Applied General Topology
© Universidad Politécnica de Valencia
Volume 4, No. 1, 2003
pp. 71–77

Dense S_δ-diagonals and linearly ordered extensions

Masami Hosobuchi

Abstract. The notion of the S_δ-diagonal was introduced by H. R. Bennett to study the quasi-developability of linearly ordered spaces. In an earlier paper, we obtained a characterization of topological spaces with an S_δ-diagonal and we showed that the S_δ-diagonal property is stronger than the quasi-G_δ-diagonal property. In this paper, we define a dense S_δ-diagonal of a space and show that two linearly ordered extensions of a generalized ordered space X have dense S_δ-diagonals if the sets of right and left looking points are countable.

Keywords: S_δ-diagonal, dense S_δ-diagonal, linearly ordered space (LOTS), generalized ordered space (GO-space), linearly ordered extension.

2000 AMS Classification: 54F05.

1. S_δ-diagonals

We review in this section the definitions of S_δ-set and S_δ-diagonal, and state our results obtained in [5].

The following definition is a generalization of a G_δ-set and was introduced by H. R. Bennett [2] to study the quasi-developability of linearly ordered (topological) spaces.

Definition 1.1. Let X be a topological space. A subset A of X is called an S_δ-*set* if there exists a countable collection $\{U(1), U(2), \ldots\}$ of open subsets of X such that, for two points $p \in A$ and $q \in X \setminus A$, there exists an n such that $p \in U(n)$ and $q \notin U(n)$.

It is easy to see that a G_δ-set is an S_δ-set. Hence the notion of S_δ-set is a generalization of G_δ-set. See [3] for a description of $\mathcal{S}$-normal spaces whose closed subsets are S_δ-sets.

Definition 1.2. Let X be a topological space. X has an S_δ-*diagonal* if the diagonal subset Δ_X of $X \times X$ is an S_δ-set of $X \times X$, where Δ_X denotes the diagonal set $\{(x,x) : x \in X\}$ in the Cartesian product $X \times X$. The symbol $(\ ,\ )$ is used to stand for a point of $X \times X$.

It is useful to show the following lemma that relates to the property $(*)$ given in [2]. $\mathbb{N}$ denotes the set of natural numbers.

Lemma 1.3. [5] *Let X be a topological space. Let $\{\mathcal{G}(n) : n \in \mathbb{N}\}$ be a family of countable collections of open subsets of X. Suppose that, for any three points x, y and z with $y \neq z$, there exists an $m \in \mathbb{N}$ such that $x \in \bigcup \mathcal{G}(m)$ and that no element of $\mathcal{G}(m)$ contains the set $\{y, z\}$, where $\bigcup \mathcal{G}(m)$ denotes $\bigcup\{U : U \in \mathcal{G}(m)\}$. Then, there exists a family $\{\mathcal{F}(n) : n \in \mathbb{N}\}$ of countable collections of open subsets of X such that, for such three points above, there exists an $m \in \mathbb{N}$ such that $x \in \bigcup \mathcal{F}(m)$ and any two distinct points of $\{x, y, z\}$ do not belong to the same member of $\mathcal{F}(m)$.*

Theorem 1.4. [5] *Let X be a topological space. X has an S_δ-diagonal if and only if there exists a family $\{\mathcal{G}(n) : n \in \mathbb{N}\}$ of countable collections of open subsets of X such that, for three points x, y and z with $y \neq z$, there exists an $m \in \mathbb{N}$ such that $x \in \bigcup \mathcal{G}(m)$ and any two distinct points of $\{x, y, z\}$ do not belong to the same member of $\mathcal{G}(m)$.*

