologies as special cases. We thus achieved our goal of unification stated at the beginning.

For references on hyperspaces up to 1993, we generally refer to [1], except when a specific reference is needed. For proximities see [13], [22] and [19].

2. Preliminaries.

Let (X, τ) denote a T_1 space. For any $E \subset X$, $cl_X E$, $intE$ and E^c stand for the closure, interior and complement of E in X, respectively. Let γ, γ_1 and γ_2 be compatible LO-proximities on X. We always assume $\gamma_1 \leq \gamma_2$ (i.e. $A \not\gamma_1 B$ implies $A \not\gamma_2 B$ (see [22]) where $\not\gamma$ stands for the negation of γ). We use the symbol γ_0 to denote the fine LO-proximity on X given by

$A\gamma_0 B$ iff $cl_X A \cap cl_X B \neq \varnothing$. ($\gamma_0$ is called the *Wallman proximity*).

In the case (X, τ) is Tychonoff, then:

$\gamma^\sharp$ denotes the fine EF-proximity on X given by
$A \not\gamma^\sharp B$ iff they can be separated by a continuous function $f : X \to [0, 1]$.
($\gamma^\sharp$ is called the *functionally indistinguishable proximity* on X).

If $\mathcal{U}$ is a compatible separated uniformity on X, then $\gamma(\mathcal{U})$ denotes the EF-proximity on X given by
$A\gamma(\mathcal{U})B$ iff $U(A) \cap B \neq \varnothing$ for each $U \in \mathcal{U}$.
($\gamma(\mathcal{U})$ is called the *uniform proximity* induced by $\mathcal{U}$).

If (X, τ) is a metrizable space with metric d, then $\gamma(d)$ is the EF-proximity on X given by
$A\gamma(d)B$ iff $D_d(A, B) = \inf\{d(a, b) : a \in A, b \in B\} = 0$.
($\gamma(d)$ is called the *metric proximity* induced by d).

$CL(X)$ (resp. $K(X)$) is the family of all nonempty closed subsets of X (resp. the family of all nonempty closed and compact subsets of X).

Δ is a nonempty subfamily of $CL(X)$ which is **closed under finite unions** and which **contains all singletons**. We call Δ a **cobase** ([23]). Let us note that in the literature Δ is usually assumed to contain merely all singletons.

In our view the above assumption simplifies the results. In fact, it allows to display transparent statements and makes theory much simpler.

For any set $E \subset X$ and a subfamily $\mathcal{E} \subset \tau$ we use the following standard notation:

$E^- = \{A \in CL(X) : A \cap E \neq \varnothing\}$;
$\mathcal{E}^- = \{A \in CL(X) : A \cap E \neq \varnothing \text{ for each } E \in \mathcal{E}\}$.

Furthermore, if γ is a compatible proximity on X, we set

$E^{++}_\gamma = \{A \in CL(X) : A \ll_\gamma E, \text{ i.e. } A \not\gamma E^c\}$.

Note that $\gamma_1 \leq \gamma_2$ is equivalent to $U^{++}_{\gamma_1} \subset U^{++}_{\gamma_2}$ for every $U \in \tau$.

We omit γ if it is clear from the context and write E^{++}_γ simply as E^{++}. Moreover

$E^+ = \{A \in CL(X) : A \subset E, \text{ i.e. } A \ll_{\gamma_0} E\}$.

Now we describe some hypertopologies on $CL(X)$.

The *upper proximal* Δ*-topology* (w.r.t. γ) $\sigma(\gamma; \Delta)^+$ is generated by the basis $\{E^{++} : E^c \in \Delta\}$.

When $\gamma = \gamma_0$ we have the *upper* Δ*-topology* $\tau(\Delta)^+ = \sigma(\gamma_0; \Delta)^+$.

The *lower Vietoris* (*or finite*) *topology* $\tau(V^-)$ has a basis
$\{\mathcal{E}^- : \mathcal{E} \subset \tau \text{ is finite}\}$.

The *lower locally finite topology* $\tau(LF^-)$ has a basis
$\{\mathcal{E}^- : \mathcal{E} \subset \tau \text{ is locally finite}\}$.

The *IL-lower locally finite topology* $\tau(IL^-)$ has a basis $\{\mathcal{E}^- : \mathcal{E} \subset IL\}$ provided $IL \subset \{\mathcal{E} \subset \tau : \mathcal{E} \text{ is locally finite}\}$ satisfies the following filter condition: $(*)$ whenever $\mathcal{E}, \mathcal{F} \in IL$, then there exists $\mathcal{G} \in IL$ such that $\mathcal{G}^- \subset \mathcal{E}^- \cap \mathcal{F}^-$ (see [17]).

The *proximal* (*finite*) Δ*-topology* (w.r.t. γ) $\sigma(\gamma; \Delta) = \sigma(\gamma; \Delta)^+ \vee \tau(V^-)$.

We omit γ if it is obvious from the context and write $\sigma(\Delta)$ for $\sigma(\gamma; \Delta)$.

The Δ*-topology* $\tau(\Delta) = \tau(\Delta)^+ \vee \tau(V^-) = \sigma(\gamma_0; \Delta)^+ \vee \tau(V^-) = \sigma(\gamma_0; \Delta)$.

The *proximal locally finite* Δ*-topology* (w.r.t. γ)
$\sigma(\gamma; LF, \Delta) = \sigma(\gamma; \Delta)^+ \vee \tau(LF^-)$.

The *proximal IL-locally finite* Δ*-topology* (w.r.t. γ)
$\sigma(\gamma; IL, \Delta) = \sigma(\gamma; \Delta)^+ \vee \tau(IL^-)$.

We omit the prefix "proximal" and replace σ by τ if $\gamma = \gamma_0$.

Well known special cases are:

(a) when $\Delta = CL(X)$,

$\tau(\Delta) = \tau(V)$ the *Vietoris* or *finite topology*;
$\sigma(\gamma; \Delta) = \sigma(\gamma)$ the *proximal topology* (w.r.t. γ);
$\tau(LF\Delta) = \tau(LF)$ the *locally finite topology*;
$\sigma(\gamma; LF, \Delta) = \sigma(\gamma; LF)$ the *proximal locally finite topology.*

(b) When $\Delta = K(X)$, $\tau(\Delta) = \tau(F)$ the *Fell topology.*

(c) Let (X, d) be a metric space, $\gamma(d)$ the metric proximity induced by d and Δ denote the cobase $\mathcal{B}$ generated by all finite unions of closed balls of nonnegative radii. Then

$\tau(\Delta) = \tau(\mathcal{B})$ the *ball topology*;
$\sigma(\gamma(d); \Delta) = \sigma(\gamma(d); \mathcal{B}) = \sigma(\mathcal{B})$ the *proximal ball topology.*

It was shown in [20] that a typical neighbourhood at $A \in CL(X)$ in the *upper Wijsman topology* $\tau(W_d)^+$ is of the form:

$$U^+ \text{ where } U^c \in \mathcal{B} \text{ and } A \not\gamma(d) U^c.$$

Thus *three parameters* are involved:

(i) $\gamma(d)$ the metric proximity in $A \not\gamma(d) U^c$,
(ii) γ_0 the fine LO-proximity in U^+, and
(iii) the *cobase* $\mathcal{B}$ which contains U^c.

We note that there are *two proximities*, namely $\gamma(d)$, γ_0 with $\gamma(d) \leq \gamma_0$ and *a cobase.* By replacing the two *proximal parameters* $\gamma(d)$, γ_0 by two *LO*-proximities γ_1, γ_2 with $\gamma_1 \leq \gamma_2$ and *the cobase* $\mathcal{B}$ by Δ we have the following definition:

Definition 2.1. Let (X, τ) be a T_1 space with compatible proximities γ_1, γ_2 with $\gamma_1 \leq \gamma_2$ and Δ a cobase. Then a typical neighbourhood of $A \in CL(X)$ in the upper Bombay topology $\sigma(\gamma_1, \gamma_2; \Delta)^+$ is:

$$U_{\gamma_2}^{++} \text{ where } U^c \in \Delta \text{ and } A \not\gamma_1 U^c \text{ (or equivalently } A \in U_{\gamma_1}^{++}).$$

(Note that since $\gamma_1 \leq \gamma_2$ $A \not\gamma_1 U^c$ implies $A \not\gamma_2 U^c$ which in turn is equivalent to $A \in U_{\gamma_2}^{++}$). γ_1, γ_2 and Δ are *the three parameters* of the upper Bombay hypertopology: γ_1, γ_2 are *the proximal parameters* and Δ is the *cobase.*

Furthermore, $\sigma(\gamma_1;\Delta)^+$ (respectively, $\sigma(\gamma_2;\Delta)^+$) represents the *first coordinate upper topology* (the *second coordinate upper topology*).

