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# Effective representations of the space of linear bounded operators

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Abstract. Representations of topological spaces by infinite sequences of symbols are used in computable analysis to describe computations in topological spaces with the help of Turing machines. From the computer science point of view such representations can be considered as data structures of topological spaces. Formally, a representation of a topological space is a surjective mapping from Cantor space onto the corresponding space. Typically, one is interested in admissible, i.e. topologically well-behaved representations which are continuous and characterized by a certain maximality condition. We discuss a number of representations of the space of linear bounded operators on a Banach space. Since the operator norm topology of the operator space is non-separable in typical cases, the operator space cannot be represented admissibly with respect to this topology. However, other topologies, like the compact open topology and the Fell topology (on the operator graph) give rise to a number of promising representations of operator spaces which can partially replace the operator norm topology. These representations reflect the information which is included in certain data structures for operators, such as programs or enumerations of graphs. We investigate the sublattice of these representations with respect to continuous and computable reducibility. Certain additional conditions, such as finite dimensionality, let some classes of representations collapse, and thus, change the corresponding graph. Altogether, a precise picture of possible data structures for operator spaces and their mutual relation can be drawn.

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## 1. Preliminaries from Computable Analysis

In this paper we will study representations of the set of linear bounded operators on Banach spaces from the point of view of computable analysis, which is the Turing machine based theory of computability on real numbers and other topological spaces. Pioneering work on this theory has been presented by Turing [23], Banach and Mazur [1], Lacombe [14] and Grzegorczyk [10]. Recent monographs have been published by Pour-El and Richards [18], Ko [11] and Weihrauch [26]. Certain aspects of computable functional analysis have already been studied by several authors, see for instance [16, 9, 24, 27, 28].

In this section we briefly summarize some notions from computable analysis. For details the reader is referred to [26]. The basic idea of the representation based approach to computable analysis is to represent infinite objects like real numbers, functions or sets, by infinite strings over some alphabet $\Sigma$ (which should at least contain the symbols 0 and 1). Thus, a *representation* of a set $X$ is a surjective mapping $\delta :\subseteq \Sigma^\omega \to X$ and in this situation we will call $(X, \delta)$ a *represented space*. Here $\Sigma^\omega$ denotes the set of infinite sequences over $\Sigma$ and the inclusion symbol is used to indicate that the mapping might be partial. If we have two represented spaces, then we can define the notion of a computable function.

**Definition 1.1** (Computable function)**.** Let $(X, \delta)$ and $(Y, \delta')$ be represented spaces. A function $f :\subseteq X \to Y$ is called $(\delta, \delta')$*–computable*, if there exists some computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta' F(p) = f\delta(p)$ for all $p \in \mathrm{dom}(f\delta)$.

Of course, we have to define computability of functions $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ to make this definition complete, but this can be done via Turing machines: $F$ is computable if there exists some Turing machine, which computes infinitely long and transforms each sequence $p$, written on the input tape, into the corresponding sequence $F(p)$, written on the one-way output tape. Later on, we will also need computable multi-valued operations $f :\subseteq X \rightrightarrows Y$, which are defined analogously to computable functions by substituting $\delta' F(p) \in f\delta(p)$ for the equation in Definition 1.1 above. If the represented spaces are fixed or clear from the context, then we will simply call a function or operation $f$ *computable*.

For the comparison of representations it will be useful to have the notion of *reducibility* of representations. If $\delta, \delta'$ are both representations of a set $X$, then $\delta$ is called *reducible* to $\delta'$, $\delta \leq \delta'$ in symbols, if there exists a computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta(p) = \delta' F(p)$ for all $p \in \mathrm{dom}(\delta)$. Obviously, $\delta \leq \delta'$ holds, if and only if the identity $\mathrm{id} : X \to X$ is $(\delta, \delta')$–computable. Moreover, $\delta$ and $\delta'$ are called *equivalent*, $\delta \equiv \delta'$ in symbols, if $\delta \leq \delta'$ and $\delta' \leq \delta$.

Analogously to the notion of computability we can define the notion of $(\delta, \delta')$*–continuity* for single and multi-valued operations, by substituting a continuous function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ for the computable function $F$ in the definitions above. On $\Sigma^\omega$ we use the *Cantor topology*, which is simply the product topology of the discrete topology on $\Sigma$. The corresponding reducibility will be called *continuous reducibility* and we will use the symbols $\leq_t$ and $\equiv_t$ in this case. Again we will simply say that the corresponding function is *continuous*, if the representations are fixed or clear from the context. If not mentioned otherwise, we will always assume that a represented space is endowed with the final topology induced by its representation.

This will lead to no confusion with the ordinary topological notion of continuity, as long as we are dealing with *admissible* representations. A representation $\delta$ of a topological space $X$ is called *admissible*, if $\delta$ is maximal among all continuous representations $\delta'$ of $X$, i.e. if $\delta' \leq_t \delta$ holds for all continuous representations $\delta'$ of $X$. If $\delta, \delta'$ are admissible representations of $T_0$–spaces $X$, $Y$ with countable bases, then a function $f :\subseteq X \to Y$ is $(\delta, \delta')$–continuous, if and only if it is continuous in the ordinary topological sense. For an extension of these notions to larger classes of spaces cf. [20, 21].

Given a represented space $(X, \delta)$, we will occasionally use the notions of a *computable sequence* and a *computable point*. A *computable sequence* is a computable function $f : \mathbb{N} \to X$, where we assume that $\mathbb{N} = \{0, 1, 2, \ldots\}$ is represented by $\delta_{\mathbb{N}}(1^n 0^\omega) := n$ and a point $x \in X$ is called *computable*, if there is a constant computable function with value $x$.

Given two represented spaces $(X, \delta)$ and $(Y, \delta')$, there is a canonical representation $[\delta, \delta']$ of $X \times Y$ and a representation $[\delta \to \delta']$ of certain functions $f : X \to Y$. If $\delta, \delta'$ are *admissible* representations of $T_0$–spaces with countable bases, then $[\delta \to \delta']$ is actually a representation of the set $\mathcal{C}(X, Y)$ of continuous functions $f : X \to Y$. If $Y = \mathbb{R}$, then we write for short $\mathcal{C}(X) := \mathcal{C}(X, \mathbb{R})$. The function space representation can be characterized by the fact that it admits evaluation and type conversion.

**Proposition 1.2** (Evaluation and type conversion)**.** *Let $(X, \delta), (Y, \delta')$ be admissibly represented $T_0$–spaces with countable bases and let $(Z, \delta'')$ be a represented space. Then:*

1. **(Evaluation)** $\mathrm{ev} : \mathcal{C}(X, Y) \times X \to Y, (f, x) \mapsto f(x)$ *is* $([[\delta \to \delta'], \delta], \delta')$*–computable,*
2. **(Type conversion)** $f : Z \times X \to Y$*, is* $([\delta'', \delta], \delta')$*–computable, if and only if the function* $\check{f} : Z \to \mathcal{C}(X, Y)$*, defined by* $\check{f}(z)(x) := f(z, x)$ *is* $(\delta'', [\delta \to \delta'])$*–computable.*

The proof of this proposition is based on a version of the smn– and utm–Theorems, and can be found in [26]. If $(X, \delta)$, $(Y, \delta')$ are admissibly represented $T_0$–spaces with countable bases, then in the following we will always assume that $\mathcal{C}(X, Y)$ is represented by $[\delta \to \delta']$. It is known that the computable points in $(\mathcal{C}(X, Y), [\delta \to \delta'])$ are just the $(\delta, \delta')$–computable functions $f : X \to Y$ [26]. If $(X, \delta)$ is a represented space, then we will always assume that the set of sequences $X^{\mathbb{N}}$ is represented by $\delta^{\mathbb{N}} := [\delta_{\mathbb{N}} \to \delta]$. The computable points in $(X^{\mathbb{N}}, \delta^{\mathbb{N}})$ are just the computable sequences in $(X, \delta)$. Moreover, we assume that $X^n$ is always represented by $\delta^n$, which can be defined inductively by $\delta^1 := \delta$ and $\delta^{n+1} := [\delta^n, \delta]$.

To make this paper as self-contained as possible, we will discuss some basic facts on computable metric spaces and computable Banach spaces in the following section. Section 3 will be devoted to a short introduction into hyperspace representations and, finally, Section 4 presents our results on representations of the set of linear bounded operators.

