

Five different proofs of extraresolvability of countable totally bounded groups

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Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

ABSTRACT. We give different proofs of extraresolvability for countably infinite topological spaces and in particular for totally bounded groups.

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1. INTRODUCTION.

For a subset A of a topological space X , the closure and the derived set of A are denoted by \bar{A} and A' , respectively. As usual, A is said to be discrete provided that $A \cap A' = \emptyset$; A is said to be nowhere dense if the interior of \bar{A} is empty. A family \mathcal{A} refines a family \mathcal{B} if every element of \mathcal{A} is contained in some element of \mathcal{B} .

In this paper every space is assumed to be T_1 and dense in itself. The following elementary fact will be used in the sequel.

Proposition 1.1. *In a dense in itself topological space every discrete subset is nowhere dense.*

In 1943 E. Hewitt called a space *resolvable* if it has two disjoint dense subsets [8]. Subsequently, discussing a result of Sierpiński [17], J. G. Ceder introduced maximally resolvable spaces: he proved that locally compact spaces and spaces with a linearly ordered base of neighborhoods are maximally resolvable [3].

A space is said to be κ -resolvable if there exists a collection (resolution) of κ -many disjoint dense subsets. The resolution may be chosen in such a way that every set intersects every non-empty open subset in at least κ -many points. A space X is said to be maximally resolvable if it is $\Delta(X)$ -resolvable, where

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$\Delta(X)$ is the minimum cardinal of a non-empty open set. Interesting results about finite, countable and maximal resolvability may be found in [6, 15, 9, 10].

In [12], the authors prove that each totally bounded topological group is maximally resolvable. The following definition is due to V.I. Malykhin (*e.g.*, see [7]).

Definition 1.2. A space X is called *extraresolvable* if there exists a family \mathcal{D} of dense subsets of X such that $|\mathcal{D}| > \Delta(X)$ and $A \cap B$ is nowhere dense whenever A and B are distinct elements of \mathcal{D} .

In [11] V.I. Malykhin proved that every countably infinite totally bounded group is extraresolvable (through an almost disjoint family of cardinality \mathfrak{c}). Recall that a family \mathcal{D} of infinite subsets of a set is said to be almost disjoint if $A \cap B$ is finite whenever A and B are distinct elements of \mathcal{D} .

Below we give a small survey with different proofs of extraresolvability for countably infinite spaces and, in particular, for totally bounded groups. Each proof presents different techniques and results which may be useful in studying this topic. Each section title is given according to the main tool of resolvability used in that section.

The neutral element of a group will always be denoted by e and every group topology is assumed to be T_0 (hence completely regular). Furthermore, we work with groups which admit a totally bounded Hausdorff topology (*e.g.*, every abelian group).

2. DISCRETE SUBSETS.

A subset M of a topological space X is said to be *strongly discrete* if for every $x \in M$ there exists an open neighborhood $V(x)$ of x such that $V(a) \cap V(b) = \emptyset$ whenever a and b are different points of M .

Definition 2.1. A point z of X is called an *lsd-point* if there exists a strongly discrete subset M such that $z \in \overline{M}$.

The proof of the following propositions is straightforward.

Proposition 2.2. *A point z is an lsd-point iff there exists a set M , points of which have a disjoint system of open neighborhoods $\Theta = \{V(x) : x \in M\}$ such that $z \notin \cup \Theta$ and $z \in \overline{M}$.*

Proposition 2.3. *Let M be a strongly discrete subset of a regular space X . If $z \notin \overline{M}$ then $M \cup \{z\}$ is strongly discrete.*

Proposition 2.4. *Let Y be a subset of an Hausdorff space X . If $Y' \neq \emptyset$, then Y contains an infinite strongly discrete subset.*

In [16] P.L. Sharma and S. Sharma proved that if every point of a T_1 space is an lsd-point, then the space is \aleph_0 -resolvable. In the following theorem we use some ideas of their construction.