(i) The Bombay topology $\sigma(\gamma_1,\gamma_2;\Delta)$ is the join of the upper Bombay topology $\sigma(\gamma_1,\gamma_2;\Delta)^+$ and the lower Vietoris topology $\tau(V^-)$ i.e. $\sigma(\gamma_1,\gamma_2;\Delta) = \sigma(\gamma_1,\gamma_2;\Delta)^+ \vee \tau(V^-)$.

(ii) The locally finite Bombay topology $\sigma(\gamma_1,\gamma_2;LF,\Delta) = \sigma(\gamma_1,\gamma_2;\Delta)^+ \vee \tau(LF^-)$.

Let $IL \subset \{\mathcal{E} : \mathcal{E} \in \tau \text{ is locally finite}\}$ satisfy the following filter condition:
$(*)$ whenever $\mathcal{E}$, $\mathcal{F} \in IL$, then there is $\mathcal{G} \in IL$ such that $\mathcal{G}^- \subset \mathcal{E}^- \cap \mathcal{F}^-$.

(iii) The IL-locally finite Bombay topology $\sigma(\gamma_1,\gamma_2;IL,\Delta) = \sigma(\gamma_1,\gamma_2;\Delta)^+ \vee \tau(IL^-)$.

Remark 2.2. (a) If in *(iii)* of above Definition each member of IL is finite, then $\sigma(\gamma_1,\gamma_2;IL,\Delta)$ equals $\sigma(\gamma_1,\gamma_2;\Delta)$, since $\tau(IL^-) = \tau(V^-)$ (see also Lemma 3.2 below).

(b) Again if in *(iii)* of Definition 2.1 we choose $\gamma_1 = \gamma_2 = \gamma$, then we have the *proximal IL-locally finite Δ topology* $\sigma(\gamma;IL,\Delta)$ from which we obtain all classical hypertopologies defined above.

(c) Let (X,d) be a metric space, $\Delta = \mathcal{B}$ the cobase generated by all closed balls, and $\gamma = \gamma(d)$ the metric proximity induced by d. Then $\sigma(\gamma,\gamma_0;\Delta) = \sigma(\gamma,\gamma_0;\mathcal{B}) = \tau(W_d)$ is the *Wijsman topology*, i.e. the Bombay topology is a generalization of the Wijsman topology. In addition, suppose (X,τ) is a T_1 space, γ a compatible LO-proximity coarser than γ_0 and Δ a cobase, then we refer the upper-Bombay topology $\sigma(\gamma,\gamma_0;\Delta)^+$ (the Bombay topology $\sigma(\gamma,\gamma_0;\Delta)$) as the *upper g-Wijsman topology* (the *g- Wijsman topology*). Here g stands for generalized (w.r.t. Δ and γ since γ_0 is kept fixed).

(d) Let $(X,\mathcal{U})$ be a separated uniform space and $\gamma = \gamma(\mathcal{U})$ the compatible EF-proximity induced by $\mathcal{U}$. Let IL be the collection of all families of open sets of the form $\{U(x) : x \in Q \subset A\}$, where $A \in CL(X)$, $U \in \mathcal{U}$ and Q is U-discrete, i.e. x, $y \in Q$, $x \neq y$ implies $x \notin U(y)$. Then the *Hausdorff Bourbaki* or *H-B uniform topology* associated to $\mathcal{U}$ on $CL(X)$ is $\tau(\mathcal{U}_H) = \tau(\mathcal{U}_H)^+ \vee \tau(\mathcal{U}_H^-) = \sigma(\gamma)^+ \vee \tau(IL^-)$ (see [20]). Moreover, if the uniformity $\mathcal{U}$ comes from a metric d we get the *Hausdorff metric topology* $\tau(H_d)$. Thus all Hausdorff-Bourbaki topologies are proximal locally finite.

(e) Many new hypertopologies can be defined by replacing the lower Vietoris topology $\tau(V^-)$ by the lower *IL*-locally finite topology. Thus we have *the IL-locally finite Fell topology, the IL-locally finite Wijsman topology, the IL-locally finite ball topology, the proximal IL-locally finite* Δ*-topology*, etc. In addition, by a proper choice of the two proximal coordinates in a Bombay topology, one can get infinitely many new hypertopologies.

3. Principal results.

In this section we plan to find necessary and sufficient conditions for one *IL*-locally finite Bombay topology to be coarser than another.

It is well known that in the comparison of hit-and-miss hypertopologies, the lower and the upper parts play their roles separately. Hence, following Lemmas hold.

Lemma 3.1. *Let (X,τ) be a T_1 space with compatible LO-proximities γ_1, γ_2, η_1, η_2 satisfying $\gamma_1 \leq \gamma_2$ and $\eta_1 \leq \eta_2$. Let Δ, Λ be cobases and IL_1, IL_2 two collections of locally finite families of open sets satisfying condition $(*)$. The following are equivalent:*

(a) $\sigma(\gamma_1,\gamma_2; IL_1,\Delta) \leq \sigma(\eta_1,\eta_2; IL_2,\Lambda)$;

(b) $\tau(IL_1^-) \leq \tau(IL_2^-)$ *and* $\sigma(\gamma_1,\gamma_2;\Delta)^+ \leq \sigma(\eta_1,\eta_2;\Lambda)^+$.

Lemma 3.2. *Let (X,τ) be a T_1 space and IL, IL_1, IL_2 collections of locally finite families of open sets satisfying condition $(*)$. Then: $\tau(IL_1^-) \leq \tau(IL_2^-)$ if and only if for each $\mathcal{E} \in IL_1$, there exists $\mathcal{F} \in IL_2$ such that $\mathcal{F}^- \subset \mathcal{E}^-$.*

Thus if all members of IL_2 are finite, then the same is true of the members of IL_1 and hence $\tau(IL^-) \leq \tau(V^-)$ if and only if all members of IL are finite.

Now, we turn our attention to the upper Bombay topologies and consider the principal result of this paper showing that it includes most of the results in the literature involving comparison of various hypertopologies.

Next Definition and Lemma play a key role.

Definition 3.3. Let (X,τ) be a T_1 space and γ a compatible *LO*-proximity on X. If $B \subset X$, set $\gamma(B) = \{F \subset X : F\gamma B\}$ ([24]), i.e. $\gamma(B)$ is the collection of all subsets F of X near to B w.r.t. γ.

Lemma 3.4. *Let (X,τ) be a T_1 space, γ and η compatible LO-proximities on X and C and D nonempty closed subsets of X. The following are equivalent:*

(a) $(D^c)^{++}_\eta \subset (C^c)^{++}_\gamma$;

(b) $C \subset D$ *and* $\gamma(C) \subset \eta(D)$.

Proof. $(a) \Rightarrow (b)$. Assume not, then either i) $C \not\subset D$ or ii) $\gamma(C) \not\subset \eta(D)$. If i) occurs, then there exists $c \in C \setminus D$. Choose $F \in (D^c)^{++}_\eta$ and consider

$F' = F \cup \{c\}$. Then $F' \in (D^c)^{++}_\eta$ but $F' \notin (C^c)^{++}_\gamma$; a contradiction. If ii) occurs, then there exists an $F \subset X$ such that $F\gamma C$ but $F \not\eta D$. Since $F\gamma C$, then $F \neq \varnothing$. Let $E = cl_X F$. Then $E \in CL(X)$, $E\gamma C$ but $E\not\eta D$. Therefore $E \in (D^c)^{++}_\eta$ but $E \notin (C^c)^{++}_\gamma$; a contradiction.

$(b) \Rightarrow (a)$. Assume not, then $(D^c)^{++}_\eta \not\subset (C^c)^{++}_\gamma$. Hence there exists

$F \in CL(X)$ such that $F \not\eta D$ but $F\gamma C$, i.e. $F \in \gamma(C)$ but $F \notin \eta(D)$; a contradiction. □

Theorem 3.5. (**Main Theorem**)
Let (X,τ) be a T_1 space with compatible LO-proximities γ_1, γ_2, η_1, η_2 satisfying $\gamma_1 \leq \gamma_2$ and $\eta_1 \leq \eta_2$ and Δ and Λ cobases. The following are equivalent:

(a) $\sigma(\gamma_1,\gamma_2;\Delta)^+ \leq \sigma(\eta_1,\eta_2;\Lambda)^+$;

(b) *for each $B \in \Delta$ and $W \in \tau$, $W \neq X$, with $B \ll_{\gamma_1} W$, there exists a $B' \in \Lambda$ such that:*
  (i) $B \subset B' \ll_{\eta_1} W$, *and*
  (ii) $\gamma_2(B) \subset \eta_2(B')$.