## 2. Computable Metric and Banach Spaces

In this section we will briefly discuss computable metric spaces and computable Banach spaces. The notion of a computable Banach space will be the central notion for all following results. Computable metric spaces have been used in the literature at least since Lacombe [15]. Restricted to computable points they have also been studied by various authors [8, 12, 17, 22]. We consider computable metric spaces as special separable metric spaces, but on all points and not only restricted to computable points [25]. Pour-El and Richards have introduced a closely related axiomatic characterization of sequential computability structures for Banach spaces [18] which has been extended to metric spaces by Mori, Tsujii, and Yasugi [27].

Before we start with the definition of computable metric spaces we just mention that we will denote the *open balls* of a metric space $(X, d)$ by $B(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$, $\varepsilon > 0$ and correspondingly the *closed balls* by $\overline{B}(x, \varepsilon) := \{y \in X : d(x, y) \leq \varepsilon\}$. Occasionally, we denote complements of sets $A \subseteq X$ by $A^c := X \setminus A$.

**Definition 2.1** (Computable metric space)**.** A tuple $(X, d, \alpha)$ is called a *computable metric space*, if

(1) $d : X \times X \to \mathbb{R}$ is a metric on $X$,
(2) $\alpha : \mathbb{N} \to X$ is a sequence which is dense in $X$,
(3) $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable (double) sequence in $\mathbb{R}$.

Here, we tacitly assume that the reader is familiar with the notion of a computable sequence of reals, but we will come back to that point below. Occasionally, we will say for short that $X$ is a *computable metric space*. Obviously, a computable metric space is especially separable. Given a computable metric space $(X, d, \alpha)$, its *Cauchy representation* $\delta_X :\subseteq \Sigma^\omega \to X$ can be defined by

$$\delta_X(01^{n_0+1}01^{n_1+1}01^{n_2+1}\ldots) := \lim_{i\to\infty} \alpha(n_i)$$

for all $n_i$ such that $(\alpha(n_i))_{i\in\mathbb{N}}$ converges and $d(\alpha(n_i), \alpha(n_j)) \leq 2^{-i}$ for all $j > i$ (and undefined for all other input sequences). In the following we tacitly assume that computable metric spaces are represented by their Cauchy representations. If $X$ is a computable metric space, then it is easy to see that $d : X \times X \to \mathbb{R}$ becomes computable (see Proposition 3.2 in [3]). All Cauchy representations are admissible with respect to the corresponding metric topology.

An important computable metric space is $(\mathbb{R}, d_{\mathbb{R}}, \alpha_{\mathbb{R}})$ with the Euclidean metric $d_{\mathbb{R}}(x,y) := |x-y|$ and some numbering of the rational numbers $\mathbb{Q}$, as $\alpha_{\mathbb{R}}\langle i,j,k\rangle := (i-j)/(k+1)$. Here, $\langle i,j\rangle := 1/2(i+j)(i+j+1)+j$ denotes *Cantor pairs* and this definition is extended inductively to finite tuples. For short we will occasionally write $\overline{k} := \alpha_{\mathbb{R}}(k)$. In the following we assume that $\mathbb{R}$ is endowed with the Cauchy representation $\delta_{\mathbb{R}}$ induced by the computable metric space given above. This representation of $\mathbb{R}$ can also be defined, if $(\mathbb{R}, d_{\mathbb{R}}, \alpha_{\mathbb{R}})$ just fulfills (1) and (2) of the definition above and this leads to a definition of computable real number sequences without circularity. Occasionally, we will also use the represented space $(\mathbb{Q}, \delta_{\mathbb{Q}})$ of rational numbers with $\delta_{\mathbb{Q}}(1^n0^\omega) := \alpha_{\mathbb{R}}(n) = \overline{n}$.

Many important representations can be deduced from computable metric spaces, but we will also need some differently defined representations. For instance, we will use two further representations $\rho_<$, $\rho_>$ of the real numbers, which correspond to weaker information on the represented real numbers. Here

$$\rho_<(01^{n_0+1}01^{n_1+1}01^{n_2+1}\ldots) = x :\iff \{q \in \mathbb{Q} : q < x\} = \{\overline{n_i} : i \in \mathbb{N}\}$$

and $\rho_<$ is undefined for all other sequences. Thus, $\rho_<(p) = x$, if $p$ is a list of all rational numbers smaller than $x$. Analogously, $\rho_>$ is defined with "$>$" instead of "$<$". We write $\mathbb{R}_< = (\mathbb{R}, \rho_<)$ and $\mathbb{R}_> = (\mathbb{R}, \rho_>)$ for the corresponding represented spaces. The computable numbers in $\mathbb{R}_<$ are called *left-computable* real numbers and the computable numbers in $\mathbb{R}_>$ *right-computable* real numbers. The representations $\rho_<$ and $\rho_>$ are admissible with respect to the *lower* and *upper topology* on $\mathbb{R}$, which are induced by the open intervals $(q,\infty)$ and $(-\infty,q)$, respectively.

Computationally, we do not have to distinguish the complex numbers $\mathbb{C}$ from $\mathbb{R}^2$. Thus, we can directly define a representation of $\mathbb{C}$ by $\delta_{\mathbb{C}} := \delta_{\mathbb{R}}^2$. If $z = a+ib \in \mathbb{C}$, then we denote by $\overline{z} := a - ib \in \mathbb{C}$ the *conjugate complex number* and by $|z| := \sqrt{a^2+b^2}$ the *absolute value* of $z$. Alternatively to this ad hoc definition of $\delta_{\mathbb{C}}$, we could consider $\delta_{\mathbb{C}}$ as the Cauchy representation of a computable metric space $(\mathbb{C}, d_{\mathbb{C}}, \alpha_{\mathbb{C}})$, where $\alpha_{\mathbb{C}}$ is a numbering of $\mathbb{Q}[i]$, defined by $\alpha_{\mathbb{C}}\langle n,k\rangle := \overline{n} + \overline{k}i$ and $d_{\mathbb{C}}(w,z) := |w-z|$ is the Euclidean metric on $\mathbb{C}$. The corresponding Cauchy representation is equivalent to $\delta_{\mathbb{R}}^2$. In the following we will consider vector spaces over $\mathbb{R}$, as well as over $\mathbb{C}$. We will use the notation $\mathbb{F}$ for a field which always might be replaced by both, $\mathbb{R}$ or $\mathbb{C}$. Correspondingly, we use the notation $(\mathbb{F}, d_{\mathbb{F}}, \alpha_{\mathbb{F}})$ for a computable metric space which can be replaced by either of the computable metric spaces $(\mathbb{R}, d_{\mathbb{R}}, \alpha_{\mathbb{R}})$, $(\mathbb{C}, d_{\mathbb{C}}, \alpha_{\mathbb{C}})$ defined above. We will also use the notation $Q_{\mathbb{F}} = \mathrm{range}(\alpha_{\mathbb{F}})$, i.e. $Q_{\mathbb{R}} = \mathbb{Q}$ and $Q_{\mathbb{C}} = \mathbb{Q}[i]$.

For the definition of a computable Banach space it is helpful to have the notion of a computable vector space, which we will define next.

**Definition 2.2** (Computable vector space)**.** A represented space $(X, \delta)$ is called a *computable vector space* (over $\mathbb{F}$), if $(X, +, \cdot, 0)$ is a vector space over $\mathbb{F}$ such that the following conditions hold:

(1) $+ : X \times X \to X,\ (x, y) \mapsto x + y$ is computable,
(2) $\cdot : \mathbb{F} \times X \to X,\ (a, x) \mapsto a \cdot x$ is computable,
(3) $0 \in X$ is a computable point.

Here, $(\mathbb{F}, \delta_{\mathbb{F}})$ is a computable vector space and if $(X, \delta)$ is a computable vector space over $\mathbb{F}$, then $(X^n, \delta^n)$ and $(X^{\mathbb{N}}, \delta^{\mathbb{N}})$ are computable vector spaces over $\mathbb{F}$. If, additionally, $(X, \delta)$, $(Y, \delta')$ are admissibly represented second countable $T_0$–spaces, then $(\mathcal{C}(Y, X), [\delta' \to \delta])$ is a computable vector space over $\mathbb{F}$. Here we tacitly assume that the vector space operations on product, sequence and function spaces are defined componentwise. The proof for the function space is a straightforward application of evaluation and type conversion. The central definition for the present investigation will be the notion of a computable normed space.