Theorem 2.5. *Let X be a countably infinite regular space. If every point of X is an lsd-point, then X is extraresolvable through a collection of cardinality \mathfrak{c} .*

Proof. Let $\{z_n : n \in \omega\}$ be a one to one numeration of the space X . Since z_0 is an lsd-point, then there exist M and $\Theta = \{V(x) : x \in M\}$ as in Proposition 2.2. Put $M_0 = M$ and $\Theta_0 = \Theta$. By Proposition 2.3, it is not restrictive to assume that $z_1 \in \overline{M_0}$.

Now we are going to describe the next step of the inductive construction, which is very familiar with the general step. For each $x \in M_0$, still choose M_x and Θ_x as in Proposition 2.2. We can assume that $\bigcup \Theta_x \subseteq V(x)$ for each $x \in M_0$. Let $M_1 = \bigcup \{M_x : x \in M_0\}$. The set M_1 is strongly discrete with disjoint system of open neighborhoods $\Theta_1 = \bigcup \{\Theta_x : x \in M_1\}$. Notice that Θ_1 refines Θ_0 . Still by Proposition 2.3, it is not restrictive to assume that $z_2 \in \overline{M_0 \cup M_1}$.

In the general case, by repeating the process for every $x \in M_n$, we get sequences $\{M_n\}$ and $\{\Theta_n\}$ satisfying the following:

- (1) $z_{n+1} \in \overline{M_0 \cup \dots \cup M_n}$,
- (2) M_n is strongly discrete with disjoint systems of open neighborhoods Θ_n ,
- (3) $M_n \subseteq M'_{n+1}$,
- (4) Θ_{n+1} refines Θ_n .

By (2) and (3), we get that $(\bigcup \Theta_{n+1}) \cap M_n = \emptyset$; consequently, by (4), the sets M_n are mutually disjoint.

For every infinite subset $A \subseteq \omega$ put $X(A) = \bigcup_{n \in A} M_n$. By (1) and (3), $X(A)$ is dense in X . Since the sets M_n are mutually disjoint, then $X(A) \cap X(B) = X(A \cap B)$. Consequently $X(A) \cap X(B)$ is nowhere dense whenever $A \cap B$ is finite (Proposition 1.1). Thus if \mathcal{A} is an almost disjoint family of cardinality \mathfrak{c} of infinite subsets of ω , then $X(\mathcal{A}) = \{X(A) : A \in \mathcal{A}\}$ is a collection of cardinality \mathfrak{c} which ensures the extraresolvability of X . \square

Theorem 2.6. *Every countably infinite totally bounded Hausdorff group is extraresolvable.*

Proof. I. V. Protasov [13] constructed a strongly discrete subset D such that $e \in D'$ in every totally bounded group topology. Consequently the identity (hence every element) is an lsd-point and the conclusion follows from Theorem 2.5. \square

3. WEAK SEQUENCES.

A weak sequence on X is a countably infinite disjoint family \mathcal{F} of finite subsets of X . We say that \mathcal{F} converges to a point x if $\{F \in \mathcal{F} : F \cap V = \emptyset\}$ is finite for each neighborhood V of x .

Proposition 3.1. *Let X be a countably infinite space such that every point admits a weak sequence converging to it. Then there exist a weak sequence converging to every point of X .*

Proof. Let $X = \{z_n : n \in \omega\}$ be a one to one numeration of X and for each n let \mathcal{F}_n be a weak sequence converging to z_n . We shall construct a new weak sequence $\{K_m : m \in \omega\}$ which converges to each element of the space.

By induction, suppose that K_0, \dots, K_{m-1} have been already defined. The set

$T = \bigcup_{i < m} K_i$ is finite, so for each $i \leq m$ there is some $F^i \in \mathcal{F}_i$ such that $(\bigcup_{i < m} F^i) \cap T = \emptyset$. Put $K_m = \bigcup_{i < m} F^i$. It remains to prove that, for every neighborhood V of some point $z_{\bar{n}}$, the set $\{K_m : K_m \cap V = \emptyset\}$ is finite. Since K_m contains elements of $\mathcal{F}_{\bar{n}}$ for each $m \geq \bar{n}$ and almost all elements of $\mathcal{F}_{\bar{n}}$ meet V , we get the conclusion. \square

