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The local triangle axiom in topology and domain theory

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Abstract. We introduce a general notion of distance in weakly separated topological spaces. Our approach differs from existing ones since we do not assume the reflexivity axiom in general. We demonstrate that our partial semimetric spaces provide a common generalization of semimetrics known from Topology and both partial metrics and measurements studied in Quantitative Domain Theory. In the paper, we focus on the local triangle axiom, which is a substitute for the triangle inequality in our distance spaces. We use it to prove a counterpart of the famous Archangelskij Metrization Theorem in the more general context of partial semimetric spaces. Finally, we consider the framework of algebraic domains and employ Lebesgue measurements to obtain a complete characterization of partial metrizability of the Scott topology.

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1. Introduction

Over recent years, a number of attempts have been made to equip semantic domains with a notion of distance between points in order to provide a mechanism for making *quantitative* statements about programs, such as speed of convergence or complexity of algorithms. This particular branch of research is now known as *Quantitative Domain Theory.* The work in this area has led to a number of concepts which generalise the classical notion of a metric space to an ordered setting. For example, Smyth [21] introduced quasi-metrics to Domain Theory, which by virtue of being non-symmetric encode topology and order on a domain at the same time. Partial metrics were defined by Matthews [14] and studied further by O'Neill [16, 17]. Although they are symmetric, they are still capable of capturing order. This work was further extended by Heckmann in [9], leading to a concept of weak partial metrics.

More recently, Keye Martin introduced the idea of a *measurement* on a domain [13]. At first glance, measurements are quite different from distance functions because they take only one argument. Nevertheless, they allow for making precise quantitative statements about domains.

The desire to understand the interplay between partial metrics and measurements, discovered by the author in [24], is a major motivation for the present work. Research in this direction led to a number of general question about the nature of distance in weakly separated spaces in Topology (continuous domains in their Scott topology are examples of such spaces). Our approach is quite distinct from existing theories of *generalized metric spaces* in Quantitative Domain Theory, e.g. [18, 5, 6, 23] and Topology [8], since we do not in general assume either the reflexivity axiom or the triangle inequality. Instead, we introduce *partial semimetric spaces*, which generalize both partial metrics and semimetrics. The theory of partial semimetrics coincides with the theory of measurements in the framework of continuous domains.

In the paper we focus on a certain condition on sequences in partial semimetric spaces substituting the triangle inequality, which we call the *local triangle axiom*. It was recognised as early as 1910 in the work of Fréchet [7] and investigated in [3, 4]. The name local axiom of triangle was used by Niemytzki [15] and the property was further analysed in [27]. On continuous domains in their Scott topology, stable partial symmetrics (defined in Section 5.1) satisfying the local triangle axiom correspond to Lebesgue measurements (Definition 6.5) introduced in a more general context by Martin in [13].

There are two major results of the paper. The first one is a metrization result (Theorem 3.2), which is a twin of the famous Archangelskij Metrization Theorem working in partial semimetric spaces. We apply it in Section 4 to prove quasi-developability of our distance spaces, improving a similar result by Künzi and Vajner [10]. In the other part of the paper we consider partial semimetric spaces and the local triangle axiom in the framework of Domain Theory (Section 6) to obtain the second major result of the paper. Our result provides a solution to an open problem stated in [9] asking for conditions, which guarantee partial metrizability of continuous domains. We show that an algebraic domain admits a partial metric for its Scott topology if and only if it admits a Lebesgue measurement (Theorem 6.12). The Heckmann problem remains open for arbitrary continuous domains.

For the reader's convenience we present a summary of symbols and terms used in the paper in Table 1.

Table 1: Symbols and terms used in the paper

symbol or term	**meaning**	**definition**	**related results**
$\langle X, \rho\rangle$	distance space	Def. 2.1.	
τ_ρ	distance topology	Sect. 2.	
τ_ρ^{op}	dual distance topology	Sect. 2.	
(cv)	a convergence prop.	Sect. 2.1.	Prop. 2.5, 2.10.
(symm)	symmetry axiom	Sect. 2.2.	
(ssd)	small self-distance axiom	Sect. 2.2.	
(r)	reflexivity axiom	Sect. 2.2.	
$(\Delta^\sharp)$	sharp triangle inequality axiom	Sect. 2.2.	
partial symmetric		Def. 2.6.	
partial semimetric		Def. 2.7.	Cor. 2.9, 2.12. Prop. 2.10, 2.11, 2.14, 2.16. Def. 6.2. Thm. 6.4.
partial metric		Sect. 2.2	Cor. 2.18.
local axiom, of triangle **(L)**		Sect. 2.5, Sect. 6.3.	Prop. 2.16, 2.17, 2.19, 6.6, 6.7. Cor. 2.18. Thm. 3.2, 4.1, 6.8, 6.12.
induced distance d_ρ	symmetrization of distance ρ and its dual.	Sect. 2.4.	Prop. 2.13. Thm. 3.2.
self-distance μ_ρ		Sect. 5.	Sect. 5.1, 6.2.
measurement		Def. 6.1.	Sect. 6.

2. Outline of a general theory of distance

First, let us consider the definition of a distance space, its intrinsic topologies and some natural axioms that can be introduced in distance spaces.

Definition 2.1. A *distance* on a set X is a map $\rho\colon X \times X \to [0, \infty)$. A pair $\langle X, \rho\rangle$ is called a *distance space*.