**Definition 2.3** (Computable normed space)**.** A tuple $(X, \| \ \|, e)$ is called a *computable normed space*, if

(1) $\| \ \| : X \to \mathbb{R}$ is a norm on $X$,
(2) $e : \mathbb{N} \to X$ is a *fundamental sequence*, i.e. its linear span is dense in $X$,
(3) $(X, d, \alpha_e)$, with
$d(x, y) := \| x - y \|$ and $\alpha_e\langle k, \langle n_0, \dots, n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$,
is a computable metric space with Cauchy representation $\delta_X$,
(4) $(X, \delta_X)$ is a computable vector space over $\mathbb{F}$.

If, in the situation of the definition, the underlying space $(X, \| \ \|)$ is actually a Banach space, i.e. if $(X, d)$ is a complete metric space, then $(X, \| \ \|, e)$ is called a *computable Banach space*. If the norm and the fundamental sequence are clear from the context, or locally irrelevant, we will say for short that $X$ is a *computable normed space* or a *computable Banach space*.

We will always assume that computable normed spaces are represented by their Cauchy representations, which are admissible with respect to the norm topology. If $X$ is a computable normed space, then $\| \ \| : X \to \mathbb{R}$ is a computable function. Of course, all computable Banach spaces are separable. In the following proposition a number of computable Banach spaces are defined.

**Proposition 2.4** (Computable Banach spaces)**.** *Let $p \in \mathbb{R}$ be a computable real number with $1 \le p < \infty$ and let $a < b$ be computable real numbers. The following spaces are computable Banach spaces over $\mathbb{F}$.*

(1) $(\mathbb{F}^n, \| \ \|_p, e)$ *and* $(\mathbb{F}^n, \| \ \|_\infty, e)$ *with*
- $\| (x_1, x_2, \dots, x_n) \|_p := \sqrt[p]{\sum_{k=1}^{n} |x_k|^p}$ *and*
  $\| (x_1, x_2, \dots, x_n) \|_\infty := \max_{k=1,\dots,n} |x_k|$,
- $e_i = e(i) = (e_{i1}, e_{i2}, \dots, e_{in})$ *with* $e_{ik} := \begin{cases} 1 & \textit{if } (\exists j)\ i = jn + k, \\ 0 & \textit{otherwise.} \end{cases}$

(2) $(\ell_p, \| \ \|_p, e)$ *with*
- $\ell_p := \{x \in \mathbb{F}^{\mathbb{N}} : \| x \|_p < \infty\}$,

- $\| (x_k)_{k\in\mathbb{N}} \|_p := \sqrt[p]{\sum_{k=0}^{\infty} |x_k|^p}$,
- $e_i = e(i) = (e_{ik})_{k\in\mathbb{N}}$ *with* $e_{ik} := \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$

(3) $(\mathcal{C}[a,b], \| \|, e)$ *with*

- $\mathcal{C}[a,b] := \{f : [a,b] \to \mathbb{F} \mid f \text{ continuous}\}$,
- $\| f \| := \max_{t\in[a,b]} |f(t)|$,
- $e_i(t) = e(i)(t) = t^i$.

We leave it to the reader to check that these spaces are actually computable Banach spaces. Unless stated otherwise, we will assume that $(\mathbb{F}^n, \| \|)$ is endowed with the maximum norm $\| \| = \| \|_\infty$. It is known that the Cauchy representation $\delta_{\mathcal{C}[a,b]}$ of $\mathcal{C}[a,b] = \mathcal{C}([a,b], \mathbb{R})$ is equivalent to $[\delta_{[a,b]} \to \delta_{\mathbb{R}}]$, where $\delta_{[a,b]}$ denotes the restriction of $\delta_{\mathbb{R}}$ to $[a,b]$ (cf. Lemma 6.1.10 in [26]). In the following we will occasionally utilize the sequence spaces $\ell_p$ to construct counterexamples. We close this section with a brief discussion of product spaces of computable normed spaces.

**Proposition 2.5** (Product spaces). *If $(X, \| \|, e)$, $(Y, \| \|', e')$ are computable normed spaces, then the product space $(X \times Y, \| \|'', e'')$, defined by*

$$\| (x,y) \|'' := \max\{\| x \|, \| y \|'\} \quad \textit{and} \quad e''\langle i,j\rangle := (e(i), e'(j)),$$

*is a computable normed space too and the canonical projections of the product space* $\mathrm{pr}_1 : X \times Y \to X$ *and* $\mathrm{pr}_2 : X \times Y \to Y$ *are computable.*

## 3. Hyperspaces of closed subsets

Since we want to use the hyperspace $\mathcal{A}(X)$ of closed subsets $A \subseteq X$ in order to represent linear bounded operators, we have to discuss representations of hyperspaces. Such representations have been studied in the Euclidean case in [6, 26] and for the metric case in [5].

**Definition 3.1** (Hyperspace of closed subsets). Let $(X, d, \alpha)$ be a computable metric space. We endow the hyperspace $\mathcal{A}(X) := \{A \subseteq X : A \text{ closed}\}$ of closed subsets with the representation $\delta^<_{\mathcal{A}(X)}$, defined by

$$\begin{aligned} & \delta^<_{\mathcal{A}(X)}(01^{\langle n_0,k_0\rangle+1}01^{\langle n_1,k_1\rangle+1}01^{\langle n_2,k_2\rangle+1}\ldots) = A \\ :\iff \quad & \{(n,k) : A \cap B(\alpha(n), \overline{k}) \neq \emptyset\} = \{(n_i,k_i) \in \mathbb{N}^2 : i \in \mathbb{N}\}, \end{aligned}$$

and with the representation $\delta^>_{\mathcal{A}(X)}$, defined by

$$\delta^>_{\mathcal{A}(X)}(01^{\langle n_0,k_0\rangle+1}01^{\langle n_1,k_1\rangle+1}01^{\langle n_2,k_2\rangle+1}\ldots) := X \setminus \bigcup_{i=0}^{\infty} B(\alpha(n_i), \overline{k_i}).$$

Whenever we have two representations $\delta, \delta'$ of some set, we can define the infimum $\delta \sqcap \delta'$ of $\delta$ and $\delta'$ by $(\delta \sqcap \delta')\langle p,q\rangle = x :\iff \delta(p) = x$ and $\delta'(q) = x$. We use the short notations $\mathcal{A}_< = \mathcal{A}_<(X) = (\mathcal{A}_<(X), \delta^<_{\mathcal{A}(X)})$, $\mathcal{A}_> = \mathcal{A}_>(X) = (\mathcal{A}_>(X), \delta^>_{\mathcal{A}(X)})$ and $\mathcal{A} = \mathcal{A}(X) = (\mathcal{A}(X), \delta^<_{\mathcal{A}(X)} \sqcap \delta^>_{\mathcal{A}(X)})$ for the corresponding represented spaces. For the computable points of these spaces special names are used.

**Definition 3.2** (Recursively enumerable and recursive sets)**.** Let $X$ be a computable metric space and let $A \subseteq X$ be a closed subset.

(1) $A$ is called *r.e. closed*, if $A$ is a computable point in $\mathcal{A}_<(X)$,
(2) $A$ is called *co-r.e. closed*, if $A$ is a computable point in $\mathcal{A}_>(X)$,
(3) $A$ is called *recursive closed*, if $A$ is a computable point in $\mathcal{A}(X)$.

These definitions generalize the classical notions of r.e. and recursive sets, since a set $A \subseteq \mathbb{N}$, considered as a closed subset of $\mathbb{R}$, is r.e., co-r.e., recursive closed, if and only if it has the same property in the classical sense as a subset of $\mathbb{N}$ [6]. We close this section with a helpful result on hyperspaces, which follows directly from results in [5] and which can be considered as an effective version of the statement that closed subsets of separable metric spaces are separable. Here and in the following, $\overline{A}$ denotes the topological closure of a subset $A \subseteq X$ of some topological space $X$.

**Proposition 3.3** (Separable closed subsets)**.** *Let $X$ be a computable metric space. The mapping $\overline{\mathrm{range}} : X^{\mathbb{N}} \to \mathcal{A}_<(X),\ (x_n)_{n\in\mathbb{N}} \mapsto \overline{\{x_n : n \in \mathbb{N}\}}$ is computable and if $X$ is complete, then it admits a computable multi-valued partial right-inverse $\subseteq \mathcal{A}_<(X) \rightrightarrows X^{\mathbb{N}}$, defined for all non-empty closed subsets.*

## 4. Representations of the Operator Space

In this section we will define and compare several representations of the set $\mathcal{B}(X, Y)$ of linear bounded operators $T : X \to Y$ on computable normed spaces $X$ and $Y$.