Theorem 3.2. *Let X be a countably infinite space such that every point admits a weak sequence converging to it. Then X is extraresolvable through an almost disjoint family of cardinality \mathfrak{c} .*

Proof. By Proposition 3.1 there exists a weak sequence $\mathcal{K} = \{K_n : n \in \omega\}$ converging to every point of X . As a consequence, the set $\mathcal{K}(A) = \bigcup\{K_n : n \in A\}$ is dense in X for every infinite subset A of ω . Since the sets K_n are mutually disjoint, $\mathcal{K}(A) \cap \mathcal{K}(B) = \mathcal{K}(A \cap B)$: this set is finite if $A \cap B$ is finite. The conclusion follows by considering the family of sets $\{\mathcal{K}(A)\}$, where A ranges over an almost disjoint family of ω of cardinality \mathfrak{c} . \square

Before deducing Theorem 2.6, we need to recall the concept of quadrosequence.

If X is a countably infinite subset of an infinite group G , a *quadrosequence* is a set of the form $D(X) = \{x_i x_n^{-1} : i < n \in \omega\}$, where $\{x_n\}$ is any one to one numeration of X (the alternative choice $\{x_n x_i^{-1} : i < n\}$ has no consequence on constructions and results). Notice that the quadrosequence $D(X)$ depends on the numeration.

A basic property of totally bounded topological groups may be translated in terms of quadrosequences [2].

Theorem 3.3. *An infinite topological group is totally bounded if and only if the identity belongs to the closure of every quadrosequence.*

The quadrosequence $D(X)$ coincides with the union of finite sets $D_n(X) = \{x_i x_n^{-1} : i < n\}$. If J is a subset of ω , put $D_J(X) = \bigcup_{n \in J} D_n(X)$.

Theorem 3.4. *Let X be a countably infinite subset of a group G .*

- (1) *The identity belongs to the closure of $D_J(X)$ for every infinite subset J of ω .*
- (2) *There exists an infinite subset Y of X such that the sets $D_n(Y)$ are mutually disjoint.*
- (3) *If Y is infinite and the sets $D_n(Y)$ are mutually disjoint, then the weak sequence $\{D_n(Y)\}$ converges to the identity in every totally bounded group topology.*

Proof. (1) If $X_J = \{x_n : n \in J\}$, then $D_J(X) \supseteq D(X_J)$ and consequently $e \in \overline{D(X_J)} \subseteq \overline{D_J(X)}$.

(2) Arguing by induction, suppose that y_0, \dots, y_n have been defined in such a way that the sets $D_k = \{y_i y_k^{-1} : i < k\}$ are mutually disjoint for every $k \leq n$. Consider the set $F = \bigcup\{D_k : k \leq n\}$. Since F is finite, there exists an element

$y_{n+1} \in X \setminus \{y_0, \dots, y_n\}$ such that $y_i y_{n+1}^{-1} \notin F$, for each $i \leq n$. The required set is $Y = \{y_n : n \in \omega\}$.

(3) By (1), the identity belongs to the closure of every infinite union of sets $D_n(Y)$; hence the weak sequence $\{D_n(Y)\}$ converges to e . \square

By items (2) and (3) of Theorem 3.4, for every $g \in G$ there exists a weak sequence, namely $gD_n(Y)$, converging to g in every totally bounded group topology. Hence Theorem 3.2 implies the following strengthening of Theorem 2.6 [11].

Theorem 3.5. *Let G be a countably infinite group. There exists an almost disjoint family of cardinality \mathfrak{c} of sets which are dense in any totally bounded group topology on G .*

Remark 3.6. The nowhere density $\text{nwd}(X)$ of a space X is the minimum cardinality of a subset which fails to be nowhere dense. A space X is called strongly extraresolvable if there exists a family \mathcal{A} of dense subsets of X such that $|\mathcal{A}| > \Delta(X)$ and $|A \cap B| < \text{nwd}(X)$ whenever A and B are distinct elements of \mathcal{A} . W.W. Comfort and S. Garcia-Ferreira proved that if $d(X) = |G| \geq \omega$, then G is strongly extraresolvable [5]. The previous Theorem 3.5 proves that a countably infinite totally bounded group is strongly extraresolvable through a family of cardinality \mathfrak{c} .