A distance function assigns to each element x of X a filter $\mathcal{N}_x$ of subsets of X by taking $U \in \mathcal{N}_x$ if and only if there exists $\varepsilon > 0$ such that $B_\rho(x, \varepsilon) \subseteq U$. The set

$$B_\rho(x, \varepsilon) := \{y \in X \mid \rho(x, y) < \rho(x, x) + \varepsilon\}$$

is called a *ball centered at x with radius ε*. In the same way one forms a collection $\mathcal{N}_x^{op}$ by replacing the ball in the definition of $\mathcal{N}_x$ by a *dual ball*

$$B_\rho^{op}(x, \varepsilon) := \{y \in X \mid \rho(y, x) < \rho(y, y) + \varepsilon\}.$$

The collection

$$\tau_\rho := \{U \mid \forall x \in U.\ U \in \mathcal{N}_x\}$$

is a topology on X called the *distance topology on* X. Dually, the family of sets

$$\tau_\rho^{op} := \{V \mid \forall x \in V.\ V \in \mathcal{N}_x^{op}\}$$

constitutes a *dual* (*distance*) *topology* on X. In general the collection $\mathcal{B}_x$ of balls centered at x is *not* a neighborhood base at x and the balls are *not* open themselves.

The set X together with an operation $\mathcal{N}\colon X \to \mathcal{P}(\mathcal{P}(X))$, which assigns to each point the filter $\mathcal{N}_x$, is an example of a *neighbourhood space* in the sense of [22]. The distance topology just defined is in fact one of the natural topologies for neighbourhood spaces, studied in more detail by Smyth (cf. [22], Proposition 2.10).

Example 2.2. To illustrate the difference between arbitrary distance spaces and metric ones, consider the four-element chain as shown in Figure 1. (Numbers denote respective self-distances; we set the distances $\rho(x, y)$ and $\rho(y, x)$ between points x, y to be $\max\{\rho(x, x), \rho(y, y)\}$). The distance is symmetric and satisfies the triangle inequality. Note that open sets are upper; in particular, the only open set containing the bottom element (denoted $\perp$) is the whole space. Indeed, the distance between $\perp$ and any other element of the space is 3 and hence any ball containing $\perp$ must be the whole space.

In contrast, any metric on a finite space induces the discrete topology.

Note that in this particular example the balls are open themselves. In arbitrary distance spaces it is usually not the case.

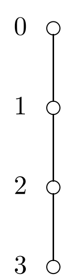

Figure 1: A non-metric distance.

2.1. **Basic properties of distance spaces.** For a subset A of X and an element $x \in X$ we define:

$$\rho(x, A) := \inf\{\rho(x, a) \mid a \in A\}.$$

Proposition 2.3. *In a distance space* $\langle X, \rho\rangle$ *a subset* $H \subseteq X$ *is closed iff for all* $x \notin H$ *we have* $\rho(x, H) > \rho(x, x)$.

Proof. Straight from the definition of the distance topology we infer that a subset H of X is closed iff for all $x \notin H$ there exists $\varepsilon > 0$ such that for all $y \in H$ we have $\rho(x,y) \geq \rho(x,x) + \varepsilon$. It is however equivalent to say that $\rho(x,H) \geq \rho(x,x) + \varepsilon > \rho(x,x)$. □

As a corollary we obtain the following interesting characterization of the specialisation preorder $\sqsubseteq_{\tau_\rho}$:

Corollary 2.4. *In a distance space $\langle X, \rho\rangle$ the following are equivalent:*

1. $x \sqsubseteq_{\tau_\rho} y$*;*
2. $x = y$ *or* $\forall \varepsilon > 0.\ \exists z \neq x.\ (\rho(x,z) < \rho(x,x) + \varepsilon$ *and* $z \sqsubseteq_{\tau_\rho} y)$*.*

Proof. ($\Rightarrow$) For $x, y \in X$,

$$\begin{array}{lll} x \not\sqsubseteq_{\tau_\rho} y & \text{iff} & x \notin \mathrm{cl}\{y\} \\ & \text{iff} & \exists \varepsilon > 0.\ \rho(x, \mathrm{cl}\{y\}) \geq \rho(x,x) + \varepsilon \\ & \text{iff} & \exists \varepsilon > 0.\ \forall z \in B_\rho(x,\varepsilon).\ z \notin \mathrm{cl}\{y\} \\ & \text{iff} & \exists \varepsilon > 0.\ \forall z \in B_\rho(x,\varepsilon).\ z \not\sqsubseteq_{\tau_\rho} y. \end{array}$$

Note that we have used Proposition 2.3 in the second equivalence.

Conversely, assume $x \neq y$ and let U be an open set in X. Then $x \in U$ implies that there exists $\varepsilon > 0$ such that $x \in B_\rho(x,\varepsilon) \subseteq U$. By assumption, there is $z \in B_\rho(x,\varepsilon) \subseteq U$ with $z \sqsubseteq_{\tau_\rho} y$. But open sets are upper with respect to the specialisation preorder and so $y \in U$. □

Proposition 2.5. *Let $\langle X, \rho\rangle$ be a distance space, (x_n) be a sequence of elements of X and $x \in X$. Then*

$$\rho(x, x_n) \to \rho(x,x) \text{ implies } x_n \to_{\tau_\rho} x.$$

Proof. Let U be any open set around x. Then there exists $\varepsilon > 0$ such that $x \in B_\rho(x,\varepsilon) \subseteq U$. Suppose that $\rho(x,x_n) \to \rho(x,x)$. Then there exists $N_\varepsilon \in \omega$ such that for all $n \geq N_\varepsilon$, $|\rho(x,x_n) - \rho(x,x)| < \varepsilon$. It is equivalent to say that either $0 \leq \rho(x,x_n) - \rho(x,x) < \varepsilon$ or $0 \geq \rho(x,x_n) - \rho(x,x) > -\varepsilon$ for all $n \geq N_\varepsilon$. In either case, we are able to conclude that $x_n \in B_\rho(x,\varepsilon) \subseteq U$ for all $n \geq N_\varepsilon$. Therefore, $x_n \to_{\tau_\rho} x$. □

Note that the corresponding equivalence:

(cv) $\rho(x,x_n) \to \rho(x,x)$ iff $x_n \to_{\tau_\rho} x$

may not hold in arbitrary partial symmetric spaces. Figure 2 provides a counterexample: we assume that all of the distances between elements, which are not given on the picture are 5. One can check that the bottom element is in every open set and hence every sequence converges to it. On the other hand, for the sequence (x_n) we have $\rho(\bot, x_i) \to 5$, for $i = 1, 2, 3, \ldots$, while $\rho(\bot,\bot) = 3$.