**Definition 4.1** (Representations of the operator space)**.** Let $(X, \| \|, e)$ and $Y$ be computable normed spaces. We define representations of $\mathcal{B}(X, Y)$:

(1) $\delta_{\mathrm{ev}}(p) = T :\Longleftrightarrow [\delta_X \to \delta_Y](p) = T$,
(2) $\delta_{\mathrm{graph}}(p) = T :\Longleftrightarrow \delta_{\mathcal{A}(X\times Y)}(p) = \mathrm{graph}(T)$,
(3) $\delta^<_{\mathrm{graph}}(p) = T :\Longleftrightarrow \delta^<_{\mathcal{A}(X\times Y)}(p) = \mathrm{graph}(T)$,
(4) $\delta_{\mathrm{seq}}(p) = T :\Longleftrightarrow \delta^{\mathbb{N}}_Y(p) = (Te_i)_{i\in\mathbb{N}}$,

for all $p \in \Sigma^\omega$ and linear bounded operators $T : X \to Y$.

Besides the representations defined we also consider some variants that are obtained by adding information on the operator bound $\| T \| = \sup_{\|x\|=1} \| Tx \|$ of the represented operator $T : X \to Y$: if $\delta$ is a representation of $\mathcal{B}(X, Y)$, then we define representations $\delta^=$, $\delta^\geqslant$ of $\mathcal{B}(X, Y)$ by

(1) $\delta^=\langle p, q\rangle = T :\Longleftrightarrow \delta(p) = T$ and $\delta_{\mathbb{R}}(q) = \| T \|$,
(2) $\delta^\geqslant\langle p, q\rangle = T :\Longleftrightarrow \delta(p) = T$ and $\delta_{\mathbb{R}}(q) \geq \| T \|$.

All the representations introduced are admissible with respect to certain topologies on the space of linear bounded operators (see the brief discussion in the Conclusion). While these representations separate into several distinct topological and computational equivalence classes, the corresponding computable linear bounded operators do essentially coincide (see Corollary 4.6). This follows from the following characterization which we have proved in [3].

**Theorem 4.2** (Computable Linear Operators)**.** *Let $X, Y$ be computable Banach spaces, let $(e_i)_{i\in\mathbb{N}}$ be a computable sequence in $X$ whose linear span is dense in $X$ and let $T : X \to Y$ be a linear operator. Then the following conditions are equivalent:*

(1) $T : X \to Y$ *is computable,*
(2) $T : X \to Y$ *is bounded and maps computable sequences in $X$ to computable sequences in $Y$,*
(3) $T : X \to Y$ *is bounded and $(Te_k)_{k\in\mathbb{N}}$ is a computable sequence in $Y$,*
(4) $\mathrm{graph}(T)$ *is an r.e. closed subset of $X \times Y$,*
(5) $\mathrm{graph}(T)$ *is a recursive closed subset of $X \times Y$.*

The equivalence in this theorem is "non-constructive" in the sense that nothing is said about the possibility of converting one type of information into another type effectively. One of the main purposes of this paper is to study these possibilities. In this sense the following result includes a uniform version of the previous theorem.

**Theorem 4.3** (Representations of the operator space)**.** *Let $X$ and $Y$ be computable Banach spaces. Then the following reductions for representations of $\mathcal{B}(X,Y)$ hold:*

(1) $\delta_{\mathrm{ev}}^{=} \le \delta_{\mathrm{ev}} \le \delta_{\mathrm{seq}} \le \delta_{\mathrm{graph}}^{<}$ *and* $\delta_{\mathrm{ev}} \le \delta_{\mathrm{graph}} \le \delta_{\mathrm{graph}}^{<}$,
(2) $\delta_{\mathrm{ev}} \equiv \delta_{\mathrm{ev}}^{\geqslant} \equiv \delta_{\mathrm{seq}}^{\geqslant} \equiv \delta_{\mathrm{graph}}^{<\geqslant} \equiv \delta_{\mathrm{graph}}^{\geqslant}$,
(3) $\delta_{\mathrm{ev}}^{=} \equiv \delta_{\mathrm{seq}}^{=} \equiv \delta_{\mathrm{graph}}^{<=} \equiv \delta_{\mathrm{graph}}^{=}$.

*Proof.* (1) "$\delta_{\mathrm{ev}}^{=} \le \delta_{\mathrm{ev}}$" holds obviously and "$\delta_{\mathrm{ev}} \le \delta_{\mathrm{seq}}$" follows by the evaluation property.

"$\delta_{\mathrm{seq}} \le \delta_{\mathrm{graph}}^{<}$" We consider the computable Banach spaces $(X, \|\ \|, e)$ and $Y$. Let $T : X \to Y$ be a linear bounded operator and let $(y_n)_{n\in\mathbb{N}}$ be the sequence in $Y$ with $y_i := Te_i$ for all $i \in \mathbb{N}$. We consider $\alpha_e\langle k, \langle n_0, \ldots, n_k\rangle\rangle = \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$. By the linearity of $T$ it follows that

$$T\alpha_e\langle k, \langle n_0, \ldots, n_k\rangle\rangle = T\left(\sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i\right) = \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)y_i.$$

Thus, given $(y_n)_{n\in\mathbb{N}}$ we can compute $T\alpha_e$, since the algebraic operations in $Y$ are computable. Using type conversion we can effectively find the sequence $f : \mathbb{N} \to X \times Y, i \mapsto (\alpha_e(i), T\alpha_e(i))$ which is dense in $\mathrm{graph}(T)$ by the continuity of $T$, since $\alpha_e$ is dense in $X$. By Proposition 3.3 the desired result follows.

"$\delta_{\mathrm{ev}} \le \delta_{\mathrm{graph}}$" follows from the Graph Theorem ([3], 8.3), which implies that $\mathrm{graph} : \mathcal{C}(X,Y) \to \mathcal{A}(X \times Y),\ T \mapsto \mathrm{graph}(T)$ is $([\delta_X \to \delta_Y], \delta_{\mathcal{A}(X\times Y)})$–computable.

"$\delta_{\mathrm{graph}} \le \delta_{\mathrm{graph}}^{<}$" follows from $\delta_{\mathcal{A}(X\times Y)} \le \delta_{\mathcal{A}(X\times Y)}^{<}$.

(2) "$\delta_{\rm ev} \leq \delta_{\rm ev}^{\geqslant}$" follows from Theorem 9.10 in [3] (see also Theorem 4.1 in [4]) which states that for every $T \in \mathcal{B}(X,Y)$, given with respect to $\delta_{\rm ev}$, we can effectively find some upper bound $s \geq \| T \|$.

"$\delta_{\rm ev}^{\geqslant} \leq \delta_{\rm ev}$" obviously holds.

"$\delta_{\rm ev}^{\geqslant} \leq \delta_{\rm seq}^{\geqslant} \leq \delta_{\rm graph}^{<\geqslant}$" follows directly from (1).

"$\delta_{\rm graph}^{<\geqslant} \leq \delta_{\rm ev}^{\geqslant}$" Given the graph$(T) \in \mathcal{A}_<(X \times Y)$ of some linear bounded operator $T : X \to Y$, we can effectively find a sequence $f : \mathbb{N} \to X \times Y$ such that range$(f)$ is dense in graph$(T)$ by Proposition 3.3. By Proposition 2.5 we can effectively find the projections $f_1 : \mathbb{N} \to X$ and $f_2 : \mathbb{N} \to Y$ of $f$ too. Given $x \in X$, a real number $s \geq \| T \|$ and a precision $m \in \mathbb{N}$ we can effectively find $n, k \in \mathbb{N}$ and numbers $q_0, \ldots, q_n \in Q_{\mathbb{F}}$ such that $s \leq 2^k$ and $\| \sum_{i=0}^n q_i f_1(i) - x \| < 2^{-m-k-1}$ since range$(f_1)$ is dense in $X$. It follows that

$$\left\| T\left(\sum_{i=0}^n q_i f_1(i)\right) - T(x)\right\| \leq \| T \| \cdot \left\| \sum_{i=0}^n q_i f_1(i) - x \right\| < 2^{-m-1}.$$

By linearity of $T$ we obtain $T(\sum_{i=0}^n q_i f_1(i)) = \sum_{i=0}^n q_i T f_1(i) = \sum_{i=0}^n q_i f_2(i)$ and thus we can evaluate $T$ effectively up to any given precision $m$. Applying this idea, we can effectively find a Cauchy sequence which rapidly converges to $T(x)$. Using type conversion we obtain the desired reducibility.