Proof. $\sigma(\gamma_1,\gamma_2;\Delta)^+ \leq \sigma(\eta_1,\eta_2;\Lambda)^+$ if and only if for each $A \in CL(X)$, $A \neq X$, $A \in U^{++}_{\gamma_2} \in \sigma(\gamma_1,\gamma_2;\Delta)^+$ - where $U^c \in \Delta$ and $A \in U^{++}_{\gamma_1}$ - there exists a $V \in \tau$ such that $V^c \in \Lambda$, $A \in V^{++}_{\eta_1}$ and $A \in V^{++}_{\eta_2} \subset U^{++}_{\gamma_2}$. Noting that $A \in V^{++}_{\eta_1}$ is equivalent to $V^c \ll_{\eta_1} A^c$ and using Lemma 3.4 we have (i) and (ii) where $W = A^c$, $B = U^c$ and $B' = V^c$. □

Corollary 3.6. *Let (X,τ) be a T_1 topological space with compatible LO-proximities γ_1, γ_2, η_1, η_2 satisfying $\gamma_1 \leq \gamma_2$ and $\eta_1 \leq \eta_2$, Δ and Λ cobases and IL_1 and IL_2 collections of locally finite families of open sets satisfying condition $(*)$. The following are equivalent:*

(a) $\sigma(\gamma_1,\gamma_2;IL_1,\Delta) \leq \sigma(\eta_1,\eta_2;IL_2,\Lambda)$;

(b) *IL_2 refines IL_1 and whenever $B \in \Delta$, $W \in \tau$, $W \neq X$, with $B \ll_{\gamma_1} W$, then there exists a $B' \in \Lambda$ such that:*
  (i) $B \subset B' \ll_{\eta_1} W$, *and*
  (ii) $\gamma_2(B) \subset \eta_2(B')$.

Remark 3.7. (a) For future reference we note that $F\gamma B$ implies that there is a net in X whose range $C \subset F$ and $C\gamma B$ (cf. Lemma 3.2 in [5]).

(b) We note that if $\eta_2 \leq \gamma_2$, then (ii) at (b) of the Main Theorem is automatically satisfied.

(c) If γ_1 is an EF-proximity, then $A \in U^{++}_{\gamma_1}$ implies the existence of $E \in CL(X)$ with $E \in U^{++}_{\gamma_1}$ and $A \in (intE)^{++}_{\gamma_1}$. So setting $W' = intE$, (i) at (b) of the Main Theorem can be written as $B \subset B' \ll_{\eta_1} W' \ll_{\gamma_1} W$.

(d) The proximal topologies $\sigma(\gamma_1;\Delta)$ and $\sigma(\gamma_2;\Delta)$ are the *proximal coordinate topologies* of the given (finite) Bombay topology $\sigma(\gamma_1,\gamma_2;\Delta)$.

Let γ and η, with $\gamma \leq \eta$, be compatible LO-proximities on a given topological space X and Δ a cobase. In the motivation we saw that the (finite) Bombay topology $\sigma(\gamma, \eta; \Delta)$ is a generalization of the g-Wijsman topology $\sigma(\gamma, \gamma_0; \Delta)$, which in turn, is a generalization of the Wijsman topology $\sigma(\gamma(d), \gamma_0; \mathcal{B})$. We conclude this section by deriving some results in the general case and we give appropriate references. We reserve the special cases:

1) (X, τ) is a metrizable space with metric d and the first proximal parameter $\gamma = \gamma(d)$ is the EF-proximity associated to d,

and

2) (X, τ) is a Tychonoff space and the first proximal parameter γ is EF or (X, τ) is a uniformizable space with separated uniformity $\mathcal{U}$ and the first proximal parameter $\gamma = \gamma(\mathcal{U})$ is the proximity induced by $\mathcal{U}$,

to next sections.

The results given below are new and follow easily from the Main Theorem or from the definitions and so we omit the proofs.

We start to compare a (finite) Bombay topology $\sigma(\gamma_1, \gamma_2; \Delta)$ with its proximal coordinate topologies.

Theorem 3.8. (cf. [1] Pages 45, 53) *Let (X, τ) be a T_1 space with compatible LO-proximities γ_1, γ_2, satisfying $\gamma_1 \leq \gamma_2$ and Δ a cobase. Then:*

(a) $\sigma(\gamma_1, \gamma_2; \Delta) \leq \sigma(\gamma_1; \Delta)$;
(b) $\sigma(\gamma_1, \gamma_2; \Delta) \leq \sigma(\gamma_2; \Delta)$.

Corollary 3.9. *Let (X, τ) be a T_1 space, γ a compatible LO-proximity and Δ a cobase. Then:*

(a) $\sigma(\gamma, \gamma_0; \Delta) \leq \sigma(\gamma; \Delta)$ (i.e. the g-Wijsman topology is coarser than its first proximal coordinate topology);
(b) $\sigma(\gamma, \gamma_0; \Delta) \leq \tau(\Delta)$ (i.e. the g-Wijsman topology is coarser than its second coordinate topology).

Next Theorem and Corollary show when a Bombay topology $\sigma(\gamma_1, \gamma_2; \Delta)$ and a g-Wijsman topology $\sigma(\gamma, \gamma_0; \Delta)$ are finer than their proximal coordinate topologies.

Theorem 3.10. (cf. [1] Pages 45, 52, 53) *Let (X, τ) be a T_1 space with compatible LO-proximities γ_1, γ_2 satisfying $\gamma_1 \leq \gamma_2$ and Δ a cobase. The following are equivalent:*

(a) $\sigma(\gamma_2; \Delta) \leq \sigma(\gamma_1, \gamma_2; \Delta)$;
(b) $\sigma(\gamma_1, \gamma_2; \Delta) = \sigma(\gamma_1; \Delta) = \sigma(\gamma_2; \Delta)$;

(c) *for each* $B \in \Delta$ *and* $W \in \tau$, $W \neq X$, *with* $B \ll_{\gamma_2} W$, *there exists a* $B' \in \Delta$ *such that* $B \subset B' \ll_{\gamma_1} W$;
(d) *whenever* $A \in CL(X)$, $B \in \Delta$ *and* $A \not\gamma_2 B$, *then* $A \not\gamma_1 B$.

Corollary 3.11. *Let* (X, τ) *be a* T_1 *space,* γ *a compatible LO-proximity on* X *and* Δ *a cobase. The following are equivalent:*
(a) $\tau(\Delta) \leq \sigma(\gamma, \gamma_0; \Delta)$;
(b) $\sigma(\gamma, \gamma_0; \Delta) = \sigma(\gamma; \Delta) = \tau(\Delta)$;
(c) *for each* $B \in \Delta$ *and* $W \in \tau$, $W \neq X$, *with* $B \subset W$, *there exists a* $B' \in \Delta$ *such that* $B \subset B' \ll_{\gamma} W$;
(d) *whenever* $A \in CL(X)$, $B \in \Delta$ *and* $A \cap B = \varnothing$, *then* $A \not\gamma B$.

Now, we compare two proximal hit-and-miss hypertopologies associated to the same cobase Δ.

Theorem 3.12. (cf. [1] Page 53 and Theorem 3.3 in [5]) *Let* (X, τ) *be a* T_1 *space with compatible LO-proximities* γ_1, γ_2 *satisfying* $\gamma_1 \leq \gamma_2$ *and* Δ *a cobase. The following are equivalent:*
(a) $\sigma(\gamma_1; \Delta) \leq \sigma(\gamma_2; \Delta)$;
(b) *for each* $B \in \Delta$ *and* $W \in \tau$, $W \neq X$, *with* $B \ll_{\gamma_1} W$, *there exists a* $B' \in \Delta$ *such that* $B \subset B' \ll_{\gamma_2} W$ *and* $\gamma_1(B) \subset \gamma_2(B')$;
(c) *whenever* $A \in CL(X)$, $B \in \Delta$ *and* $A \not\gamma_1 B$, *then there exists a* $B' \in \Delta$ *such that i)* $B \subset B'$, $A \not\gamma_2 B'$ *and ii) whenever* C *is the range of a net satisfying* $C \subset (B')^c$ *and* $C \gamma_1 B$ *then* $C \gamma_2 B'$.