4. TALAGRAND'S THEOREM.

In this section we shortly present the proof of Theorem 3.5 given in [11].

We identify a subset $A \subseteq X$ with its characteristic function $\chi_A \in 2^X$, where 2^X has the product topology. If \mathcal{F} denotes a family of subsets of X , then the subspace $\mathfrak{B}(\mathcal{F}) = \{\chi_F : F \in \mathcal{F}\} \subseteq 2^X$ is called the binary space of \mathcal{F} .

M. Talagrand [18] proved that for a free filter \mathcal{F} on a set X the following conditions are equivalent:

- (a) $\mathfrak{B}(\mathcal{F})$ is meager (as a subset of 2^X).
- (b) There exists a sequence K_n of mutually disjoint finite subsets of X such that the set $\{n : K_n \cap F = \emptyset\}$ is finite for each $F \in \mathcal{F}$.

An accurate reading of the proof of Talagrand's theorem shows that it suffices to assume that the family \mathcal{F} of non-empty subsets of X satisfy the following condition: if $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$.

A subset L of an infinite group G is called large if there exists a finite subset $K \subseteq G$ such that $KL = LK = G$. In [1] the authors proved that the binary space $\mathfrak{B}(\mathcal{L})$ of all large subsets of a countably infinite group G is meager. So, according to the general form of Talagrand's theorem, there exists a sequence $\{K_n\}$ of mutually disjoint finite subsets of G such that $\{n : K_n \cap L = \emptyset\}$ is finite for every large subset L of G . As each non-empty open subset of a totally bounded group is large, we deduce the following proposition.

Proposition 4.1. *Let G be a countably infinite group. There exists a weak sequence $\{K_n\}$ which converges at each point of G with respect to any Hausdorff totally bounded group topology.*

The third proof of Theorem 2.6 follows as in Theorem 3.2 by considering the family of sets $G(A) = \bigcup_{n \in A} K_n$, where A ranges over an almost disjoint family of cardinality \mathfrak{c} .

5. QUADROSEQUENCES.

The proof given here uses some ideas which are already present in Section 3. One interesting point is the result provided in Proposition 5.1.

We denote by $X^m = \{x_1, \dots, x_m\}$ a one to one numeration of a set with m elements of a group G and by $D(X^m)$ the set $\{x_i x_k^{-1} : i < k, k \leq m\}$.

Proposition 5.1. *Let $G = \{g_n : n \in \omega\}$ be a one to one numeration of a countably infinite group G . There exist infinite subsets X_n , $n \in \omega$, such that the subsets $T_n = \{g_1, \dots, g_n\} \cdot D(X_n)$ are mutually disjoint.*

Proof. Let us consider a general step of a pyramidal inductive construction. Let us assume that some initial parts $X_k^{m_k} = \{x_{k,1}, \dots, x_{k,m_k}\}$ have already been defined for each $k \leq n$ in such a way that the sets

$$T_k^{m_k} = \{g_1, \dots, g_k\} \cdot D(X_k^{m_k})$$

are pairwise disjoint. The $n + 1$ th step consists in adding a new point to every $X_k^{m_k}$ and starting with the first two points of X_{n+1} .

- *Adding a new point to some $X_k^{m_k}$, for $k \leq n$.*

In this case, an added point $x = x_{m_k+1}$ must satisfy to the following conditions:

$$g_i y x^{-1} \notin T_r^{m_r}, \quad i \leq k, \quad r \neq k, \quad r \leq n, \quad y \in X_k^{m_k}.$$

The sets $T_r^{m_r}$ vary during the process of adding these points.

Such a point x does exist since the number of excluded conditions is finite.