"$\delta_{\rm ev}^{\geqslant} \leq \delta_{\rm graph}^{\geqslant}$" and "$\delta_{\rm graph}^{\geqslant} \leq \delta_{\rm graph}^{<\geqslant}$" follow from (1).

(3) This follows directly from (2). $\square$

It should be mentioned that we have used completeness only for the reduction "$\delta_{\rm graph}^{<\geqslant} \leq \delta_{\rm ev}^{\geqslant}$" (implicitly, by application of Proposition 3.3) and thus for the results in (2) and (3), while the statement of (1) remains true for computable normed spaces $X, Y$. In case of a finite-dimensional $Y = \mathbb{F}^m$, one can prove that the inverse graph$^{-1}$ of the graph map graph : $\mathcal{C}(X, \mathbb{F}^m) \to \mathcal{A}(X \times \mathbb{F}^m)$ is $(\delta_{\mathcal{A}(X\times\mathbb{F}^m)}, [\delta_X \to \delta_{\mathbb{F}^m}])$–computable, see Theorem 14.6 in [3]. As a direct consequence we obtain the following corollary.

**Corollary 4.4.** *Let $X$ be a computable normed space and $m \geq 1$ a natural number. Then $\delta_{\rm ev} \equiv \delta_{\rm graph}$ for the corresponding representations of $\mathcal{B}(X, \mathbb{F}^m)$.*

Now the question arises as to which of the reductions given in Theorem 4.3(1) are strict reductions. At least for certain spaces all of these reductions are strict as the following theorem shows. Figure 1 summarizes the results.

$$\begin{tikzcd}
 & & \delta_{\rm seq} \arrow[dr] & \\
\delta_{\rm ev}^{=} \arrow[r] & \delta_{\rm ev} \arrow[ur] \arrow[dr] & & \delta_{\rm graph}^{<} \\
 & & \delta_{\rm graph} \arrow[ur] &
\end{tikzcd}$$

**Figure 1. Representations of the operator space $\mathcal{B}(\ell_p, \ell_p)$.**

Here an arrow means $\leq$ and $\not\geq_t$. The transitive closure of the diagram is complete, i.e. all missing arrows in the closure indicate $\not\leq_t$.

**Theorem 4.5.** *Let $p \geq 1$ be some computable real number. Then the following holds for representations of the space $\mathcal{B}(\ell_p, \ell_p)$ of linear bounded operators:*

(1) $\delta^{<}_{\rm graph} \not\leq_t \delta_{\rm graph} \not\leq_t \delta_{\rm seq} \not\leq_t \delta_{\rm ev} \not\leq_t \delta^{=}_{\rm ev}$,

(2) $\delta^{<}_{\rm graph} \not\leq_t \delta_{\rm seq} \not\leq_t \delta_{\rm graph} \not\leq_t \delta_{\rm ev}$.

*In case of the operator space $\mathcal{B}(\ell_p, \mathbb{F}^m)$ with $m \geq 1$ the same statements hold with the exception of $\delta_{\rm graph} \not\leq_t \delta_{\rm seq}$ and $\delta_{\rm graph} \not\leq_t \delta_{\rm ev}$.*

*Proof.* For the first two results we only have to consider the case $\mathcal{B}(\ell_p, \ell_p)$.

"$\delta_{\rm graph} \not\leq_t \delta_{\rm ev}$" follows from the fact that the inverse graph$^{-1}$ of the map graph : $\mathcal{C}(\ell_p, \ell_p) \to \mathcal{A}(\ell_p \times \ell_p)$ is not $([\delta_{\ell_p} \to \delta_{\ell_p}], \delta_{\mathcal{A}(\ell_p \times \ell_p)})$–continuous, see Theorem 8.5 in [3].

"$\delta_{\rm graph} \not\leq_t \delta_{\rm seq}$" Let us assume that $\delta_{\rm graph} \leq_t \delta_{\rm seq}$ is realized by a continuous function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$. Let us consider the zero operator $T : \ell_p \to \ell_p$, i.e. $Tx = 0$ for all $x \in \ell_p$. Given a sequence $q \in {\rm dom}(\delta_{\rm graph})$ with $\delta_{\rm graph}(q) = T$ the function $F$ produces a sequence $t = F(q)$ such that $\delta_{\rm seq}(t) = T$. Moreover, the mapping $T \mapsto Te_0$ is obviously $(\delta_{\rm seq}, \delta_{\ell_p})$–continuous and realized by some continuous function $G :\subseteq \Sigma^\omega \to \Sigma^\omega$. We consider $H := G \circ F$. Since $H$ is continuous too and $\delta_{\ell_p} H(q) = Te_0 = 0$, there is some finite prefix $w$ of $q$ such that $\delta_{\ell_p} H(w\Sigma^\omega) \subseteq B(0, 1)$. The word $w$ contains only finitely many items of positive information $U_i = B((x_i, y_i), r_i)$, $i = 0, \ldots, n$ on graph$(T)$, i.e. graph$(T) \cap U_i \neq \varnothing$, and also only finitely many items of negative information $U'_i = B((x'_i, y'_i), r'_i) \subseteq {\rm graph}(T)^c$ with $i = 0, \ldots, m$. Since $U'_i \subseteq {\rm graph}(T)^c$, we obtain $0 \notin B(y'_i, r'_i)$ for all $i = 0, \ldots, m$ and since any $y'_i$ is of the form $y'_i = \sum_{j=0}^{k_i} a_{ij} e_j$, the value $j := \max\{k_0, \ldots, k_m\} + 1$ exists and we obtain $ae_j \notin \bigcup_{i=0}^m B(y'_i, r'_i)$ for all $a \in \mathbb{F}$ (since $e_{ik} = 1$ if $i = k$ and $e_{ik} = 0$ otherwise and hence $\| ae_j - y'_i \|_p \geq \| y'_i \|_p \geq r'_i$ for all $i \leq m$). Correspondingly, any $x_i$ is of the form $x_i = \sum_{j=0}^{l_i} b_{ij} e_j$, the value $\iota := \max\{l_0, \ldots, l_n\} + 1$ exists and there exists some $\varepsilon$ with $0 < \varepsilon < \min\{r_0, \ldots, r_n\}$ and hence $x_i + \varepsilon e_{\iota'} \in B(x_i, r_i)$ for all $i = 0, \ldots, n$ and $\iota' \geq \iota$. Now we define a matrix operator $T' : \ell_p \to \ell_p$ which corresponds to the zero matrix except for row number $j$, which is

$$\left(1, \underbrace{0, \ldots, 0}_{\iota-1\text{–times}}, -\frac{b_{00}}{\varepsilon}, -\frac{b_{10}}{\varepsilon}, \ldots, -\frac{b_{n0}}{\varepsilon}, 0, 0, \ldots\right).$$

In particular $T'e_0 = e_j$, $T'e_{\iota+i} = -\frac{1}{\varepsilon} b_{i0} e_j$ for $i = 0, \ldots, n$ and $T'e_i = 0$ for all $i \notin \{0, \iota, \iota+1, \ldots, \iota+n\}$. Then by the choice of $j$ we obtain $U'_i \subseteq {\rm graph}(T')^c$ for all $i = 0, \ldots, m$ and we claim that also $U_i \cap {\rm graph}(T') \neq \varnothing$ for all $i = 0, \ldots, n$: the last statement follows since $x_i + \varepsilon e_{\iota+i} \in B(x_i, r_i)$ for all $i = 0, \ldots, n$ and $T'(x_i + \varepsilon e_{\iota+i}) = b_{i0} e_j - \varepsilon \frac{1}{\varepsilon} b_{i0} e_j = 0 \in B(y_i, r_i)$. Altogether, $w$ is also a prefix of a name $q'$ of $T'$, i.e. $\delta_{\rm graph}(q') = T'$ but $T'e_0 = e_j \notin B(0, 1)$ in contrast to the choice of $w$. A contradiction!

For the remaining results it suffices to consider the case $\ell'_p = \mathcal{B}(\ell_p, \mathbb{F})$. This follows from the fact that $I : \mathcal{B}(\ell_p, \mathbb{F}) \to \mathcal{B}(\ell_p, \ell_p), f \mapsto (x \mapsto (f(x), 0, 0, 0, \ldots))$

is an isometric embedding such that $I$, as well as $I^{-1}$, are $(\delta,\delta)$–computable for all representations $\delta \in \{\delta^<_{\rm graph}, \delta_{\rm graph}, \delta_{\rm seq}, \delta_{\rm ev}, \delta^=_{\rm ev}\}$. Hence the case $\mathcal{B}(\ell_p,\ell_p)$ can be reduced to the case $\mathcal{B}(\ell_p,\mathbb{F})$. Analogous considerations yield the results in the case $\mathcal{B}(\ell_p,\mathbb{F}^m)$ with $m>1$.