Corollary 3.13. *Let* (X, τ) *be a* T_1 *space,* γ *a compatible LO-proximity on* X *and* Δ *a cobase. The following are equivalent:*
(a) $\sigma(\gamma; \Delta) \leq \tau(\Delta)$;
(b) *for each* $B \in \Delta$ *and* $W \in \tau$, $W \neq X$, *with* $B \ll_{\gamma} W$, *there exists a* $B' \in \Delta$ *such that* $B \subset B' \ll_{\gamma_0} W$ *and* $\gamma(B) \subset \gamma_0(B')$;
(c) *whenever* $A \in CL(X)$, $B \in \Delta$ *and* $A \not\gamma B$, *then there exists a* $B' \in \Delta$ *such that i)* $B \subset B'$, $A \cap B' = \varnothing$ *and ii) whenever* $\{x_\lambda : \lambda \in \Sigma\}$ *is a net whose range* C *is contained in* $(B')^c$ *and* $C \gamma B$, *then* $\{x_\lambda : \lambda \in \Sigma\}$ *has a cluster point.*

Theorem 3.14. (cf. [1] Page 53 and Theorem 3.11 in [5]) *Let* (X, τ) *be a* T_1 *space with compatible LO-proximities* γ_1, γ_2 *satisfying* $\gamma_1 \leq \gamma_2$ *and* Δ *a cobase. The following are equivalent:*
(a) $\sigma(\gamma_2; \Delta) \leq \sigma(\gamma_1; \Delta)$;
(b) $\sigma(\gamma_2; \Delta) = \sigma(\gamma_1; \Delta)$;
(c) *for each* $B \in \Delta$ *and* $W \in \tau$, $W \neq X$, *with* $B \ll_{\gamma_2} W$, *there exists a* $B' \in \Delta$ *such that* $B \subset B' \ll_{\gamma_1} W$;
(d) *whenever* $A \in CL(X)$, $B \in \Delta$ *and* $A \not\gamma_2 B$, *then* $A \not\gamma_1 B$.

Corollary 3.15. *Let* (X, τ) *be a* T_1 *space,* γ *a compatible LO-proximity on* X *and* Δ *a cobase. The following are equivalent:*

(a) $\tau(\Delta) \leq \sigma(\gamma;\Delta)$;
(b) $\tau(\Delta) = \sigma(\gamma;\Delta)$;
(c) *for each* $B \in \Delta$ *and* $W \in \tau$, $W \neq X$, *with* $B \subset W$, *there exists a* $B' \in \Delta$ *such that* $B \subset B' \ll_\gamma W$;
(d) *whenever* $A \in CL(X)$, $B \in \Delta$ *and* $A \cap B = \varnothing$, *then* $A \not\gamma B$.

Next Theorem compares two Bombay topologies which have the same second proximity parameter but different cobases.

Theorem 3.16. (cf. [1] Page 39) *Let* (X,τ) *be a* T_1 *space and* γ, δ *and* η *compatible LO-proximities on* X *such that* $\gamma \leq \eta$ *as well as* $\delta \leq \eta$. *Let* Δ *and* Λ *be cobases. The following are equivalent:*
(a) $\sigma(\gamma,\eta;\Delta) \leq \sigma(\delta,\eta;\Lambda)$;
(b) *for each* $B \in \Delta$ *and* $W \in \tau$, $W \neq X$, *with* $B \ll_\gamma W$, *there exists a* $B' \in \Lambda$ *such that* $B \subset B' \ll_\delta W$.

Corollary 3.17. *Let* (X,τ) *be a* T_1 *space,* γ *and* δ *compatible LO-proximities on* X *and* Δ *and* Λ *cobases. The following are equivalent:*
(a) $\sigma(\gamma,\gamma_0;\Delta) \leq \sigma(\delta,\gamma_0;\Lambda)$;
(b) *for each* $B \in \Delta$ *and* $W \in \tau$, $W \neq X$, *with* $B \ll_\gamma W$, *there exists a* $B' \in \Lambda$ *such that* $B \subset B' \ll_\delta W$.

Corollary 3.18. *Let* (X,τ) *be a* T_1 *space,* Δ *and* Λ *cobases. The following are equivalent:*
(a) $\tau(\Delta) \leq \tau(\Lambda)$;
(b) *for each* $B \in \Delta$ *and* $W \in \tau$, $W \neq X$, *with* $B \subset W$, *there exists a* $B' \in \Lambda$ *such that* $B \subset B' \subset W$.

4. The metric case.

A large part of the literature is in the setting of a metric space. Let (X,τ) be a metrizable space with metric d, $\gamma = \gamma(d)$ the d-metric proximity, $\mathcal{B}(d)$ the family of finite unions of all d-closed balls and $TB(d)$ denote the family of all closed d-totally bounded subsets of X (we omit d if it is obvious from the context and write respectively $\mathcal{B}$ and TB for $\mathcal{B}(d)$ and $TB(d)$).

Remark 4.1. Let (X,τ) be a metrizable space with metric d and $\gamma = \gamma(d)$ the d-metric proximity. Let α, δ and ε be positive reals with $\varepsilon < \delta < \alpha$ and $S_d(x,\varepsilon)$ and $B_d(x,\varepsilon)$ be the open and the closed d-balls centered at x of radius ε. We omit the subscript d if it is clear from the context. Then:

(a) If X is a metrizable space with metric d, then $TB = TB(d)$ is a cobase and it is even hereditarily closed.

(b) Whenever α, δ and ε are positive reals with $\varepsilon < \delta < \alpha$, then
$$S(x,\varepsilon) \subset B(x,\varepsilon) \ll_\gamma S(x,\delta) \subset B(x,\delta) \ll_\gamma S(x,\alpha) \subset B(x,\alpha).$$

(c) A set D is said to be *strictly d-included* in E ($D \subset\subset_d E$) iff there is a finite set of points $\{x_1, \cdots, x_n\}$ of E and positive reals $\varepsilon_k < \alpha_k$, $k = 1, \cdots, n$, such that

$$(\mathbf{SDI}) \quad D \subset \bigcup_{k=1}^{n} S(x_k, \varepsilon_k) \subset \bigcup_{k=1}^{n} S(x_k, \alpha_k) \subset E \text{ ([1] Page 38).}$$

So, using (b) it can be shown that open balls can be replaced by closed balls in (**SDI**) as well as (**SDI**) is in turn equivalent to

$$D \subset \bigcup_{k=1}^{n} S(x_k, \varepsilon_k) \ll_\gamma \bigcup_{k=1}^{n} S(x_k, \alpha_k) \subset E.$$

(d) If A is a nonempty subset of X, B' is a finite unions of balls, $W \in \tau$ and $A \subset B' \ll_\gamma W$, then there exists a finite union of balls B'' such that $A \subset B' \subset\subset_d B'' \ll_\gamma W$ and thus by above $A \ll_\gamma B'' \ll_\gamma W$. Indeed, if $B' \ll_\gamma W$, then $D(B', W^c) = \inf\{d(b, y) : b \in B', y \in W^c\} = \varepsilon > 0$. Since $B' = \bigcup_{k=1}^{n} S(x_k, \varepsilon_k)$, select $t = \dfrac{\varepsilon}{2}$ and set $B'' = \bigcup_{k=1}^{n} S(x_k, \varepsilon_k + t)$.

Thus whenever $A \in CL(X)$ and $W \in \tau$ with $W \neq X$, the following are equivalent:

(1) there exists a $B' \in \mathcal{B}$ such that $A \subset B' \ll_\gamma W$;
(2) $A \subset\subset_d W$;
(3) there exists a $B'' \in \mathcal{B}$ such that $A \ll_\gamma B'' \ll_\gamma W$.

(e) We note that from Remark 3.7 (a) if C and C' are nonempty subsets of X with $C \subset C'$, then the inclusion $\gamma(C) \subset \gamma_0(C')$ occurs if and only if whenever there is a sequence of points $\{x_n : n \in IN\}$ in $(C')^c$ with $\lim_{n\to\infty} d(x_n, C) = 0$, then the sequence $\{x_n : n \in IN\}$ has a cluster point (see [1] Page 52).

(f) Let X be a metrizable space with compatible metrics d and e inducing the metric proximities $\gamma = \gamma(d)$ and $\eta = \eta(e)$, respectively. Let $\Delta = \mathcal{B} = \mathcal{B}(d)$ and $\Lambda = \mathcal{B}' = \mathcal{B}(e)$. The following are equivalent:

(1) $\tau(W_d) \leq \tau(W_e)$;
(2) for each $B \in \mathcal{B}$ and $W \in \tau$ with $B \ll_\gamma W$ there exists a $B' \in \mathcal{B}'$ such that $B \subset B' \ll_\eta W$;
(3) for each $B \in \mathcal{B}$ and $W \in \tau$ with $B \ll_\gamma W$ there exists a $B'' \in \mathcal{B}'$ such that $B \subset\subset_e B'' \subset\subset_e W$ (cf. [3], [15]);
(4) each proper open d-sphere is strictly e-included in every its open enlargement (cf. Theorem 2.1.10 in [1]).

(g) From (d) we have that strict d-inclusion is equivalent to *weak total boundedness*. We recall that a closed subset E of X is said to be *weakly totally bounded* or *w-TB* in a open set W iff there exists a $B \in \mathcal{B}$ such that $E \ll_\gamma B \ll_\gamma W$ (see (16.1) in [10]). Moreover:

(g_1) In any infinite dimensional Banach space, the closed unit ball is not totally bounded but it is weakly totally bounded in any open ball centered at 0 and radius greater than 1;

(g_2) let l_2 be the Hilbert space of square summable sequences, θ the origin and $\{e_n : n \in IN\}$ the standard orthonormal base for l_2. Let (X, d) be the metric subspace of l_2 where $X = \{\theta\} \cup \{e_n : n \in IN\}$. Then $\{e_{2n+1} : n \in IN\}$ is not w-TB in $X \setminus \{e_{2n} : n \in IN\}$.