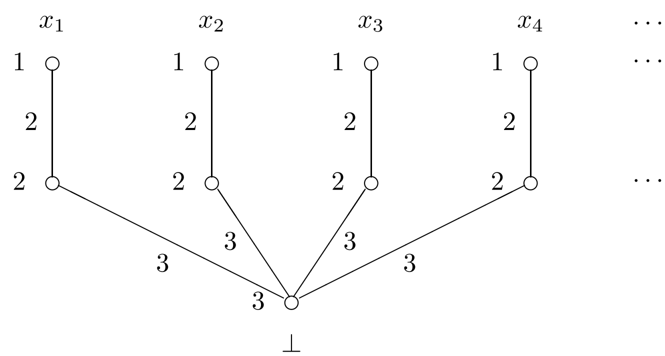

Figure 2: (cv) does not hold in general.

It is known that **(cv)** holds if the T_2 axiom is assumed [8], Lemma 9.3 p.481.

2.2. **Distance axioms.** On a distance space a number of further axioms can be introduced. We consider *symmetry* of the distance map

(**symm**) $$\forall x, y \in X.\ \rho(x, y) = \rho(y, x)$$

and the axiom of *small self-distances*

(**ssd**) $$\forall x, y \in X.\ \rho(x, x) \le \rho(x, y) \text{ and } \rho(x, x) \le \rho(y, x),$$

which obviously reduces to

$$\forall x, y \in X.\ \rho(x, x) \le \rho(x, y)$$

in the presence of symmetry. The latter condition is a generalisation of the *reflexivity axiom*

(**r**) $$\forall x, y \in X.\ \rho(x, x) = 0$$

known from the theory of metric spaces.

In Topology, consult for example [8], a *symmetric* is a distance map, which satisfies **(symm)** and **(r)**. On the other hand, a symmetric distance function satisfying the *sharp triangle inequality*

($\Delta^\sharp$) $$\forall x, y, z \in X.\ \rho(x, y) \le \rho(x, z) + \rho(z, y) - \rho(z, z)$$

is named a *weak partial metric* [9]. A *partial metric* [14] is a weak partial metric with **(ssd)**. Lastly, one considers mostly distance topologies, which satisfy the T_0 separation axiom. An elegant formulation of the ($\mathbf{T_0}$) axiom in terms of the distance mapping will be given later for a wide class of spaces (cf. Corollary 2.12). The following definition is a compromise on the existing terminology.

Definition 2.6. A distance function $\rho\colon X \times X \to [0, \infty)$ on a set X is a *partial symmetric* whenever it satisfies **(symm)**, **(ssd)** and the distance topology τ_ρ is T_0.

2.3. **Partial semimetric spaces.** In this section we introduce a particularly interesting subclass of partial symmetric spaces. Both partial metric spaces considered by Matthews [14], O'Neill [16, 17], Schellekens [19, 20], Heckmann [9] and Waszkiewicz [24] and semimetric spaces in Topology are our major examples of partial semimetric spaces.

Definition 2.7. Let $\langle X, \rho \rangle$ be a partial symmetric space. The map ρ is a *partial semimetric* if for any $x \in X$, the collection $\{\mathrm{int}(B_\rho(x, \varepsilon)) \mid \varepsilon > 0\}$ is a base for the filter $\mathcal{N}_x$.

In other words, we require that the family $\{B_\rho(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$ forms a (not necessarily open) neighborhood base at x with respect to the distance topology τ_ρ on X.

To summarize the key differences between "classical" distance maps and ours, we collect their properties in the following table. A '+' indicates that the respective condition is satisfied. All maps in the table satisfy **(symm)**, **(ssd)** and **($\mathbf{T_0}$)**.

Distance	Balls form a neighbourhood base	**(r)**
partial symmetric		
partial semimetric	+	
symmetric		+
semimetric	+	+

Proposition 2.8. *Let $\langle X, \rho \rangle$ be a distance space. The following are equivalent:*

(1) *$\{B_\rho(x, \varepsilon) \mid \varepsilon > 0\}$ forms a (not necessarily open) neighborhood base at x;*

(2) *$\forall x \in X. \forall H \subseteq X.$ $\rho(x, H) \leq \rho(x, x)$ iff $x \in \mathrm{cl}(H)$.*

Proof. ($\Rightarrow$) Let $x \in X$ and $\varepsilon > 0$.

$$\begin{aligned} x \in \mathrm{cl}(H) \quad &\text{iff} \quad \forall \varepsilon > 0.\ B_\rho(x, \varepsilon) \cap H \neq \varnothing \\ &\text{iff} \quad \forall \varepsilon > 0.\ \exists z \in H.\ \rho(x, z) < \rho(x, x) + \varepsilon \\ &\text{iff} \quad \rho(x, H) \leq \rho(x, x). \end{aligned}$$

Conversely, Since $X \setminus B_\rho(x, \varepsilon) = \{z \mid \rho(x, z) \geq \rho(x, x) + \varepsilon\}$, we have

$$\rho(x, X \setminus B_\rho(x, \varepsilon)) \geq \rho(x, x) + \varepsilon > \rho(x, x).$$

Therefore, $x \notin \mathrm{cl}(X \setminus B_\rho(x, \varepsilon))$ and so $x \in \mathrm{int}(B_\rho(x, \varepsilon))$. □

As an immediate corollary, we note that a partial semimetric can be introduced by an appropriate closure operator familiar from the theory of metric spaces. This was stated in [8], Theorem 9.7, for semimetrics.