2. Two linearly ordered extensions and notation

Recall that a generalized ordered space (GO-space) is a triple $(X, \tau, <)$, where $<$ is a linear ordering of the set X and τ a Hausdorff topology on X having a base of order-convex sets. We will denote by λ the order topology on $(X, <)$. It is known that $\lambda \subset \tau$. A space of the form $(X, \lambda, <)$ is called a linearly ordered topological space (LOTS). Every LOTS is a GO-space, but not conversely. In fact it is known that the class of GO-spaces coincides with the class of subspaces of LOTS. Given a GO-space X there are two well-known linearly ordered extensions of X. One of these is X^* and was defined by D. J. Lutzer [7]. The other one is $L(X)$ and was studied in [8]. We review here the definitions of those linearly ordered extensions. The intervals in a GO-space or a LOTS are written in the form [a,b], [a,b[,]a,b] and]a,b[. For example, $[a, b] = \{x : a \leq x \leq b\}$, $[a, b[= \{x : a \leq x < b\}$ and so on. For a GO-space X, we set $R = \{x \in X : [x, \rightarrow [\in \tau - \lambda\}$ and $L = \{x \in X :]\leftarrow, x] \in \tau - \lambda\}$, where λ denotes the order topology as mentioned above. R (resp. L) is called the *set of right* (resp. *left*) *looking points*. Then X^* is defined as follows:

$$X^* = (X \times \{0\}) \cup \{(x, k) : x \in R, k < 0, k \in \mathbb{Z}\} \cup \{(x, k) : x \in L, k > 0, k \in \mathbb{Z}\} \subset X \times \mathbb{Z},$$

where $\mathbb{Z}$ denotes the set of integers. On the other hand, $L(X)$ is defined as follows:

$$L(X) = (X \times \{0\}) \cup \{(x, -1) : x \in R\} \cup \{(x, 1) : x \in L\} \subset X \times \{-1, 0, 1\}.$$

X^* and $L(X)$ are linearly ordered topological spaces equipped with the lexicographic order topologies. We, furthermore, need some technical notation for the proof of the theorems in Section 5. For a convex open subset U of a GO-space X, we define a convex open subset $\widetilde{U}$ of $E(X)$, where $E(X)$ denotes either X^* or $L(X)$. Then eight cases can occur. In the following, the intervals must be considered in $E(X)$.

(1) If a is the minimum point of U, then we define $\tilde{U_1} = [(a,0), \rightarrow [\subset E(X)$.
(2) Let $a = \inf U$ and $a \in X \setminus U$. If $E(X) = X^*$, then $\tilde{U_1} = \{(x,k) \in X^* : a < x\} =](a,+\infty), \rightarrow [\subset X^*$, where $(a,+\infty) \in X \times (\mathbb{Z} \cup \{+\infty\})$ and the interval is taken in X^*. Likewise, if $E(X) = L(X)$, then $\tilde{U_1} = \{(x,k) \in L(X) : a < x\} =](a,1), \rightarrow [\subset L(X)$. Note that $(a,1)$ may not belong to $L(X)$.
(3) If there is a gap $u = (A,B)$ such that u is the left end-point of U, then we define $\tilde{U_1} =](u,0), \rightarrow [\subset E(X)$.
(4) If none of Cases $1-3$ occurs, then we define $\tilde{U_1} = E(X)$.
(5) If b is the maximum point of U, then we define $\tilde{U_2} =]\leftarrow,(b,0)] \subset E(X)$.
(6) Let $b = \sup U$ and $b \in X \setminus U$. If $E(X) = X^*$, then $\tilde{U_2} = \{(x,k) \in X^* : x < b\} =]\leftarrow,(b,-\infty)[\subset X^*$ (cf. (2)). If $E(X) = L(X)$, then $\tilde{U_2} = \{(x,k) \in L(X) : x < b\} =]\leftarrow,(b,-1)[\subset L(X)$. Note that $(b,-1)$ may not belong to $L(X)$.
(7) If there is a gap $v = (A,B)$ such that v is the right end-point of U, then we define $\tilde{U_2} =]\leftarrow,(v,0)[\subset E(X)$.
(8) If none of Cases $5-7$ occurs, then we define $\tilde{U_2} = E(X)$.