"$\delta_{\rm ev} \not\leq_t \delta^=_{\rm ev}$" In order to prove this, it suffices to show that the operator norm mapping $\|\ \|: \mathcal{B}(\ell_p,\mathbb{F}) \to \mathbb{R}$, defined for linear bounded $T:\ell_p \to \mathbb{F}$, is not $(\delta_{\rm ev},\delta_{\mathbb{R}})$–continuous. Let $q$ be such that $\frac{1}{p}+\frac{1}{q}=1$ and $q=\infty$ if $p=1$. For any sequence $a=(a_k)_{k\in\mathbb{N}} \in \ell_q$ we define the functional $\lambda_a:\ell_p \to \mathbb{F}, (x_k)_{k\in\mathbb{N}} \mapsto \sum_{k=0}^\infty a_k x_k$. In the following we will use the map $L :\subseteq \mathbb{F}^{\mathbb{N}} \times \mathbb{R} \to \mathcal{B}(\ell_p,\mathbb{F}), (a,s) \mapsto \lambda_a$ with $\operatorname{dom}(L) := \{(a,s) \in \ell_q \times \mathbb{R} : \|\ a\ \|_q \leq s\}$ and with $q$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $q=\infty$ if $p=1$. For all $(a,s) \in \operatorname{dom}(L)$ we obtain $\|\ L(a,s)\ \| = \|\ \lambda_a\ \| = \|\ a\ \|_q \leq s$ and thus $L$ is $([\delta^{\mathbb{N}}_{\mathbb{F}},\delta_{\mathbb{R}}],\delta_{\rm ev})$–continuous. If we assume that the operator norm mapping $\|\ \|$ is $(\delta_{\rm ev},\delta_{\mathbb{R}})$–continuous, then this implies that $\|\ \| \circ L :\subseteq \mathbb{F}^{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ is $([\delta^{\mathbb{N}}_{\mathbb{F}},\delta_{\mathbb{R}}],\delta_{\mathbb{R}})$–continuous. In particular, this implies that $N :\subseteq \mathbb{F}^{\mathbb{N}} \to \mathbb{R}, a \mapsto \|\ a\ \|_q$ is continuous for all $a$ with $\|\ a\ \|_q \leq 1$. But this is obviously not the case (in general, no finite prefix of the sequence $a$ determines the value $\|\ a\ \|_q$ up to a given precision).

"$\delta_{\rm seq} \not\leq_t \delta_{\rm ev}$" Let us assume that $\delta_{\rm seq} \leq_t \delta_{\rm ev}$. Let $L$, $\lambda$ and $q$ be as defined above. Since $\lambda_a e_i = a_i$ for all $a=(a_n)_{n\in\mathbb{N}} \in \ell_q$, it follows by the evaluation property and Theorem 9.10 in [3] (cf. Theorem 4.1 in [4]) that $L$ admits a $(\delta_{\rm ev},[\delta^{\mathbb{N}}_{\mathbb{F}},\delta_{\mathbb{R}}])$–continuous multi-valued right-inverse $L^{-1} :\subseteq \mathcal{B}(\ell_p,\mathbb{F}) \rightrightarrows \mathbb{F}^{\mathbb{N}} \times \mathbb{R}$. We consider the function $\lambda :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathcal{B}(\ell_p,\mathbb{F}),\ a \mapsto \lambda_a$ with $\operatorname{dom}(\lambda) := \mathbb{R}^{\mathbb{N}} \cap \ell_q$. Since $\lambda_a e_i = a_i$ for all $a=(a_n)_{n\in\mathbb{N}} \in \ell_q$, it also follows that $\lambda$ is $(\delta^{\mathbb{N}}_{\mathbb{R}},\delta_{\rm seq})$–computable and thus $(\delta^{\mathbb{N}}_{\mathbb{R}},\delta_{\rm ev})$–continuous by assumption. Thus, $L^{-1} \circ \lambda :\subseteq \mathbb{R}^{\mathbb{N}} \rightrightarrows \mathbb{F}^{\mathbb{N}} \times \mathbb{R}$ is $(\delta^{\mathbb{N}}_{\mathbb{R}},[\delta^{\mathbb{N}}_{\mathbb{F}},\delta_{\mathbb{R}}])$–continuous too and hence the projection $S :\subseteq \mathbb{R}^{\mathbb{N}} \rightrightarrows \mathbb{R}$ on the second component is continuous in the ordinary sense too. Thus $S$ is a continuous operation such that there exists some $s \in S(a)$ for all $a \in \mathbb{R}^{\mathbb{N}}$ with $\|\ a\ \|_q < \infty$ and for all such $s$ the inequality $\|\ a\ \|_q \leq s$ holds. Such a continuous operation can obviously not exist (since it would have to determine an upper bound of $\|\ a\ \|_q$, knowing only a finite prefix of the sequence $a$). A contradiction!

"$\delta_{\rm seq} \not\leq_t \delta_{\rm graph}$" This follows from $\delta_{\rm seq} \not\leq_t \delta_{\rm ev}$ since $\delta_{\rm graph} \equiv \delta_{\rm ev}$ in the case of $\mathcal{B}(\ell_p,\mathbb{F})$ by Corollary 4.4.

"$\delta^<_{\rm graph} \not\leq_t \delta_{\rm graph}$" This follows from $\delta_{\rm seq} \not\leq_t \delta_{\rm graph}$ and $\delta_{\rm seq} \leq \delta^<_{\rm graph}$. The latter holds by Theorem 4.3(1).

"$\delta^<_{\rm graph} \not\leq_t \delta_{\rm seq}$" Let us assume that $\delta^<_{\rm graph} \leq_t \delta_{\rm seq}$. We consider the function $\lambda :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathcal{B}(\ell_p,\mathbb{F}),\ a \mapsto \lambda_a$ with $\operatorname{dom}(\lambda) := \mathbb{R}^{\mathbb{N}} \cap \ell_1$, the functionals $\lambda_a : \ell_p \to \mathbb{F}$ as above and the function

$$A :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{R},\ (a_i)_{i\in\mathbb{N}} \mapsto a_0 2^{\frac{1}{p}} - \sum_{j=0}^{\infty} a_{j+1} 2^{-\frac{j+1}{p}},$$

with $\mathrm{dom}(A) := \mathbb{R}^{\mathbb{N}} \cap \ell_1$. Let $A' :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be defined by $A'(a_i)_{i\in\mathbb{N}} := (A(a_i)_{i\in\mathbb{N}}, a_1, a_2, \ldots)$. We claim that $\lambda \circ A' :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathcal{B}(\ell_p, \mathbb{F})$ is $(\delta_{\mathbb{R}}^{\mathbb{N}}, \delta_{\mathrm{graph}}^{<})$–computable and thus $(\delta_{\mathbb{R}}^{\mathbb{N}}, \delta_{\mathrm{seq}})$–continuous by assumption too. But this implies that $A$ is continuous since $A(a) = \lambda_{A'(a)} e_0 = \lambda \circ A'(a)(e_0)$. But obviously $A$ is not continuous, since the value of the sum does substantially depend on coefficients $a_j$ for large $j$. A contradiction! It remains to prove the claim. Therefore, we define a sequence $e' = (e'_i)_{i\in\mathbb{N}}$ in $\ell_p$ by $e'_{0j} := 2^{-\frac{j+1}{p}}$ and $e'_{i+1} := e_{i+1}$ for all $i, j \in \mathbb{N}$. Then $\| \ e'_i \ \|_p = 1$ for all $i \in \mathbb{N}$ and $e' : \mathbb{N} \to \ell_p$ is a fundamental sequence, i.e. its linear span is dense in $\ell_p$. Now let $a = (a_i)_{i\in\mathbb{N}} \in \ell_1$ be a sequence and let $a' := A'(a)$. Then $\lambda_{a'} e'_i = \lambda_a e_i = a_i$ for all $i \geq 1$ and

$$\lambda_{a'} e'_0 = \sum_{j=0}^{\infty} a'_j e'_{0j} = A(a) 2^{-\frac{1}{p}} + \sum_{j=1}^{\infty} a_j 2^{-\frac{j+1}{p}} = a_0.$$