Corollary 2.9. *Let $\langle X, \rho \rangle$ be a symmetric distance space. The following are equivalent:*

(1) *ρ is a partial semimetric.*
(2) $\forall x \in X.\, \forall H \subseteq X.\;\; \rho(x, H) = \rho(x, x)$ iff $x \in \mathrm{cl}(H)$.

One can check that if the distance topology is Hausdorff, then X is partially semimetrizable iff it is partially symmetrizable and first countable (see the discussion before Definition 9.5 of [8]). In the absence of separation axioms, the property **(cv)** is necessary for this characterization to hold.

Proposition 2.10. *Let $\langle X, \rho\rangle$ be a partial symmetric space. Then the following are equivalent:*

(1) *The map ρ is a partial semimetric;*
(2) *X is first countable and* **(cv)** *holds.*

Proof. If ρ is a partial semimetric, then the collection

$$\{\mathrm{int}(B_\rho(x, 1/2^{n+1})) \mid n \in \omega\}$$

is a neighborhood base at $x \in X$, which amounts to first countability of X. Now, if for some sequence (x_n) of elements of X and some $x \in X$, we have $x_n \to_{\tau_\rho} x$, then for every $\varepsilon > 0$, (x_n) is cofinally in $\mathrm{int}(B_\rho(x, \varepsilon))$ and so in $B_\rho(x, \varepsilon)$. Therefore, using the **(ssd)** axiom,

$$\forall \varepsilon > 0.\, \exists N \in \omega.\, \forall n \geq N.\;\; |\rho(x, x_n) - \rho(x, x)| < \varepsilon.$$

That is, $\rho(x, x_n) \to \rho(x, x)$.

Conversely, if X is first countable and **(cv)** holds, then a point $x \in X$ is in $\mathrm{int}(B_\rho(x, \varepsilon))$ for any $\varepsilon > 0$. Suppose not; then there exists $\varepsilon_0 > 0$ and a sequence (x_n) with $x_n \to_{\tau_\rho} x$ such that $x_n \notin B_\rho(x, \varepsilon_0)$. (Choose x_n to be in the n^{th} member of a decreasing countable base for x and not in $B_\rho(x, \varepsilon_0)$). So $\rho(x, x_n) \not\to \rho(x, x)$, which contradicts **(cv)**. Now, if $x \in \mathrm{int}(B_\rho(x, \varepsilon))$, then the collection $\{B_\rho(x, \varepsilon) \mid \varepsilon > 0\}$ forms a neighborhood base at x. □

In a partial semimetric space the specialisation preorder reflects the distance in a simple way.

Proposition 2.11. *Let $\langle X, \rho\rangle$ be a partial semimetric space. Then for all $x, y \in X$,*

$$x \sqsubseteq_{\tau_\rho} y \text{ iff } \rho(x, y) = \rho(x, x).$$

Proof.

$$\begin{aligned} x \sqsubseteq_{\tau_\rho} y \quad &\text{iff} \quad x \in \mathrm{cl}(\{y\}) \\ &\text{iff} \quad x \in \{z \mid \rho(z, y) = \rho(z, z)\} \\ &\text{iff} \quad \rho(x, y) = \rho(x, x). \end{aligned}$$

We use Proposition 2.8 (2) and **(ssd)** in the second equivalence. □

As a result we are able to characterize the $(\mathbf{T_0})$ axiom in terms known from the theory of partial metric spaces.

Corollary 2.12. *The distance topology in a partial semimetric space* $\langle X, \rho \rangle$ *is* T_0 *iff*

$$x = y \text{ iff } \rho(x, y) = \rho(x, x) = \rho(y, y).$$

□

2.4. **Derived distance functions and their topologies.**

Convention	In the rest of this paper we assume that the $(\mathbf{T_0})$ axiom holds in all distance spaces that we consider.

In a distance space $\langle X, \rho \rangle$ one can introduce a number of other maps, which are derived from the distance. Whenever the map ρ satisfies **(ssd)**, it is convenient to form the corresponding *quasi-distance* $q_\rho \colon X \times X \to [0, \infty)$ by

$$q_\rho(x, y) := \rho(x, y) - \rho(x, x).$$

Whenever possible, we will drop the index ρ from q_ρ for the sake of clarity. The **(ssd)** axiom assures that the function q is well-defined. The usefulness of the quasi-distance stems from the fact that it satisfies the reflexivity axiom **(r)** and still induces the same topology as the distance ρ. Whenever the map ρ satisfies more axioms, the name for the quasi-distance q will change accordingly. For instance, a *quasi-semimetric* is a quasi-distance formed from a partial semimetric.

The quasi-distance has a dual, namely the map $q_\rho^{op} \colon X \times X \to [0, \infty)$ defined by $q_\rho^{op}(x, y) := \rho(x, y) - \rho(y, y)$. The dual quasi-distance induces the dual distance topology.