Theorem 4.2. (cf. **3.9**) *Let (X, d) be a metric space, $\gamma = \gamma(d)$ the metric proximity induced by d and $\mathcal{B}$ the cobase generated by all closed d-balls. Then:*

(a) $\tau(W_d) \leq \sigma(\mathcal{B})$;
(b) $\tau(W_d) \leq \tau(\mathcal{B})$.

Theorem 4.3. (cf. **3.11**) *Let (X, d) be a metric space, $\gamma = \gamma(d)$ the metric proximity induced by d and $\mathcal{B}$ the cobase generated by all closed d-balls. In the following (a), (b), (c) and (d) are equivalent and each implies (e) which is equivalent to (f).*

(a) $\tau(\mathcal{B}) \leq \tau(W_d)$;
(b) $\tau(W_d) = \sigma(\mathcal{B}) = \tau(\mathcal{B})$;
(c) *for each $B \in \mathcal{B}$ and $W \in \tau$, $W \neq X$, with $B \subset W$ implies $B \subset\subset_d W$;*
(d) *X is ball Atsuji* (i.e. disjoint closed sets, one of which is a ball, are far).
(e) $\sigma(\mathcal{B}) \leq \tau(W_d)$;
(f) *for each positive real ε and each $B \in \mathcal{B}$, B is strictly d-included in its ε-enlargement, i.e. $B \subset\subset_d S(B, \varepsilon)$.*

Theorem 4.4. (cf. **3.13**) *Let (X, d) be a metric space, $\gamma = \gamma(d)$ the metric proximity induced by d and $\mathcal{B}$ the cobase generated by all closed d-balls. The followings are equivalent:*

(a) $\sigma(\gamma; \mathcal{B}) \leq \tau(\mathcal{B})$;
(b) *for each $B \in \mathcal{B}$ and $W \in \tau$, $W \neq X$, with $B \ll_\gamma W$, there exists a $B' \in \mathcal{B}$ such that $B \subset B' \subset W$ and $\gamma(B) \subset \gamma_0(B')$;*
(c) *for each $B \in \mathcal{B}$ and $W \in \tau$, $W \neq X$, with $B \ll_\gamma W$, there exists a $B' \in \mathcal{B}$ such that $B \subset B' \subset W$ and for each sequence $\{x_n : n \in IN\}$ in $(B')^c$ with $\lim_{n\to\infty} d(x_n, B) = 0$ has a cluster point.*

Theorem 4.5. (cf. Theorem 3.1 in [8]) *Let X be a metrizable space with metric d, $\gamma = \gamma(d)$ the metric proximity induced by d and Δ a cobase. The following are equivalent:*

(a) $\tau(W_d) \leq \tau(\Delta)$;

(b) *for every* $x \in X$ *and every reals* α, β *with* $0 < \alpha < \beta$ *and* $S(x,\beta) \neq X$, *there exists a* $B' \in \Delta$ *such that* $S(x,\alpha) \subset B' \ll_{\gamma_0} S(x,\beta)$;

(c) $\tau(W_d) \leq \sigma(\gamma;\Delta)$.

Lemma 4.6. *Let* X *be a metrizable space with metric* d *and* $A \in CL(X)$. *The following are equivalent:*

(a) A *is totally bounded;*

(b) *for each positive real* ε *and each closed subset* E *of* A, $E \subset\subset_d S(E,\varepsilon)$;

(c) *for each positive real* ε *and each closed subspace* E *of* A, E *is w-TB in* $S(E,\varepsilon)$ (i.e. there exists $B \in \mathcal{B}$ such that $E \ll B \ll S(E,\varepsilon)$).

Proof. It suffices to show that $(c) \Rightarrow (a)$, since $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ follow from the definition of total boundedness and Remark 4.1 (d) respectively.

Suppose (c) holds but not (a). Then, without any loss of generality, we may suppose that $A = \{a_n : n \in IN\}$ is bounded and there is an $\varepsilon > 0$ such that $d(a_p,a_q) > \varepsilon$ for all $p \neq q$.

Now we use the *"milky way "* technique. Set $a_{1,n} = a_n$ for all $n \in IN$. Since $\{a_{1,n} : n \in IN\}$ is bounded, it has a subsequence $\{a_{2,n} : n \in IN\}$ such that $d(a_{1,1},a_{2,n}) \to \alpha_1 \geq \varepsilon$. Inductively, for each $r \in IN$, $\{a_{r,n} : n \in IN\}$ has a subsequence $\{a_{r+1,n} : n \in IN\}$ such that $d(a_{r,r},a_{r+1,n}) \to \alpha_r \geq \varepsilon$. We note that for all infinite subsets $D \subset \{a_{s,n} : s > r, n \in IN\}$ $d(a_{r,r},D) = \alpha_r$.

Set $E = \{a_{2r,2r} : r \in IN\}$ and $\eta = \inf\{\alpha_r : r \in IN\}$.

Clearly, $(*)$ $0 < \varepsilon \leq \eta < +\infty$.

We claim that E is not w-TB in $S(E,\eta)$. For if not, there is a $p \in IN$ and a $t > 0$ such that $S(a_{2p,2p};t) \ll S(E,\eta)$ and $S(a_{2p,2p};t)$ contains an infinite subset $F \subset \{a_{2n,2n} : n \in IN\}$. From what we have done so far, it follows that $t < \eta$ and $d(a_{2p,2p},F) = \alpha_{2p} \leq t$. So, $\alpha_{2p} \leq t < \eta \leq \inf\{\alpha_r : r \in IN\}$ which contradicts $(*)$. □

Corollary 4.7. *Let* X *a metrizable space with metric* d *and* $B \in CL(X)$. *The following are equivalent:*

(a) B *is compact;*

(b) *for each closed subset* E *of* B *and each* $W \in \tau$ *with* $E \subset W$, $E \subset\subset_d W$;

(c) *for each closed subset* E *of* B *and each* $W \in \tau$ *with* $E \subset W$, E *is w-TB in* W.

Proof. $(a) \Rightarrow (b)$ is clear because since γ is an EF-proximity and B is compact, then $B \subset W$ is equivalent to $B \ll_\gamma W$. Hence apply (d) in Remark 4.1.

Again, $(b) \Leftrightarrow (c)$ follows from (d) in Remark 4.1.

$(c) \Rightarrow (a)$ It suffices to prove that if every closed subset of B is w-TB in a larger open set, then every sequence $\{x_n : n \in IN\}$ in B has a convergent subsequence. By above Lemma $\{x_n : n \in IN\}$ has a Cauchy subsequence $\{y_k : k \in IN\}$. If $\{y_k : k \in IN\}$ does not converge, then the closed set $\{y_{2k+1} : k \in IN\}$ is not w-TB in the open set $X \setminus \{y_{2k} : k \in IN\}$. □

From (d) in Remark 4.1, Main Theorem, Lemma 4.6 and Corollary 4.7 we have:

Theorem 4.8. (cf. Proposition 4.1 in [8]) *Let X be a metrizable space with metric d, $\gamma = \gamma(d)$ the metric proximity induced by d, $\mathcal{B} = \mathcal{B}(d)$ the family of finite unions of all closed d-balls and Δ a cobase. The following are equivalent:*

(a) $\sigma(\gamma; \Delta) \leq \tau(W_d)$;
(b) *for every $B \in \Delta \setminus \{X\}$ and every positive real ε there exists a $B' \in \mathcal{B}$ such that $B \subset B' \ll_\gamma S(B, \varepsilon)$;*
(c) *for every $B \in \Delta \setminus \{X\}$ and every positive real ε, $B \subset\subset_d S(B, \varepsilon)$.*

Thus if Δ is hereditarily closed, then $\sigma(\gamma; \Delta) \leq \tau(W_d)$ if and only if every $B \in \Delta$ is totally bounded.

Corollary 4.9. (cf. Theorem (5.5) in [2]) *Let (X, d) be a metric space and γ the metric proximity induced by d. The following are equivalent:*

(a) $\tau(W_d) = \sigma(\gamma)$;
(b) *for each positive real ε and each proper closed set E, $E \subset\subset_d S(E, \varepsilon)$;*
(c) *for each positive real ε and each closed subspace B of X, B is w-TB in $S(B, \varepsilon)$;*
(d) *X is totally bounded.*

Theorem 4.10. (cf. Proposition 4.2 in [8]) *Let X be a metrizable space with metric d, $\gamma = \gamma(d)$ the metric proximity induced by d, $\mathcal{B} = \mathcal{B}(d)$ the family of finite unions of all closed d-balls and Δ a cobase. The following are equivalent:*

(a) $\tau(\Delta) \leq \tau(W_d)$;
(b) *for every $B \in \Delta$ and every $W \in \tau$, $W \neq X$, with $B \subset W$, there exists a $B' \in \mathcal{B}$ such that $B \subset B' \ll_\gamma W$;*
(c) *whenever $B \in \Delta$ and $W \in \tau$, $W \neq X$, with $B \subset W$, then $B \subset\subset_d W$.*

Thus if Δ is hereditarily closed, then $\tau(\Delta) \leq \tau(W_d)$ if and only if every $B \in \Delta$ is compact.