We set $\tilde{U} = \tilde{U}_1 \cap \tilde{U}_2$. $\tilde{U}$ is called *the convex open set associated with* U. Let U be an open set of a GO-space X. Then U is decomposed into a union of open convex subsets $\{U_\alpha : \alpha \in A\}$. In this case, we define $\widetilde{U} = \bigcup\{\tilde{U}_\alpha : \alpha \in A\}$, where $\tilde{U}_\alpha$ is the open set associated with U_α. Then $\tilde{U}$ is an open subset of $E(X)$, and called the open set associated with U.

3. S_δ-diagonals in linearly ordered extensions

The following theorems are proved in our paper [5]. Let X be a GO-space. It is easily seen that X^* contains X as a closed subset and $L(X)$ contains X as a dense subset. See [7, 8] for further information about X^* and $L(X)$. In both cases, X and $X \times \{0\}$ are identified by the correspondence of x to $(x,0)$.

Theorem 3.1. [5] *Let X be a generalized ordered space with an S_δ-diagonal. If $R \cup L$ is countable, then X^* has an S_δ-diagonal.*

To prove a similar theorem concerning $L(X)$, it is necessary to assume the existence of sequences in X that witnesses first-countability for points of $R \cup L$:

Theorem 3.2. [5] *Let X be a GO-space with an S_δ-diagonal. Assume that, for every point $s \in L$, there exists a decreasing sequence $\{x(s,n) : n \in \mathbb{N}\}$ in X such that* $\inf\{x(s,n)\} = s$ *and, for every point $s \in R$, there exists an increasing sequence $\{y(s,n) : n \in \mathbb{N}\}$ in X such that* $\sup\{y(s,n)\} = s$. *If $R \cup L$ is countable, then $L(X)$ has an S_δ-diagonal.*

4. Dense S_δ-diagonals

The following definition gives an analogy to the dense G_δ-diagonal in [1].

Definition 4.1. A Hausdorff space X has a *dense S_δ-diagonal* if there exists a dense subset D of Δ_X such that D is an S_δ-subset of $X \times X$, where Δ_X denotes the diagonal subset of the Cartesian product $X \times X$.

We show the following theorem that is analogous to a result concerning spaces that have a dense G_δ-diagonal [1].

Theorem 4.2. *Let X be a Hausdorff space. Then X has a dense S_δ-diagonal if and only if X has a dense subset Y such that Y is an S_δ-subset of X and Y has an S_δ-diagonal.*

Proof. If $D \subset \Delta_X$ is a dense S_δ-set in $X \times X$, then $D \cap \Delta_X$ is a dense S_δ-set in Δ_X. Now the map $h : \Delta_X \to X$ defined by $h(x, x) = x$ is a homeomorphism, and the homeomorphic image of a dense S_δ-set is a dense S_δ-set.

Conversely, suppose Y is a dense S_δ-subset of X. Then $h^{-1}(Y)$ is a dense S_δ-subset of Δ_X. The rest is easily verified. □

5. Theorems concerning dense S_δ-diagonals of linearly ordered extensions

Theorem 5.1. *Let $X = (X, \tau)$ be a GO-space with a dense S_δ-diagonal. If $R \cup L$ is countable, then X^* has a dense S_δ-diagonal.*

We first show the following lemma.

Lemma 5.2. *Let X be a GO-space and X^* the linearly ordered extension of X. For a subspace Y of X, set $Z = Y \cup (X^* \setminus X)$. If Y is dense in X and an S_δ-subset of X, then Z is a dense subspace of X^* and an S_δ-subset of X^*.*

Proof. To see that Z is dense in X^*, let $x \in X^* \setminus Z = X \setminus Y$ and V be a neighborhood of x in X^*, where X is identified with $X \times \{0\}$ as usual. Since $V \cap X$ is a neighborhood of x in X, it follows that $V \cap X \cap Y \neq \varnothing$. Since $V \cap X \cap Y \subset V \cap Z$, it follows that $V \cap Z \neq \varnothing$. Hence Z is a dense subspace of X^*. To show the last part, let $\{U(n) : n \in \mathbb{N}\}$ be a countable collection of open subsets of X such that, for $y \in Y$ and $x \in X \setminus Y$, there exists an $m \in \mathbb{N}$ such that $y \in U(m)$ and $x \notin U(m)$. For every $n \in \mathbb{N}$, let $\tilde{U}(n)$ be the open subset associated with $U(n)$ as in Section 3. Set $\tilde{U}(0) = X^* \setminus X$. Then it is obvious that $\tilde{U}(0)$ is an open subset of X^*. We show that the countable collection $\{\tilde{U}(n) : n \geq 0\}$ of open subsets of X^* assures that Z is an S_δ-subset of X^*. Let $z \in Z$ and $x \in X^* \setminus Z = X \setminus Y$.