The symmetrization of the quasi-distance and its dual is the map

$$d_\rho \colon X \times X \to [0, \infty),$$

$$\forall x, y \in X.\ d_\rho(x, y) := q_\rho(x, y) + q_\rho^{op}(x, y) = 2\rho(x, y) - \rho(x, x) - \rho(y, y).$$

The function d_ρ is always symmetric and reflexive and hence is a symmetric. It is called the *induced symmetric*. We do not know whether it is true in general that the induced symmetric derived from a partial semimetric is a *semimetric*. However, there are two notable cases when the prefix "*semi-*" is retained: firstly, whenever a partial semimetric topology is Hausdorff, secondly, in the presence of the *local triangle axiom* studied in the next section (the latter claim follows from Proposition 2.16 and the observation that a partial semimetric is a semimetric if and only if it satisfies the reflexivity axiom).

Lastly, for a distance function, the *self-distance mapping* (called also the *weight function*) is non-trivial in general and, as we shall see in Section 5, exhibits many interesting properties.

Proposition 2.13. *For a distance space* $\langle X, \rho \rangle$ *with* **(ssd)**, *the induced topology* τ_{d_ρ} *is the join of the distance topology* τ_ρ *and its dual* τ_ρ^{op}. *Moreover,* τ_{d_ρ} *is semimetrizable whenever both* τ_ρ *and* τ_ρ^{op} *are partially semimetrizable.*

Proof. Both statements follow from the fact that

$$B_{d_\rho}(x,\varepsilon) \subseteq B(x,\varepsilon) \cap B_\rho^{op}(x,\varepsilon) \subseteq B_{d_\rho}(x,2\varepsilon)$$

for any $x \in X$ and $\varepsilon > 0$. □

Proposition 2.14. *For a sequence (x_n) and an element $z \in X$ in a partial semimetric space $\langle X,\rho\rangle$, the following are equivalent:*

(a) $\lim d_\rho(x_n,z) = 0$*;*
(b) $\lim \rho(x_n,z) = \rho(z,z)$ *and* $\lim \rho(x_n,z) = \lim \rho(x_n,x_n)$*;*
(c) $(x_n) \to_{\tau_\rho} z$ *and* $\lim \rho(x_n,z) = \lim \rho(x_n,x_n)$*.*

Proof. Note that all limits are taken with respect to the Euclidean topology on the real line. The equivalence of (a) and (b) follows easily since

$$d_\rho(x_n,z) = (\rho(x_n,z) - \rho(z,z)) + (\rho(x_n,z) - \rho(x_n,x_n))$$

and both terms on the right-hand side of the equation are non-negative by **(ssd)**. The equivalence of (b) and (c) is clear by Proposition 2.10. □

2.5. **The local triangle axiom.** In this subsection we study a certain condition on the convergence of sequences in a distance space $\langle X,\rho\rangle$, namely:

(**L**) $$q_\rho(x,y_n) \to 0,\ q_\rho(y_n,z_n) \to 0 \text{ imply } q_\rho(x,z_n) \to 0,$$

for any two sequences $(y_n),(z_n)$ and element x from X. The property **(L)** of a distance space is called here the *local triangle axiom*.

We give several characterization of the local triangle axiom. We show that the condition implies partial semimetrizability of the underlying topology. We discuss its dependence on the other axioms and the similarity to the triangle inequality. For symmetric spaces in Topology a similar condition was recognised as early as 1910 in the work of Fréchet [7] and investigated in [3, 4]. The local triangle axiom was defined by Niemytzki [15] and analysed in [27]. A more modern-style proof of metrizability of a topological space, which admits a symmetric satisfying the local triangle axiom, is given by Archangelskij [2] and explained in detail in [8]. The precise formulation of the Archangelskij Metrization Theorem follows:

Theorem 2.15 (Archangelskij)**.** *Let the set X be a T_1 topological space symmetrizable with respect to the symmetric d. If for all $x \in X$ and all sequences $(y_n),(z_n) \subseteq X$ we have*

$$d(x,y_n) \to 0 \text{ and } d(y_n,z_n) \to 0 \text{ imply } d(x,z_n) \to 0,$$

then X is metrizable. □

In Section 3 we prove a counterpart of Archangelskij's theorem, which works in partial symmetric spaces. Here, we demonstrate some useful characterizations of the local triangle axiom and its basic implications.

In fact any partial symmetric with property **(L)** is a partial semimetric.

Proposition 2.16. *Let $\langle X,\rho\rangle$ be a partial symmetric space. If the condition* (**L**) *holds in X, then the map ρ is a partial semimetric.*

Proof. For any $x \in X$ denote by $\mathcal{B}_x$ the collection of all ρ-balls centered at x. For all $x \in X$ the family $\mathcal{B}_x$ constitutes a neighbourhood base at x if and only if for all $x \in X$ and for all $B \in \mathcal{B}_x$ there exists $C \in \mathcal{B}_x$ such that if $y \in C$, there is some $D \in \mathcal{B}_y$ with $D \subseteq B$. (See for example [26], Theorem 4.5 p.33.) Equivalently, we have

$$\begin{aligned}&\forall x \in X.\ \forall \varepsilon > 0.\ \exists \delta_1 > 0.\ \forall y \in X.\ \exists \delta_2 > 0.\ \forall z \in X.\\ &q_\rho(x,y) < \delta_1,\ q_\rho(y,z) < \delta_2 \text{ imply } q_\rho(x,z) < \varepsilon,\end{aligned}$$

or in other words,

$$\begin{aligned}&\forall x \in X.\ \forall \varepsilon > 0.\ \forall (y_n), (z_m) \subseteq X.\\ &\lim_n q_\rho(x, y_n) = 0,\ \lim_n \lim_m q_\rho(y_n, z_m) = 0 \text{ imply } \lim_m q_\rho(x, z_m) = 0,\end{aligned}$$

which is in general a weaker condition than **(L)**. □

Alternative proof: By Corollary 2.9 it is enough to show that for any subset H of X, the set $H' := \{x \mid q_\rho(x,H) = 0\}$ is closed. Suppose not. Then $q_\rho(x, H') = 0$ for some $x \notin H'$. By definition of the distance between a point and a set, there exist sequences $(y_n) \subseteq H'$ and $(z_n) \subseteq H$ such that $q_\rho(x, y_n) \to 0$ and $q_\rho(y_n, z_n) \to 0$. By assumption, $q_\rho(x, z_n) \to 0$. Hence $q_\rho(x, H) = 0$ and consequently, $x \in H'$, a contradiction. □