Corollary 4.11. (cf. Corollary (5.6) in [2]) *Let (X, d) be a metric space and γ the metric proximity induced by d. The following are equivalent:*

(a) $\tau(W_d) = \tau(V)$;
(b) *whenever E is a proper closed subset of X and $W \in \tau$ is such that $E \subset W$, then $E \subset\subset_d W$;*
(c) *for each closed subspace B of X and each $W \in \tau$ with $B \subset W$, B is w-TB in W;*
(d) *X is compact.*

Theorem 4.12. (cf. Proposition 3.1 in [7]) *Let X be a metrizable space with metric d, $\gamma = \gamma(d)$ the metric proximity induced by d, $TB = TB(d)$ the cobase of all closed d-totally bounded set of X. The following are equivalent:*

(a) $\tau(TB) \leq \tau(W_d)$;
(b) *X is complete.*

Proof. The Main Theorem shows that the range of any subsequence of a Cauchy sequence (which is totally bounded) is far from every disjoint closed set. □

Theorem 4.13. *Let X be a metrizable space and $\gamma = \gamma(d)$ the metric proximity induced by d. The following are equivalent:*

(a) $\tau(V) = \sigma(\gamma)$;
(b) $\gamma = \gamma_0$;
(c) *X is Atsuji.*

Proof. Use the Main Theorem noting that $\tau(V) = \sigma(\gamma_0) = \sigma(\gamma)$ if and only if $\tau(V)^+ = \sigma(\gamma_0)^+ = \sigma(\gamma)^+$. □

5. The Tychonoff case.

As we noted above, quite a bit of the literature on hyperspaces, especially that concerning the Wijsman topology, is based on metric spaces. Here we show that many results are valid in (generalized) uniform spaces (see [6], [10], [11] and [5]).

In this section (X, τ) denotes a Tychonoff space, γ a compatible EF-proximity, whereas γ_0 and $\gamma^\sharp$ denote respectively the Wallman (the fine LO-) proximity and the functionally indistinguishable (the fine EF-) proximity on X (see Preliminaries). It is a well known fact that $\gamma^\sharp = \gamma_0$ if and only if X is normal (Urysohn's Lemma).

$\Pi(\gamma)$ denotes the family of all uniformities $\mathcal{U}$ compatible with γ, $\mathcal{U}^\star$ is the coarsest totally bounded member of $\Pi(\gamma)$ and $\mathcal{U}^\sharp$ the fine uniformity. $\Pi(\tau)$ denotes the family of all uniformities $\mathcal{U}$ compatible with τ. It is a well kown fact that $\Pi(\gamma) \subset \Pi(\tau)$.

Without loss of generality we may assume that each $\mathcal{U} \in \Pi(\tau)$ consists of members which are **symmetric and closed**.

For each $\mathcal{U} \in \Pi(\tau)$, $\gamma(\mathcal{U})$ is the uniform proximity induced by $\mathcal{U}$, whereas $\mathcal{B}(\mathcal{U})$ denotes the collection of finite unions of (generalized) closed balls w.r.t. $\mathcal{U}$, where for each $x \in X$ and $U \in \mathcal{U}$, $U[x]$ is the (generalized) U ball in X centered at x.

$TB(\mathcal{U})$ denotes the collection of all closed $\mathcal{U}$-totally bounded subsets of X.

The result below cannot be extended to LO-proximity spaces (see Lemma 6.1 in [5]).

Theorem 5.1. (cf. Theorem 2.7 in [11], [1] Page 50) *Let (X, τ) be a Tychonoff space and γ a compatible EF-proximity on X. Then*
$\sup\{\sigma(\gamma, \gamma_0; \mathcal{B}(\mathcal{U})) : \mathcal{U} \in \Pi(\gamma)\} = \sigma(\gamma)$.

Proof. It suffices to prove $\sigma(\gamma) \leq \sup\{\sigma(\gamma, \gamma_0; \mathcal{B}(\mathcal{U})) : \mathcal{U} \in \Pi(\gamma)\}$.

Suppose $A \in W_\gamma^{++} \in \sigma(\gamma)$, $W \in \tau$, $W \neq X$ and $A \in CL(X)$. Then there are a closed entourage $U \in \mathcal{U}^\star$ and a finite subset F of X such that $A \subset U[F] \subset U^2[F] \ll_\gamma W$. Thus $\sigma(\gamma) \leq \sigma(\gamma, \gamma_0; \mathcal{B}(\mathcal{U}^\star))$ and hence the claim holds. □

Corollary 5.2. (cf. [11]) *Let (X, τ) be a Tychonoff space. Then*
$\sup\{\sigma(\gamma, \gamma_0; \mathcal{B}(\mathcal{U})) : \mathcal{U} \in \Pi(\tau)\} = \sigma(\gamma^\sharp)$.

Corollary 5.3. (cf. [21]) *Let (X,τ) be a Tychonoff space. The following are equivalent:*

(a) $\sup\{\sigma(\gamma,\gamma_0;\mathcal{B}(\mathcal{U})) : \mathcal{U}\in\Pi(\tau)\} = \tau(V)$*;*
(b) *X is normal;*
(c) $\gamma^\sharp = \gamma_0$.

Corollary 5.4. (cf. [2]) *Let X be a metrizable space with metric d and $\gamma=\gamma(d)$ the metric proximity induced by d. Then:*

(a) $\sup\{\tau(W_e)$*: e varies in the set of all metrics uniformly equivalent to d*$\} = \sigma(\gamma)$*;*
(b) $\sup\{\tau(W_e)$*: e varies in the set of all metrics topologically equivalent to d*$\} = \tau(V)$.

Theorem 5.5. (cf. [7] and Lemma 6.12 in [5]) *Let (X,τ) be a Tychonoff space, γ a compatible EF-proximity on X, $\mathcal{U}\in\Pi(\gamma)$ and $TB(\mathcal{U})$ the family of all totally bounded closed subsets of X w.r.t. $\mathcal{U}$. Then*
$\sigma(\gamma;TB(\mathcal{U}))\leq\sigma(\gamma,\gamma_0;\mathcal{B}(\mathcal{U}))$.

Proof. Let $B\in TB(\mathcal{U})$ and $W\in\tau$ with $B\ll_\gamma W\neq X$. Then there are an entourage $U\in\mathcal{U}$ and a finite set $F\subset X$ such that $B\subset U[F]\subset U^2[F]\subset W$. Observe that $U[F]\in\mathcal{B}(\mathcal{U})$ and $U[F]\subset U^2[B]\subset W$ implies $U[F]\ll_\gamma W$. The result follows from the Main Theorem. □

Let $\mathcal{U}$ be a separated uniformity on X and $U\in\mathcal{U}$. We recall that that $U'\in\mathcal{U}$ is *composably contained* in U iff there is a $U''\in\mathcal{U}$ such that $U'\circ U''\subset U$ (see Definition (3.3) in [10]).

Next result generalizes Theorem 3.1 in [8] to Tychonoff spaces.

Theorem 5.6. (cf. Theorem 3.1 in [8]) *Let (X,τ) be a Tychonoff space, $\mathcal{U}\in\Pi(\tau)$, the compatible EF-proximity $\gamma=\gamma(\mathcal{U})$ and Δ a cobase. The following are equivalent:*

(a) $\sigma(\gamma,\gamma_0;\mathcal{B}(\mathcal{U}))\leq\tau(\Delta)$*;*
(b) *for each $x\in X$, for each $U\in\mathcal{U}$ with $U[x]\neq X$ and for each $U'\in\mathcal{U}$ composably contained in U there is a $B'\in\Delta$ such that $U'[x]\subset B'\ll_\gamma U[x]$;*
(c) $\sigma(\gamma,\gamma_0;\mathcal{B}(\mathcal{U}))\leq\sigma(\gamma;\Delta)$.

Proof. It suffices to use the Main Theorem observing that if $U'\in\mathcal{U}$ is composably contained in $U\in\mathcal{U}$, then $U'[x]\ll_\gamma U[x]$ for each $x\in X$. □

Using the notion of composably contained we have the next definition which extends to uniform setting the concept of strict inclusion.