Case 1. Let $z \in X^* \setminus X$. Then it is easy to see that $z \in \tilde{U}(0)$ and $x \notin \tilde{U}(0)$.

Case 2. Let $z \in Y$. Since $x \in X \setminus Y$, there exists an $m \in \mathbb{N}$ such that $z \in U(m)$ and $x \notin U(m)$. By the definition of $\tilde{U}(m)$, it follows that $z \in \tilde{U}(m)$ and $x \notin \tilde{U}(m)$. This completes the proof. □

Now we shall prove Theorem 5.1.

Proof of Theorem 5.1. By Theorem 4.2, there exists a dense subspace Y of X such that Y is an S_δ-subset of X and that Y has an S_δ-diagonal. Let $\{\mathcal{G}(n) : n \in \mathbb{N}\}$ be a family of countable collections of open subsets of Y such that, for three points x, y and z of Y with $y \neq z$, there exists an $m \in \mathbb{N}$ such that $x \in \bigcup \mathcal{G}(m)$ and no element of $\mathcal{G}(m)$ contains $\{y, z\}$. The existence of the above family is guaranteed by Theorem 1.4. For an open subset V of Y, there exists an open set V_X of X such that $V_X \cap Y = V$. Let $\tilde{V}_X$ be the open subset of X^* associated with V_X as explained in Section 2. Set $V_Z = \tilde{V}_X \cap Z$, where Z is as mentioned in Lemma 5.2.

It is clear that V_Z is open in Z and that $V_Z \cap Y = V$. For every $n \in \mathbb{N}$, set $\tilde{\mathcal{G}}(n) = \{V_Z : V \in \mathcal{G}(n)\}$ and $\tilde{\mathcal{G}}(0) = \{\{x\} : x \in X^* \setminus X\}$. Let $S = R \cup L = \{s_i : i \in \mathbb{N}\}$ be an enumeration of the countable set S. Let $(s_i, k) \in X^* \setminus X$. Set $\tilde{\mathcal{G}}_+(s_i, k) = \{](s_i, k), \rightarrow [\, \cap Z\}$ and $\tilde{\mathcal{G}}_-(s_i, k) = \{]\leftarrow, (s_i, k)[\, \cap Z\}$, where these intervals are considered in X^*. By virtue of Lemma 5.2 and Theorem 4.2, it is sufficient to show that a family of those countable collections of open subsets of Z witnesses the S_δ-diagonal of Z. To see this, let x, y and z be three points of Z with $y \neq z$. We may assume without loss of generality that $y < z$.

Case 1. If $\{x, y, z\} \subset Y$, then there exists an $m \in \mathbb{N}$ such that $x \in \bigcup \mathcal{G}(m)$ and $\{y, z\} \not\subset V$ for any $V \in \mathcal{G}(m)$. Hence it follows that $x \in \bigcup \tilde{\mathcal{G}}(m)$ and that $\{y, z\} \not\subset V_Z$ for any $V_Z \in \tilde{\mathcal{G}}(m)$.

Case 2. Let $x \in Y$ and y or z belong to $Z \setminus Y$.