Now we consider $\alpha_{e'} : \mathbb{N} \to \ell_p$ with $\alpha_{e'}\langle k, \langle n_0, \ldots, n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i) e'_i$. By linearity of $\lambda_{A'(a)}$ it follows that

$$\lambda_{A'(a)} \alpha_{e'}\langle k, \langle n_0, \ldots, n_k\rangle\rangle = \lambda_{A'(a)} \left( \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i) e'_i \right) = \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i) a_i.$$

Thus, given $a = (a_i)_{i\in\mathbb{N}}$ we can use type conversion to effectively determine the sequence $f : \mathbb{N} \to \ell_p \times \mathbb{F}, i \mapsto (\alpha_{e'}(i), \lambda_{A'(a)} \alpha_{e'}(i))$ which is dense in $\mathrm{graph}(\lambda_{A'(a)})$ by continuity of $\lambda_{A'(a)}$ since $\alpha_{e'}$ is dense in $\ell_p$. But this proves by Proposition 3.3 that $\lambda \circ A'$ is $(\delta_{\mathbb{R}}^{\mathbb{N}}, \delta_{\mathrm{graph}}^{<})$–computable. $\square$

While this theorem shows that the aforementioned representations have to be distinguished with respect to computability and continuity, we can deduce from Theorem 4.3 (or, alternatively, from Theorem 4.2) that the corresponding classes of computable operators coincide. However, the class of computable operators with computable norm is strictly smaller in case of $X = Y = \ell_p$ or $X = \ell_p$ and $Y = \mathbb{F}$, as one can easily show (cf. Corollary 9.5 and Example 13.2 in [3]).

**Corollary 4.6.** *Let $X, Y$ be computable Banach spaces. Then the subsets of $\delta$–computable operators of $\mathcal{B}(X, Y)$ coincide for $\delta \in \{\delta_{\mathrm{ev}}, \delta_{\mathrm{graph}}, \delta_{\mathrm{seq}}, \delta_{\mathrm{graph}}^{<}\}$. The class of $\delta_{\mathrm{ev}}^{=}$–computable operators is strictly smaller in general.*

In the case of the space $\mathcal{B}(\ell_p, \mathbb{F}^m)$, Corollary 4.4 leads to a modification of Figure 1 which is displayed in Figure 2.

$$\delta_{\mathrm{ev}}^{=} \longrightarrow \delta_{\mathrm{ev}} \equiv \delta_{\mathrm{graph}} \longrightarrow \delta_{\mathrm{seq}} \longrightarrow \delta_{\mathrm{graph}}^{<}$$

**Figure 2. Representations of the operator space $\mathcal{B}(\ell_p, \mathbb{F}^m)$.**

Thus, especially in the case of the dual space all the representations considered can be ordered linearly with respect to computable reducibility.

In the case where both spaces $X, Y$ are finite-dimensional, the representations considered are equivalent, as the following result shows.

**Theorem 4.7.** *Let $n, m \geq 1$ be natural numbers. Then*

$$\delta_{\mathrm{ev}}^{=} \equiv \delta_{\mathrm{ev}} \equiv \delta_{\mathrm{graph}} \equiv \delta_{\mathrm{seq}} \equiv \delta_{\mathrm{graph}}^{<}$$

*holds for the corresponding representations of $\mathcal{B}(\mathbb{F}^n, \mathbb{F}^m)$.*

*Proof.* By Theorem 4.3 and Corollary 4.4 we obtain

$$\delta_{\mathrm{graph}}^{<=} \equiv \delta_{\mathrm{ev}}^{=} \leq \delta_{\mathrm{ev}} \equiv \delta_{\mathrm{graph}} \leq \delta_{\mathrm{seq}} \leq \delta_{\mathrm{graph}}^{<}$$

and thus it suffices to prove $\delta_{\mathrm{graph}}^{<} \leq \delta_{\mathrm{graph}}^{<=}$. In order to prove this it suffices to show that the operator norm $\|\ \| : \mathcal{B}(\mathbb{F}^n, \mathbb{F}^m) \to \mathbb{R}$ is $(\delta_{\mathrm{graph}}^{<}, \delta_{\mathbb{R}})$–computable. Given $\mathrm{graph}(T) \in \mathcal{A}_<(\mathbb{F}^n \times \mathbb{F}^m)$ for some bounded linear operator $T : \mathbb{F}^n \to \mathbb{F}^m$, we can effectively find some function $f : \mathbb{N} \to \mathbb{F}^n \times \mathbb{F}^m$ such that $\mathrm{range}(f)$ is dense in $\mathrm{graph}(T)$ by Proposition 3.3. Especially, we can obtain the projections $f_1 : \mathbb{N} \to \mathbb{F}^n$ and $f_2 : \mathbb{N} \to \mathbb{F}^m$ of $f$. Using the fact that

$$\{(x_1, \ldots, x_n) \in (\mathbb{F}^n)^n : (x_1, \ldots, x_n) \text{ linearly independent}\}$$

is an r.e. open subset of $(\mathbb{F}^n)^n$ (cf. [29, 7, 30]) we can effectively find numbers $i_1, \ldots, i_n \in \mathbb{N}$ such that $(b_1, \ldots, b_n) := (f_1(i_1), \ldots, f_1(i_n))$ is a basis of $\mathbb{F}^n$. Now we can effectively determine the function

$$g : \mathbb{F}^n \to \mathbb{F}^n,\ (a_1, \ldots, a_n) \mapsto \sum_{j=1}^{n} a_j b_j$$

by type conversion. Since $g$ is linear and bijective we can effectively determine $g^{-1}$ (this follows for instance from Corollary 14.4 in [3]) and thus we can compute the representation $e_i = \sum_{j=1}^{n} a_{ij} b_j$ of the unit vectors $e_1, \ldots, e_n \in \mathbb{F}^n$ by $(a_{i1}, \ldots, a_{in}) := g^{-1}(e_i)$. By linearity of $T$ we obtain

$$y_i := Te_i = T\left(\sum_{j=1}^{n} a_{ij} b_j\right) = \sum_{j=1}^{n} a_{ij} T b_j = \sum_{j=1}^{n} a_{ij} f_2(i_j)$$

for $i = 1, \ldots, n$ and thus $A := (y_1, \ldots, y_n) \in \mathbb{F}^{m \times n}$ is the matrix which represents $T$. Now we obtain

$$\| T \| = \| A \| := \max_{i=1,\ldots,m} \sum_{j=1}^{n} |y_{ij}|$$

where $y_j =: (y_{1j}, \ldots, y_{mj})$ for $j = 1, \ldots, n$ (see e.g. Example 23.3.b in [19] for the equality $\| T \| = \| A \|$). Thus, given the graph of $T$ we can actually compute the norm $\| T \|$, which implies the desired result. $\square$

We close this section with a brief discussion of the inversion operator. On the one hand, it is well-known that inversion $T \mapsto T^{-1}$ is continuous with respect to the operator norm topology on the subset of bijective bounded linear operators of $\mathcal{B}(X, Y)$ (cf. Banach's Inversion Stability Theorem 5.6.12 in [13]). On the other hand, it is known that we cannot admissibly represent $\mathcal{B}(X, Y)$

with respect to the operator norm topology in the non-separable case. This follows for instance from the fact that admissibly represented metric spaces are necessarily separable (cf. Lemma 8.1.1. in [26]). Moreover, one can show that the inversion operator $T \mapsto T^{-1}$ is not continuous with respect to $\delta_{\mathrm{ev}}$ in general (this follows, for instance, from Corollary 6.5 in [3]). However, it is easy to see that the inversion operator is computable with respect to $\delta_{\mathrm{graph}}$ (and analogously, one can prove that it is computable with respect to $\delta^{<}_{\mathrm{graph}}$). In the light of Corollary 4.6 this is an interesting observation since it shows that there exist representations of $\mathcal{B}(X,Y)$ which provide both the ordinary class of computable operators *and* a computable inversion operator.