Definition 5.7. (cf. [3], [15] and [1] Page 38) Let (X,τ) a Tychonoff space, $\mathcal{U}\in\Pi(\tau)$ and D, $E\subset X$. We say that D is *strictly $\mathcal{U}$-included* in E ($D\subset\subset_{\mathcal{U}} E$) iff there exist a finite set $F\subset D$ and entourages U, $U'\in\mathcal{U}$ with U' composably contained in U such that

$$(\star) \quad D \subset U'[F] \subset U[F] \subset E.$$

Remark 5.8. If $\gamma = \gamma(\mathcal{U})$, then in above definition condition $(\star)$ is equivalent to the following one

$$(\star') \quad D \subset U'[F] \ll_\gamma U[F] \subset E.$$

Thus whenever γ is a compatible EF-proximity on X and $\mathcal{U} \in \Pi(\gamma)$, a set D is stricly $\mathcal{U}$-included in E iff there exist a finite subset F and entourages U, $U' \in \mathcal{U}$ such that $D \subset U'[F] \ll_\gamma U[F] \subset E$.

We also note that if $\mathcal{U}$ is the metric uniformity induced by d, then for each $U \in \mathcal{U}$ there exists some positive ε such that $U = \{(x,y) \in X \times X : d(x,y) \leq \varepsilon\}$. Thus $U[x] = B(x,\varepsilon)$ for each $x \in X$ and $D \subset\subset_{\mathcal{U}} E$ is equivalent to $D \subset\subset_d E$ (cf. (c) in Remark 4.1).

The next result generalizes Theorems 4.1 and 4.2 in [8] to the EF-proximities setting. We omit the proof since it is easily derived from definitions, the Main Theorem and the above Remark.

Theorem 5.9. *Let (X,τ) be a Tychonoff space, γ a compatible EF-proximity on X, $\mathcal{U} \in \Pi(\gamma)$ and Δ a cobase. In the following (a) and (b) are equivalent and each implies (c) which is equivalent to (d).*

- (a) $\tau(\Delta) \leq \sigma(\gamma, \gamma_0; \mathcal{B}(\mathcal{U}))$;
- (b) *whenever $B \in \Delta \setminus \{X\}$ and $W \in \tau$ with $B \subset W$, then $B \subset\subset_{\mathcal{U}} W$.*
- (c) *for every $B \in \Delta \setminus \{X\}$ and $U \in \mathcal{U}$, $B \subset\subset_{\mathcal{U}} U[B]$;*
- (d) $\sigma(\gamma; \Delta) \leq \sigma(\gamma, \gamma_0; \mathcal{B}(\mathcal{U}))$.

Corollary 5.10. (cf. **3.11**) *Let (X,τ) be a Tychonoff space, γ a compatible EF-proximity on X and $\mathcal{U} \in \Pi(\gamma)$. In the following (a), (b), (c), (d) and (e) are equivalent and each implies (f) which is equivalent to (g).*

- (a) $\tau(\mathcal{B}(\mathcal{U})) \leq \sigma(\gamma, \gamma_0; \mathcal{B}(\mathcal{U}))$;
- (b) $\sigma(\gamma, \gamma_0; \mathcal{B}(\mathcal{U})) = \tau(\mathcal{B}(\mathcal{U})) = \sigma(\gamma; \mathcal{B}(\mathcal{U}))$;
- (c) *whenever $B \in \mathcal{B}(\mathcal{U})$ and $W \in \tau$ with $B \subset W$, then $B \subset\subset_{\mathcal{U}} W$;*
- (d) *for every $B \in \mathcal{B}(\mathcal{U})$ and $W \in \tau$, $W \neq X$, with $B \subset W$, there exists a $B' \in \mathcal{B}(\mathcal{U})$ such that $B \subset B' \ll_\gamma W$;*
- (e) *X is $\mathcal{B}(\mathcal{U})$-Atsuji space w.r.t. γ (i.e. disjoint closed sets, one of which is a member of $\mathcal{B}(\mathcal{U})$, are far w.r.t. γ).*
- (f) $\sigma(\gamma; \mathcal{B}(\mathcal{U})) \leq \sigma(\gamma, \gamma_0; \mathcal{B}(\mathcal{U}))$;
- (g) *for every $B \in \mathcal{B}(\mathcal{U})$ and $U \in \mathcal{U}$, $B \subset\subset_{\mathcal{U}} U[B]$.*

We end this section with the following results derived from (d) in Remark 2.2 and Lemma 3.2.

Theorem 5.11. (cf. [6]) *Let (X,τ) be a Tychonoff space, γ a compatible EF-proximity on X and $\mathcal{U} \in \Pi(\gamma)$. Then the proximal topology $\sigma(\gamma)$ and the Hausdorff-Bourbaki topology $\tau(\mathcal{U}_H)$ are equal if and only if the uniformity $\mathcal{U}$ equals the coarsest member $\mathcal{U}^\star \in \Pi(\gamma)$, and thus $\mathcal{U}$ is totally bounded.*

Proof. The result follows from the fact that $\mathcal{U}$ is totally bounded iff maximal U-discrete sets are finite for all $U \in \mathcal{U}$. □

Corollary 5.12. (cf. [2]) *Let (X, τ) be a Tychonoff space, γ a compatible EF-proximity on X and $\mathcal{U} \in \Pi(\gamma)$. The following are equivalent:*

(a) $\sigma(\gamma, \gamma_0; \mathcal{B}(\mathcal{U})) = \tau(\mathcal{U}_H)$;
(b) *$\mathcal{U}$ is totally bounded.*

6. Bounded Hypertopologies.

In the literature on hyperspaces of a metric space (X, d), there are several bounded hypertopologies such as (i) the bounded Vietoris $\tau(bV)$, (ii) the bounded proximal $\sigma(b\gamma)$ w.r.t. γ, (iii) the bounded Hausdorff metric topology (or the Attouch-Wets topology) $\tau(AW_d)$, etc. In this section, with the help of a slight generalization of the concept of **abstract boundedness** due to Hu ([16], also see [12] and [22]) we show that all bounded hypertopologies can be subsumed under the locally finite Bombay topologies.

Definition 6.1. A nonempty collection $\Delta \subset CL(X)$ is called a *boundedness* in a topological space X iff Δ is **closed under finite unions** and is **closed hereditary**. We also assume that Δ **contains the singletons**.

Remark 6.2. (a) The followings are examples of boundedness:

(i) $B(X)$ the family of all closed d-bounded subsets of a metric space (X, d). This can be extended easily to a uniform space $(X, \mathcal{U})$ by declaring a closed set A is $\mathcal{U}$-bounded iff there is a finite set $\{x_1, \cdots, x_n\} \subset X$ and entourages $U_k \in \mathcal{U} \setminus \{X \times X\}$, $k = 1, \cdots, n$, such that $A \subset \bigcup_{k=1}^{n} U_k[x_k]$ or using notation of Section 4, there is a $B \in \mathcal{B}(\mathcal{U})$ such that $A \subset B$. Let $\Delta = B(\mathcal{U})$ denote the family of all $\mathcal{U}$-bounded subsets of X and $\gamma = \gamma(\mathcal{U})$ the EF-proximity induced by $\mathcal{U}$.

(ii) The family $K(X)$ of all nonempty compact subsets of a Hausdorff topological space.

(b) Let $(X, \mathcal{U})$ be a Hausdorff uniform space, Δ a boundedness and $\gamma = \gamma(\mathcal{U})$. Then the Δ-*bounded Hausdorff Bourbaki filter* (or Δ-*Attouch-Wets filter*) $\mathcal{U}_\Delta$ is generated by sets of the form:

$[B, U] = \{(A_1, A_2) \in CL(X) \times CL(X) : A_1 \cap B \subset U[A_2]$ and $A_2 \cap B \subset U[A_2]\}$, where $B \in \Delta$ and $U \in \mathcal{U}$.

The associated topology $\tau(\mathcal{U}_\Delta)$ is the Δ-*bounded Hausdorff topology* (or the Δ-*Attouch-Wets topology*).

We note that if $X \in \Delta$ we get the *Hausdorff Bourbaki-uniformity.* If $\mathcal{U}$ is induced by a metric d and $\Delta = TB(d)$, then the $\tau(\mathcal{U}_\Delta)$ is the *bounded Hausdorff topology* (i.e. the *Attouch-Wets topology*) $\tau(AW_d)$. It was shown in [20] that if $IL = IL_{\mathcal{U},\Delta}$, then the family of sets

$\{U[x] : x \in \mathcal{Q}\}$ where $\mathcal{Q}$ is a maximal discrete subset of $B \in \Delta$ for $U \in \mathcal{U}$

describes $\tau(IL_{\mathcal{U},\Delta})$. Thus $\tau(\mathcal{U}_\Delta) = \tau(IL^-_{\mathcal{U},\Delta}) \vee \sigma(\delta; \Delta)^+$.