(i) We assume that $y \in Z \setminus Y$. Then we can write $y = (s_i, k)$, where $k \neq 0$. If $x < y$, then $x \in\,]\leftarrow, y[\, \cap Z$ and $\{y, z\} \not\subset\,]\leftarrow, y[$. Hence, by the definition, it follows that $x \in \bigcup \tilde{\mathcal{G}}_-(s_i, k)$ and that $\{y, z\} \not\subset V$ for $V \in \tilde{\mathcal{G}}_-(s_i, k)$. If $y < x$, then $x \in\,]y, \rightarrow [\, \cap Z$ and $\{y, z\} \not\subset\,]y, \rightarrow [$. Hence it follows that $x \in \bigcup \tilde{\mathcal{G}}_+(s_i, k)$ and that $\{y, z\} \not\subset V$ for $V \in \tilde{\mathcal{G}}_+(s_i, k)$.

(ii) Let $z \in Z \setminus Y$. Then the proof is analogous to (i).

Case 3. Let $x \in Z \setminus Y$. Then it follows that $x \in \bigcup \tilde{\mathcal{G}}(0)$ and $\{y, z\} \not\subset V$ for any $V \in \tilde{\mathcal{G}}(0)$. Therefore, by virtue of Lemma 1.3 and Theorem 1.4, X^* has a dense S_δ-diagonal. This completes the proof. □

Theorem 5.3. *Let $X = (X, \tau)$ be a GO-space with a dense S_δ-diagonal. If $R \cup L$ is countable, then $L(X)$ has a dense S_δ-diagonal.*

Proof. By Theorem 4.2, there exists a dense subspace Y of X such that Y is an S_δ-subset of X and Y has an S_δ-diagonal. Since X is dense in $L(X)$, it follows that Y is a dense subspace of $L(X)$. To prove that $L(X)$ has a dense S_δ-diagonal, it is sufficient to show, by Theorem 4.2, that Y is an S_δ-subset of $L(X)$. Let $\{U(n) : n \in \mathbb{N}\}$ be a countable collection of open subsets of X such that, for $y \in Y$ and $x \in X \setminus Y$, there exists an $m \in \mathbb{N}$ such that $y \in U(m)$ and $x \notin U(m)$. For every $n \in \mathbb{N}$, let $\tilde{U}(n)$ be the open subset of $L(X)$ associated with $U(n)$. For $s_i \in S = R \cup L$ and $\varepsilon \in \{-1, 1\}$, set $\tilde{U}_+(s_i, \varepsilon) = \,](s_i, \varepsilon), \rightarrow [$ and $\tilde{U}_-(s_i, \varepsilon) = \,]\leftarrow, (s_i, \varepsilon)[$, where the intervals are considered in $L(X)$. The countable collection $\{\tilde{U}(n),\ \tilde{U}_+(s_i,\varepsilon),\ \tilde{U}_-(s_i,\varepsilon) : n \in \mathbb{N},\ i \in \mathbb{N},\ \varepsilon = \pm 1\}$ of open subsets of $L(X)$ guarantees that Y is an S_δ-subset of $L(X)$. To see this, let $y \in Y$ and $z \in L(X) \setminus Y$.

Case 1. Let $z \in X \setminus Y$. Then there exists an $m \in \mathbb{N}$ such that $y \in U(m)$ and $z \notin U(m)$. Hence it follows that $y \in \tilde{U}(m)$ and $z \notin \tilde{U}(m)$.

Case 2. Let $z \in L(X) \setminus X$. We can write $z = (s_i, \varepsilon)$, where $\varepsilon \in \{-1, 1\}$. If $y < z$, then it follows that $y \in \tilde{U}_-(s_i,\varepsilon)$ and $z \notin \tilde{U}_-(s_i,\varepsilon)$. If $z < y$, then it follows that $y \in \tilde{U}_+(s_i,\varepsilon)$ and $z \notin \tilde{U}_+(s_i,\varepsilon)$. Hence Y is an S_δ-subset of $L(X)$. This completes the proof of Theorem 5.3. □

6. Examples

Example 6.1. Theorems 3.1 and 3.2 do not hold without the assumption of the countability of the set $R \cup L$. Let us consider the Sorgenfrey line $X = (\mathbb{R}, \mathcal{S})$. In this case, the right looking points $R = \mathbb{R}$ is uncountable. Since X has a G_δ-diagonal, X has an S_δ-diagonal. However, X^* does not have an S_δ-diagonal. To prove this, it is sufficient to see that X^* does not have a quasi-G_δ-diagonal [5]. We easily see that there does not exist a family of countable collections of open subsets of X^* that separates two points of the form $(x, 0)$ and $(x, 1)$, where $x \in X$.