Proposition 2.17. *For a distance space $\langle X, \rho\rangle$, the following are equivalent:*

(1) *ρ satisfies* (L)*;*
(2) $\forall x \in X.\ \forall \varepsilon > 0.\ \exists \delta > 0.\ \forall y, z \in X.$
$\rho(x,z) < \rho(x,x) + \delta, \rho(z,y) < \rho(z,z) + \delta$ imply $\rho(x,y) < \rho(x,x) + \varepsilon$.

Proof.

(2) holds

iff

$$\begin{aligned}&\forall x \in X.\ \forall \varepsilon > 0.\\ &\neg[\exists (y_k), (z_k) \subseteq X.\ \rho(x, z_k) \to \rho(x,x), \rho(z_k, y_k) \to \rho(z,z),\\ &\qquad\qquad \rho(x, y_k) \geq \rho(x,x) + \varepsilon],\end{aligned}$$

iff

$$\begin{aligned}&\forall x \in X.\\ &\neg[\exists (y_k), (z_k) \subseteq X.\ \rho(x, z_k) \to \rho(x,x), \rho(z_k, y_k) \to \rho(z,z),\\ &\qquad\qquad \rho(x, y_k) \nrightarrow \rho(x,x)],\end{aligned}$$

iff

$$\begin{aligned}&\forall x \in X.\ \forall (y_k), (z_k) \subseteq X.\\ &\rho(x, z_k) \to \rho(x,x), \rho(z_k, y_k) \to \rho(z,z) \text{ imply } \rho(x, y_k) \to \rho(x,x)\end{aligned}$$

iff

(1) holds.

□

Therefore, we can easily show that every partial metric satisfies the local triangle axiom. From the proof of the following result it is also obvious that the converse claim will not hold in general.

Corollary 2.18. *Let $\langle X, p\rangle$ be a partial metric space. Then the map p satisfies* **(L)**.

Proof. We will prove a more general statement. We claim that a mapping $\rho\colon X \times X \to [0,\infty)$ satisfies $(\Delta^\sharp)$ if and only if

$$\forall x,y,z \in X.\ \forall \varepsilon_1, \varepsilon_2 > 0. \quad \begin{array}{l}\rho(x,z) < \rho(x,x) + \varepsilon_1 \text{ and } \rho(z,y) < \rho(z,z) + \varepsilon_2 \\ \text{imply } \rho(x,y) < \rho(x,x) + \varepsilon_1 + \varepsilon_2.\end{array}$$

The formula above implies **(L)** (which is easily seen by Proposition 2.17 (2)).

To prove the claim, let $x,y,z \in X$, $\varepsilon_1, \varepsilon_2 > 0$ and assume the hypothesis of the implication above. Then by the sharp triangle inequality we have

$$\rho(x,y) \le \rho(x,z) + \rho(z,y) - \rho(z,z) < \rho(x,x) + \varepsilon_1 + \varepsilon_2,$$

as required. Conversely, take any $\varepsilon > 0$. We have $\rho(x,z) < \rho(x,x) + (\rho(x,z) - \rho(x,x) + \varepsilon)$ and $\rho(z,y) < \rho(z,z) + (\rho(z,y) - \rho(z,z) + \varepsilon)$. By assumption, $\rho(x,y) < \rho(x,x) + (\rho(x,z) - \rho(x,x) + \varepsilon) + (\rho(z,y) - \rho(z,z) + \varepsilon) = \rho(x,z) + \rho(z,y) - \rho(z,z) + 2\varepsilon$. Hence the sharp triangle inequality follows from the arbitrariness of ε and the claim is now proved. □

The following result generalizes a similar result for symmetric spaces obtained by H. W. Martin in [12].

Proposition 2.19. *Let $\langle X, \rho\rangle$ be a partial symmetric space. Then the following are equivalent:*

(1) *The map ρ satisfies* **(L)**.
(2) *If K is compact, H is closed and $K \cap H = \varnothing$, then $q_\rho(K,H) > 0$.*

Proof. We will show that $q_\rho(K,H) = 0$ implies $K \cap H \neq \varnothing$. Let $q_\rho(K,H) = 0$. Hence there is a sequence (k_n) of elements of K such that $q_\rho(k_n, H) < 1/2^{n+1}$ for every $n \in \omega$. This means that there also exists a sequence (h_n) of elements from H such that $q_\rho(k_n, h_n) \to 0$. By compactness of K, there exists a convergent subsequence (k_{n_m}) of (k_n) with $k_{n_m} \to x \in K$. Proposition 2.16 guarantees that the map ρ is a partial semimetric and hence $q_\rho(x, k_{n_m}) \to 0$. Then for the corresponding subsequence (h_{n_m}) of h_n we have $q_\rho(k_{n_m}, h_{n_m}) \to 0$. By assumption, $q_\rho(x, h_{n_m}) \to 0$ and hence $x \in \mathrm{cl}(H) = H$. This means that $K \cap H \neq \varnothing$.