- *Forming a new set $X_{n+1}^2 = \{a, b\}$.*

In this case the elements $a = x_{n+1,1}$ and $b = x_{n+1,2}$ must satisfy to the following conditions:

$$g_i a b^{-1} \notin T_k^{m_k}, \quad i \leq n + 1, \quad k \leq n.$$

This construction ends the proof. □

Arguing as in Theorem 3.4, item (2), one obtains the following:

Lemma 5.2. *Let F be a finite subset of G and let X be a countably infinite subset of G . There exists an infinite subset Y of X such that the sets $F \cdot D_m(Y)$ are mutually disjoint.*

Now we deduce Theorem 3.5 from Proposition 5.1.

Proof. By Theorem 3.4, item (1), $g \in \overline{g \cdot \bigcup \{D_m(X_n) : m \in J\}}$ for every infinite subset J of ω . Consequently, if A is a subset of ω , the set

$$\mathcal{T}(A) = \bigcup_{n \in A} \left(\bigcup_{m \in A} \{g_1, \dots, g_n\} \cdot D_m(X_n) \right)$$

is dense whenever A is infinite.

By Lemma 5.2, in Proposition 5.1 it is not restrictive to assume that, for each $n \in \omega$, the family

$$\{\{g_1, \dots, g_n\} \cdot D_m(X_n) : m \in \omega\}$$

is disjoint. Since the sets T_n are mutually disjoint too, the set

$$\mathcal{T}(A) \cap \mathcal{T}(B) = \mathcal{T}(A \cap B) = \bigcup_{n \in A \cap B} \left(\bigcup_{m \in A \cap B} \{g_1, \dots, g_n\} \cdot D_m(X_n) \right)$$

is finite whenever $A \cap B$ is finite. The required collection of sets is obtained by considering the almost disjoint family $\{\mathcal{T}(A)\}$, where A ranges over an almost disjoint family of ω of cardinality \mathfrak{c} . \square

6. PROTASOV METHOD.

In this section we use a method due to Protasov, by applying his argument to the countable case [14, 12].

As in Section 4, a subset L of a group G is said to be large if there exists a finite subset K of G such that $KL = LK = G$ (e.g., see [1]). Sets of the form gK and Kg are called right and left circles of radius K and center g , respectively. The following criterium of [1] will be useful in the sequel.

Proposition 6.1. *The set $G \setminus S$ fails to be large if and only if S contains (right or left) circles of any finite radius.*

Proposition 6.2. *Let G be a countably infinite group. There exists a weak sequence $\mathcal{F} = \{F_n\}$ such that, whenever $A \subseteq \omega$ is infinite and $\omega \setminus A$ is infinite, both sets $\mathcal{F}(A) = \bigcup_{n \in A} F_n$ and $G \setminus \mathcal{F}(A)$ fail to be large.*

Proof. Let $G = \{g_n\}$ be a one to one numeration of G and let $G_n = \{g_k : k < n\}$, for each $n \in \omega$. Arguing by induction, we shall construct a weak sequence $\{F_n\}$ in such a way that F_n contains a circle of radius G_n for each n . Let us assume that F_0, \dots, F_{n-1} have been already defined. Since $T = \bigcup_{i < n} F_i$ is finite, the set $T \cdot G_n^{-1} \neq G$. By choosing $t_n \notin T \cdot G_n^{-1}$, we have $t_n G_n \cap T = \emptyset$. Put $F_n = t_n G_n$. By construction, the sets F_n are mutually disjoint and every F_n contains a circle of radius G_n . Let A be an infinite subset of ω . Every finite set K is contained in some G_n with $n \in A$ and therefore $\mathcal{F}(A) = \bigcup \{F_n : n \in A\}$ contains circles of any finite radius, so that $G \setminus \mathcal{F}(A)$ fails to be large. If $\omega \setminus A$ is infinite, then $G \setminus \mathcal{F}(A) \supseteq \mathcal{F}(\omega \setminus A)$ also contains circles of any finite radius, and consequently $\mathcal{F}(A)$ fails to be large too. \square

Since in a totally bounded group a set containing a non-empty open set is large, we have that the sets $\mathcal{F}(A)$ of Proposition 6.2 are dense in every totally bounded group topology. The extraresolvability, as in Theorem 3.5, follows once more by considering the family $\{\mathcal{F}(A)\}$, where A ranges over an almost disjoint family of ω of cardinality \mathfrak{c} .

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