Example 6.2. Theorem 3.2 does not hold without the existence of the sequences for points of $R \cup L$. To show a counterexample, let Y be the set of countable ordinals $[0, \omega_1[$ with the discrete topology. The right looking points of Y comprise the set of limit ordinals. Let Y^* be the linear extension of Y defined in Section 2. Let $X = Y^* \cup \{(\omega_1, 0)\}$, where X is ordered as $(\omega_1, 0) > \alpha$ for all $\alpha \in Y^*$, and given the discrete topology. Then $R = \{(\omega_1, 0)\}$ is a singleton and $L(X) = Y^* \cup \{(\omega_1, -1)\} \cup \{(\omega_1, 0)\}$, where $\alpha < (\omega_1, -1) < (\omega_1, 0)$ for all $\alpha \in Y^*$. There does not exist an increasing sequence in Y^* that converges to $(\omega_1, -1)$. Furthermore, $L(X)$ does not have a quasi-G_δ-diagonal, because the points of X and the point $\{(\omega_1, -1)\}$ are not separated by a family of countable collections of open subsets of $L(X)$. Hence $L(X)$ does not have an S_δ-diagonal.

Example 6.3. A generalized ordered space does not necessarily have a dense S_δ-diagonal. To show this, consider the linearly ordered space Z that was constructed by H. R. Bennett and D. J. Lutzer [4]. They proved that Z is not first-countable at any point. Z is defined as follows:

$$Z = \{(\alpha_1, \alpha_2, \dots, \alpha_n, \omega_1, \omega_1, \dots) : \alpha_i < \omega_1,\ 1 \le i \le n,\ \alpha_i = \omega_1,\ i > n,\ n \ge 1\},$$

with the lexicographic order. Since Z is densely-ordered, a dense subset Y of Z is a LOTS. If Y has a quasi-G_δ-diagonal, Y is quasi-developable. Since a quasi-developable space is first-countable, Y does not have a quasi-G_δ-diagonal. Therefore, Z does not have a dense S_δ-diagonal.

Acknowledgements. The author is grateful to the referees for their helpful comments.

References

[1] A. V. Arhangel'skiĭ and Lj. D. Kočinac, *On a dense G_δ-diagonal*, Publ. L'Institut Math. **47 (61)** (1990), 121–126.
[2] H. R. Bennett, *LOTS with S_δ-diagonals*, Topology Proc. **12** (1987), 211–216.
[3] H. R. Bennett, M. Hosobuchi and D. J. Lutzer, *A note on perfect generalized ordered spaces*, Rocky Mountain J. Math. **29 (4)** (1999), 1195–1207.
[4] H. R. Bennett and D. J. Lutzer, *Point countability in generalized ordered spaces*, Topology Appl. **71** (1996), 149–165.
[5] M. Hosobuchi, *S_δ-diagonals and generalized ordered spaces*, J. Tokyo Kasei Gakuin Univ. (Nat. Sci. Tech.) **41** (2001), 1–7.
[6] D. J. Lutzer, *A metrization theorem for linearly orderable spaces*, Proc. Amer. Math. Soc. **22** (1969), 557–558.
[7] D. J. Lutzer, *On generalized ordered spaces*, Dissertationes Math. **89** (1971), 1–32.
[8] T. Miwa and N. Kemoto, *Linearly ordered extensions of GO spaces*, Topology Appl. **54** (1993), 133–140.

Received January 2002 Revised December 2002

Masami Hosobuchi
Tokyo Kasei Gakuin University,
Department of Housing and Planning,
2600 Aihara, Machida,
Tokyo 194-0292,
Japan.

E-mail address: mhsbc@kasei-gakuin.ac.jp