Conversely, for any two sequences $(x_n), (y_n)$ and an element x of X, suppose that $q_\rho(x, x_n) \to 0$ and $q_\rho(x_n, y_n) \to 0$. Define a compact subset K of X and a closed subset H of X in the following way:

$$K := \{x\} \cup \{x_n \mid n \in \omega\},$$
$$H := \mathrm{cl}_{\tau_\rho}\{y_n \mid n \in \omega\}.$$

There are two cases to consider: either $K \cap H = \varnothing$ or $K \cap H \neq \varnothing$. In the former case, by assumption (2), we infer that there exists $\varepsilon > 0$ such that $q_\rho(K,H) > \varepsilon$. Therefore,

$$\inf_n\{q_\rho(x_n, y_n)\} \ge q_\rho(K,H) > \varepsilon,$$

which is impossible since $q_\rho(x_n, y_n) \to 0$. In the latter case we distinguish three simple subcases. If no x_n's belong to $K \cap H$, then $x \in K \cap H$ and hence $(y_n) \to_{\tau_\rho} x$, which by Proposition 2.10 is equivalent to saying that $q_\rho(x, y_n) \to 0$. If infinitely many x_n's belong to $K \cap H$, then since H is closed, also $x \in K \cap H$ and then by the same argument as above, $q_\rho(x, y_n) \to 0$. Finally, if a finite number of x_n's belong to $K \cap H$, say $x_1, \ldots, x_{k-1} \subseteq K \cap H$, then consider the remaining sequence $(x_n)_{n \geq k}$ and repeat the proof with $K := \{x\} \cup \{x_n \mid n \geq k\}$. □

3. Metrizability of the induced distance space

In this section we present the first of the two major results of this paper. We prove a counterpart for the Archangelskij Metrization Theorem (Theorem 2.15) working in partial symmetric spaces.

Lemma 3.1. *If $\langle X, \rho\rangle$ is a semimetric space that satisfies* **(L)***, then the induced distance space $\langle X, d_\rho\rangle$ satisfies* **(L)**.

Proof. Suppose that for some sequences $(y_n), (z_n) \subseteq X$ and an element $x \in X$ we have $\lim d_\rho(x, y_n) = 0$ and $\lim d_\rho(y_n, z_n) = 0$. Then by Proposition 2.14.(b), one sees that $\lim \rho(x, y_n) = \rho(x, x)$, $\lim \rho(y_n, z_n) = \lim \rho(y_n, y_n)$ and $\rho(x, x) = \lim \rho(z_n, z_n)$. The two former equalities imply that $\lim \rho(x, z_n) = \rho(x, x)$ by **(L)** for the map ρ. Together with the third equality, we get $\lim d_\rho(x, z_n) = 0$ by yet another application of Proposition 2.14. □

Theorem 3.2. *For any partial symmetric space $\langle X, \rho\rangle$ with* **(L)** *the induced distance d_ρ is metrizable.*

Proof. By Lemma 3.1 and Proposition 2.16, the induced distance is a semimetric and satisfies **(L)**.

We will show that $\langle X, d_\rho\rangle$ is Hausdorff. Since it is first-countable by Proposition 2.10, it will be enough to demonstrate that limits of sequences are unique in X. Hence, let (y_n) be a sequence of elements of X and suppose that (y_n) has two limits x and z in X. Then again by Proposition 2.10 applied to the mapping d_ρ we have $d_\rho(x, y_n) \to 0$ and $d_\rho(z, y_n) \to 0$. Using **(L)** we conclude that $d_\rho(x, z) = 0$. The last equality is equivalent to $\rho(x, z) = \rho(x, x)$ and $\rho(x, z) = \rho(z, z)$. The characterization of the order in partial semimetric spaces from Proposition 2.11 implies that $x \sqsubseteq_{\tau_\rho} z$ and $z \sqsubseteq_{\tau_\rho} x$. Hence, $x = z$ using the T_0 axiom.

Taking all of the proved properties together, one can see that $\langle X, d_\rho\rangle$ is a Hausdorff semimetric space, which satisfies the local triangle axiom. Therefore, Archangelskij's Theorem applies and we conclude that $\langle X, d_\rho\rangle$ is metrizable. □

4. Quasi-developability of a distance space

As an application of the metrizability theorem proved in the last section, we consider the problem of quasi-developability of partial symmetric spaces. Let us first introduce the necessary terminology.

Let $\langle X, \tau\rangle$ be a topological space, $x \in X$ and $\mathcal{C}$ be any collection of subsets of X. Denote $\mathrm{card}\{C \in \mathcal{C} \mid x \in C\}$ by $\mathrm{ord}(x, \mathcal{C})$.

A sequence $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \ldots$ of collections of open subsets of a topological space $\langle X, \tau\rangle$ is called a *quasi-development* for X provided that if $x \in U \in \tau$, then there exists $n \in \omega$ and G such that $x \in G \in \mathcal{G}_n$ and $\mathrm{St}(x, \mathcal{G}_n) \subseteq U$, where $\mathrm{St}(x, \mathcal{G}_n) := \bigcup\{V \in \mathcal{G}_n \mid x \in V\}$. A topological space $\langle X, \tau\rangle$ is *quasi-developable* if it admits a quasi-development.

Proposition 1 of [10] states that every partial metric space is quasi-developable. Since we have proved in Corollary 2.18 that every partial metric satisfies **(L)** and noted that the converse does not hold in general, the following theorem improves Künzi and Vajner's result. We adapt the idea of the proof and the notation from [10].