**Corollary 4.8** (Inversion)**.** *Let $X, Y$ be computable Banach spaces and consider the inversion mapping*

$$\iota :\subseteq \mathcal{B}(X,Y) \to \mathcal{B}(Y,X),\ T \mapsto T^{-1}$$

*with* $\mathrm{dom}(\iota) := \{T \in \mathcal{B}(X,Y) : T \text{ bijective}\}$. *Then*

(1) *$\iota$ is $(\delta_{\mathrm{graph}}, \delta_{\mathrm{graph}})$– and $(\delta^{<}_{\mathrm{graph}}, \delta^{<}_{\mathrm{graph}})$–computable,*

(2) *$\iota$ is neither $(\delta_{\mathrm{ev}}, \delta_{\mathrm{ev}})$– nor $(\delta^{=}_{\mathrm{ev}}, \delta^{=}_{\mathrm{ev}})$–continuous in general.*

## 5. Conclusion

In this paper we have studied several representations of the set of linear bounded operators on computable Banach spaces. Such representations are important if we actually want to compute with linear bounded operators. From the computer science point of view, these representations reflect different data structures. For instance, $\delta_{\mathrm{ev}}$ represents linear bounded operators by their "programs". Theorem 4.3(2) shows that adding the information on some upper bound of the operator bound (i.e. $\delta^{\geqslant}_{\mathrm{ev}}$) does not lead to a strictly stronger representation while Theorem 4.5 shows that adding the operator bound itself actually leads to a stronger type of information. Representing linear bounded operators by a sequence of values on a fundamental sequence (i.e. $\delta_{\mathrm{seq}}$) and representing operators by positive and negative information on their graphs (i.e. $\delta_{\mathrm{graph}}$) both lead to mutually incomparable but weaker representations than $\delta_{\mathrm{ev}}$. But both types of information are strictly stronger than just positive information on the operator graph (i.e. $\delta^{<}_{\mathrm{graph}}$).

All representations considered on the set of linear bounded operators are admissible with respect to certain topologies. This can be deduced from known closure principles of admissible representations (see [21]). While $\delta_{\mathrm{ev}}$ is admissible with respect to the compact open topology on $\mathcal{B}(X,Y)$, $\delta^{=}_{\mathrm{ev}}$ is admissible with respect to the weakest topology which is included in the compact open topology and which makes the operator norm $\|\ \| : \mathcal{B}(X,Y) \to \mathbb{R}$ continuous. Correspondingly, $\delta_{\mathrm{graph}}$ is admissible with respect to the weakest topology on $\mathcal{B}(X,Y)$ such that the graph mapping $\mathrm{graph} : \mathcal{B}(X,Y) \to \mathcal{A}(X \times Y)$ becomes continuous, where $\mathcal{A}(X \times Y)$ is endowed with the Fine topology (see [2] for the definitions) and $\delta_{\mathrm{seq}}$ is admissible with respect to the weakest topology on

$\mathcal{B}(X,Y)$ such that the evaluation $\mathrm{seq} : \mathcal{B}(X,Y) \to Y^{\mathbb{N}},\ T \mapsto (Te_i)_{i\in\mathbb{N}}$ becomes continuous, where $(e_i)_{i\in\mathbb{N}}$ is the fundamental sequence of $X$ and $Y^{\mathbb{N}}$ is endowed with the product topology. All these topologies are possible substitutes for the operator norm topology which is usually considered in functional analysis but which cannot be handled in computable analysis (see the discussion in the previous section).

While Corollary 4.6 shows that the class of computable operators is quite stable with respect to different representations, we can deduce from Corollary 4.8 that these representations actually behave differently with respect to important operations that one might wish to compute. Thus, it will depend on the context which representation is the most adequate function space representation in a concrete application.

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## References


[1] S. Banach and S. Mazur, *Sur les fonctions calculables*, Ann. Soc. Pol. de Math. **16** (1937), 223.

[2] G. Beer, *Topologies on Closed and Closed Convex Sets*, (Kluwer Academic, Dordrecht, 1993).

[3] V. Brattka, *Computability of Banach space principles*, Informatik Berichte 286 (FernUniversität Hagen, Fachbereich Informatik, Hagen, June 2001). `http://www.informatik.fernuni-hagen.de/thi1/vasco.brattka/publications/banach.html`

[4] V. Brattka, *Computing uniform bounds*, in: V. Brattka, M. Schröder, and K. Weihrauch (eds.), *CCA 2002 Computability and Complexity in Analysis*, vol. 66 of Electronic Notes in Theoretical Computer Science (Elsevier, Amsterdam 2002).

[5] V. Brattka and G. Presser, *Computability on subsets of metric spaces*, Theoret. Comp. Sci., to appear.

[6] V. Brattka and K. Weihrauch, *Computability on subsets of Euclidean space I: Closed and compact subsets*, Theoret. Comp. Sci. **219** (1999), 65–93.

[7] V. Brattka and M. Ziegler, *Computability of linear equations*, in: R. Baeza-Yates, U. Montanari, and N. Santoro (eds.), *Foundations of Information Technology in the Era of Network and Mobile Computing*, vol. 223 of IFIP International Federation for Information Processing (Kluwer Academic Publishers, Dordrecht 2002), 95–106.

[8] G. Ceĭtin, *Algorithmic operators in constructive metric spaces*, Tr. Mat. Inst. Steklov **67** (1962), 295–361.

[9] X. Ge and A. Nerode, *Effective content of the calculus of variations I: semi-continuity and the chattering lemma*, Ann. Pure Appl. Logic **78** (1996), 127–146.

[10] A. Grzegorczyk, *On the definitions of computable real continuous functions*, Fund. Math. **44** (1957), 61–71.

[11] K.-I. Ko, *Complexity Theory of Real Functions*, Birkhäuser, Boston 1991.

[12] B. A. Kušner, *Lectures on Constructive Mathematical Analysis*, (American Mathematical Society, Providence 1984).

[13] S. Kutateladze, *Fundamentals of Functional Analysis*, (Kluwer Academic Publishers, Dordrecht 1996).
[14] D. Lacombe, *Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles I-III*, Comptes Rendus **240,241** (1955), 2478–2480, 13–14, 151–153.
[15] D. Lacombe, *Quelques procédés de définition en topologie récursive*, in: A. Heyting (ed.), *Constructivity in mathematics*, (North-Holland, Amsterdam 1959), 129–158.
[16] G. Metakides, A. Nerode, and R. Shore, *Recursive limits on the Hahn-Banach theorem*, in: M. Rosenblatt (ed.), *Errett Bishop: Reflections on Him and His Research*, vol. 39 of Contemporary Mathematics (American Mathematical Society, Providence 1985), 85–91.
[17] Y. N. Moschovakis, *Recursive metric spaces*, Fund. Math. **55** (1964), 215–238.
[18] M. B. Pour-El and J. I. Richards, *Computability in Analysis and Physics*, (Springer, Berlin 1989).
[19] E. Schechter, *Handbook of Analysis and Its Foundations*, (Academic Press, San Diego 1997).
[20] M. Schröder, *Admissible representations of limit spaces*, in: J. Blanck, V. Brattka, and P. Hertling (eds.), *Computability and Complexity in Analysis*, vol. 2064 of Lect. Not. Comp. Sci. (Springer, Berlin 2001), 273–295.
[21] M. Schröder, *Extended admissibility*, Theoret. Comp. Sci. **284** (2002), 519–538.
[22] D. Spreen, *On effective topological spaces*, J. Symbolic Logic **63** (1998), 185–221.
[23] A. M. Turing, *On computable numbers, with an application to the "Entscheidungsproblem"*, Proc. London Math. Soc. **42** (1936), 230–265.
[24] M. Washihara, *Computability and tempered distributions*, Mathematica Japonica **50** (1999), 1–7.
[25] K. Weihrauch, *Computability on computable metric spaces*, Theoret. Comp. Sci. **113** (1993), 191–210.
[26] K. Weihrauch, *Computable Analysis*, (Springer, Berlin 2000).
[27] M. Yasugi, T. Mori, and Y. Tsujii, *Effective properties of sets and functions in metric spaces with computability structure*, Theoret. Comp. Sci. **219** (1999), 467–486.
[28] N. Zhong, *Computability structure of the Sobolev spaces and its applications*, Theoret. Comp. Sci. **219** (1999), 487–510.
[29] M. Ziegler and V. Brattka, *Computing the dimension of linear subspaces*, in: V. Hlaváč, K. G. Jeffery, and J. Wiedermann (eds.), *SOFSEM 2000: Theory and Practice of Informatics*, vol. 1963 of Lect. Not. Comp. Sci. (Springer, Berlin 2000), 450–458.
[30] M. Ziegler and V. Brattka, *A computable spectral theorem*, in: J. Blanck, V. Brattka, and P. Hertling (eds.), *Computability and Complexity in Analysis*, vol. 2064 of Lect. Not. Comp. Sci. (Springer, Berlin 2001), 378–388.

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