It is now obvious that Δ-bounded hypertopologies are a part and parcel of Δ-Bombay topologies and the usual metric d-boundedness is also included in our definition of Δ, and by above it follows that:

Theorem 6.3. *The Δ-bounded Hausdorff topology* (i.e. the Δ-Attouch-Wets topology) $\tau(\mathcal{U}_\Delta)$ *and the bounded Hausdorff topology* (i.e. the Attouch-Wets topology) $\tau(AW_d)$ *are proximal IL-locally finite.*

Moreover, using the Main Theorem and the decomposition property of the locally finite Bombay topology we have:

Theorem 6.4. *Let (X, τ) be a T_1 topological space with compatible LO-proximities γ_1, γ_2, η_1, η_2 satisfyng $\gamma_1 \leq \gamma_2$ and $\eta_1 \leq \eta_2$, Δ, Λ two boundedness and IL_1 and IL_2 locally finite collections of open subsets of X. The following are equivalent:*

(a) $\sigma(\gamma_1, \gamma_2; IL_1, \Delta) \leq \sigma(\eta_1, \eta_2; IL_2, \Lambda)$;

(b) *IL_2 refines IL_1 and whenever $B \in \Delta$, $W \in \tau$, $W \neq X$, with $B \ll_{\gamma_1} W$, then there exists a $B' \in \Lambda$ such that:*
 (i) $B \subset B' \ll_{\eta_1} W$, *and*
 (ii) $\gamma_2(B) \subset \eta_2(B')$.

Let (X, d) be a metric space and as usual $\gamma = \gamma(d)$ the metric proximity, $\mathcal{U}$ the separated uniformity associated to d, $\mathcal{B}$ the cobase generated by all closed d-balls and $B(X)$ the family of all closed d-bounded subsets of X. We have: (see [1] Page 115)

(1) the *bounded proximal topology* $\sigma(\delta; B(X)) = \tau(V^-) \vee \sigma(\delta; B(X))^+$;

(2) the *dual bounded proximal topology* $\sigma(\gamma; IL, \mathcal{B}) = \tau(IL^-_{\mathcal{U},\mathcal{B}}) \vee \sigma(\gamma, \gamma_0; \mathcal{B})^+$, which is clearly a IL-locally finite bounded Wijsman topology.

We point out that from above Remark (b) if in (2) if we replace $\mathcal{B}$ with $B(X)$, then $\sigma(\gamma; IL, B(X)) = \tau(IL^-_{\mathcal{U},B(X)}) \vee \sigma(\gamma, \gamma_0; B(X))^+$.

Thus from Theorem 6.4 and above Remark the following results are obvious and certainly they have generalizations:

Theorem 6.5. *Let X be a metrizable space with metric d, $\gamma = \gamma(d)$ the metric proximity induced by d, $\mathcal{B}$ the cobase generated by all proper closed d balls and $B(X)$ the family of all closed d bounded subsets of X. Then:*

(a) $\sigma(\gamma, \gamma_0; \mathcal{B}) \leq \sigma(\gamma; \mathcal{B}) \leq \sigma(\gamma; B(X)) \leq \sigma(\gamma)$;
(b) $\sigma(\gamma, \gamma_0; \mathcal{B}) \leq \sigma(\gamma; \mathcal{B}) \leq \sigma(\gamma; B(X)) \leq \tau(AW_d)$.

Theorem 6.6. (cf. [1], Page 93) *Let X be a metrizable space, d and e compatible metrics, γ and η the metric proximities induced by d and e respectively, $B(X)$ the family of all closed d-bounded sets and $B'(X)$ the family of all closed e-bounded sets. The following are equivalent:*

(a) $\tau(AW_d) = \tau(AW_e)$;
(b) $\delta = \delta'$, $B(X) = B'(X)$ (i.e. every closed d-bounded subset is e-bounded and vice versa) *and they have the same uniformly continuous functions on bounded sets.*

Theorem 6.7. *Let X be a metrizable space with metric d, $\gamma = \gamma(d)$. As usual, let $B(X)$ and $TB = TB(d)$ denote respectively the family of all closed d-bounded sets and the family of all closed d-totally bounded sets. The following are equivalent:*

(a) $B(X) \subset TB$;
(b) $\tau(AW_d) = \tau(W_d)$;
(c) $\tau(AW_d) = \sigma(\gamma; B(X))$.

7. Uniformizable Bombay hypertopologies.

Let (X, τ) be a T_1 space. This section is devoted to the characterization of the uniformizable Bombay topologies associated with a cobase Δ and compatible LO-proximities γ, η satysfying $\gamma < \eta$. We do not consider the case $\gamma = \eta$ because from (a) and (b) in Remark 2.2 we get the proximal Δ topologies $\sigma(\gamma; \Delta)$ (w.r.t. γ) and the uniformizable proximal Δ topologies $\sigma(\gamma; \Delta)$ have been treated in [1], [6] and [9] for example. Furthermore, we omit the proofs as they are similar to those in [9]. First we need the following definitions.

Definition 7.1. Let (X, τ) be a T_1 space, γ a compatible LO-proximity and $\Delta \subset CL(X)$.

(a) Δ is called γ-*Urysohn* iff whenever $D \in \Delta$ and $A \in CL(X)$ are far w.r.t. γ, there exists an $S \in \Delta$ such that $D \ll_\gamma S \ll_\gamma A^c$ (see also [4])

(b) Δ is called *Urysohn* iff (a) above is true w.r.t. the LO-proximity γ_0, i.e. whenever $D \in \Delta$ and $A \in CL(X)$ are disjoint, there exists an $S \in \Delta$ such that $D \subset intS \subset S \subset A^c$.

Lemma 7.2. (cf. Theorem 1.6 in [9]) *Let X be a T_1 space with compatible LO-proximities γ, η satysfying $\gamma < \eta$ and Δ a cobase. If Δ is γ-Urysohn, then the relation π defined on the power set of X by:*

($\star$) *$A \not\pi B$ iff $clA \not\gamma clB$ and either $clA \in \Delta$ and $clB \ll_\eta (clA)^c$ or $clB \in \Delta$ and $clA \ll_\eta (clB)^c$*

is a compatible EF-proximity on X. Moreover, $\pi \leq \gamma$ and Δ is γ-Urysohn iff Δ is π-Urysohn.

Remark 7.3. (a) Observe that even if the starting proximities γ and η are just LO, the new compatible proximity π is always EF. Thus the base space X is automatically completely regular.

Note that we have a procedure that allows us to construct an EF-proximity on a Tychonoff space X by using as seeds LO-proximities γ, η with $\gamma < \eta$ and a cobase Δ which is γ-Urysohn.

(b) Let X be an infinite set, τ the cofinite topology, $\Delta = CL(X)$ and γ_l the coarsest compatible LO-proximity on X defined by

$$A \not\gamma_l B \text{ iff } A \not\gamma_0 B \text{ and either } A \text{ or } B \text{ is finite.}$$

Then X is T_1 and it is easy to check that Δ is not γ-Urysohn for each compatible LO-proximity γ with $\gamma_l \leq \gamma \leq \gamma_0$.

Lemma 7.4. (cf. Theorem 2.2 in [9]) *Let X be a T_1 space with compatible LO-proximities γ and η satisfying $\gamma < \eta$ and Δ a cobase. If Δ is γ-Urysohn, then the upper Bombay topology $\sigma(\delta, \eta; \Delta)^+$ equals the upper proximal Δ-topology $\sigma(\pi; \Delta)^+$ w.r.t. π, where π is the EF-proximity on X constructed in above Lemma 7.2.*

Theorem 7.5. (cf. Theorem 2.1 in [9]) *Let X be a T_1 space with compatible LO-proximities γ, η satisfying $\gamma < \eta$ and Δ a cobase hereditarily closed. The following are equivalent:*

(a) *Δ is γ-Urysohn;*
(b) *the Bombay topology $\sigma(\gamma, \eta; \Delta)$ is Tychonoff.*

Let X be a Tychonoff space with a compatible uniformity $\mathcal{U}$ (as usual we assume that all elements $U \in \mathcal{U}$ are **symmetric and closed**), $\mathcal{B}(\mathcal{U})$ the cobase generated by all (generalized) closed balls w.r.t. $\mathcal{U}$ (see preliminaries in Section 4) and $\gamma = \gamma(\mathcal{U})$ denote the uniform proximity induced by $\mathcal{U}$. Then it is easy to show that $\mathcal{B}(\mathcal{U})$ is γ-Urysohn. It therefore follows that:

Corollary 7.6. (cf. [14]) *Let $\mathcal{U}$ be a compatible uniformity on X whose elements are* **symmetric and closed**, *$\mathcal{B}(\mathcal{U})$ the cobase generated by all (generalized) $\mathcal{U}$-balls and γ the uniform proximity induced by $\mathcal{U}$. Then the Wijsman topology $\tau(W(\mathcal{U}))$ is Tychonoff.*

Problem 7.7. *Is it true that in a Tychonoff space every EF-proximity can be constructed using two LO-proximities as in Lemma 7.2 ?*

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