Theorem 4.1. *Let $\langle X, \rho\rangle$ be a partial symmetric space with* **(L)**. *Then the topological space $\langle X, \tau_\rho\rangle$ is quasi-developable.*

Proof. For simplicity, denote the induced semimetric by d, instead of the standard d_ρ. For each $k, n \in \omega$, set

$$A_{nk} := \{x \in X \mid \rho(x, x) \in [(k-1)2^{-n}, k2^{-n})\}.$$

Let $\bigcup_{t\in\omega} \mathcal{B}_t$ be a base for the metrizable induced topology τ_d such that each collection $\mathcal{B}_t$ is discrete. For each $t, l \in \omega$, define

$$C_{tl} := \{x \in X \mid B_d(x, 2^{-l}) \text{ hits at most one element of } \mathcal{B}_t\}.$$

For each $k, l, t \in \omega$ set

$$\mathcal{R}_{klt} := \{\mathrm{int} B_\rho(B \cap C_{tl} \cap A_{(l+1)k}, 2^{-(l+1)}) \mid B \in \mathcal{B}_t\}.$$

We claim that $\bigcup_{k,l,t} \mathcal{R}_{klt}$ is a quasi-development for τ_ρ.

For an arbitrary $x \in X$ and a natural number h, denote by $m := m(x, h)$ the natural number chosen according to Proposition 2.17 (that is, for $x \in X$ and $\varepsilon := 2^{-h}$ we take $\delta := 2^{-m}$). By Proposition 2.16, the map ρ is a partial semimetric. Therefore, $x \in \mathrm{int} B_\rho(x, 2^{-m})$. Using the fact that $\tau_\rho \subseteq \tau_d$, there exists $t_0 \in \omega$ such that $x \in B_0 \subseteq \mathrm{int} B_\rho(x, 2^{-m}) \subseteq B_\rho(x, 2^{-m})$ for some $B_0 \in \mathcal{B}_{t_0}$. Furthermore, since $\mathcal{B}_{t_0}$ is discrete, there is $l_0 \in \omega$ with $l_0 \geq m$ such that

$$(\dagger) \qquad B_d(x, 2^{-l_0}) \cap B \neq \varnothing \text{ and } B \in \mathcal{B}_{t_0} \text{ imply } B = B_0.$$

Finally, there exists $k_0 \in \omega$ such that $x \in A_{(l_0+1)k_0}$.

Note first that $x \in B_0 \cap C_{t_0 l_0} \cap A_{(l_0+1)k_0}$. Hence

$$x \in \mathrm{int} B_\rho(x, 2^{-(l_0+1)}) \subseteq \mathrm{int} B_\rho(B_0 \cap C_{t_0 l_0} \cap A_{(l_0+1)k_0}, 2^{-(l_0+1)}).$$

We claim that

$$B_\rho(B_0 \cap C_{t_0 l_0} \cap A_{(l_0+1)k_0}, 2^{-(l_0+1)}) \subseteq B_\rho(x, 2^{-h}),$$

which will show that $\bigcup_{k,l,t} \mathcal{R}_{klt}$ is a basis for τ_ρ. Let $y \in B_\rho(B_0 \cap C_{t_0 l_0} \cap A_{(l_0+1)k_0}, 2^{-(l_0+1)})$. Then $y \in B_\rho(B_0, 2^{-(l_0+1)})$. That is, $\rho(z, y) < \rho(z, z) + 2^{-(l_0+1)}$ for some $z \in B_0$. Hence $\rho(z, y) < \rho(z, z) + 2^{-m}$. On the other hand, since $B_0 \subseteq B_\rho(x, 2^{-m})$, we have $\rho(x,z) < \rho(x,x) + 2^{-m}$. From the last two inequalities we conclude that $\rho(x,y) < \rho(x,x) + 2^{-h}$, again using Proposition 2.17. That is, we have shown that $y \in B_\rho(x, 2^{-h})$.

It remains to prove that $\mathbf{ord}(x, \mathcal{R}_{k_0(l_0+1)t_0}) = 1$. Let $x \in B_\rho(B_0 \cap C_{t_0(l_0+1)} \cap A_{(l_0+2)k_0}, 2^{-(l_0+2)})$. Then there is $y \in B_0 \cap C_{t_0(l_0+1)} \cap A_{(l_0+2)k_0}$ such that $\rho(y,x) < \rho(y,y) + 2^{-(l_0+2)}$.

We will demonstrate that

$$\rho(x,y) < \rho(x,x) + 2^{-(l_0+1)}.$$

If $\rho(x,x) > \rho(y,y)$, then $\rho(x,y) = \rho(y,x) < \rho(x,x) + 2^{-(l_0+1)}$, using symmetry of ρ. Otherwise, since $x, y \in A_{(l_0+2)k_0}$, we have $\rho(y,y) < \rho(x,x) + 2^{-(l_0+2)}$ and so $\rho(x,y) = \rho(y,x) < \rho(y,y) + 2^{-(l_0+2)} < \rho(x,x) + 2^{-(l_0+1)}$.

Hence $d(x,y) = 2\rho(x,y) - \rho(x,x) - \rho(y,y) < 2^{-l_0}$, which means that $y \in B_d(x, 2^{-l_0}) \cap B$. By (†), we thus have $B = B_0$. We have shown that $\mathbf{ord}(x, \mathcal{R}_{k_0(l_0+1)t_0}) = 1$. Now, it is immediate that the collection $\bigcup_{k,l,t} \mathcal{R}_{klt}$ is a quasi-development for τ_ρ. □

5. The self-distance mapping in distance spaces

The self-distance map $\mu_\rho \colon X \to [0,\infty)^{op}$ (called also a *weight function*) associated with a partial semimetric ρ proves to be an important object to study. We start with some basic properties of the mapping. Later, in Section 5, we will see that in Domain Theory weight functions correspond to *measurements* in the sense of Martin [13]. In Section 6.4 we use measurements to build partial metrics on algebraic domains.

Note that the codomain of the self-distance map is the set of non-negative real numbers with the opposite of the natural order.

In this section we consider the interplay between self-distance maps and the specialisation order in partial semimetric spaces.

Proposition 5.1. *The self-distance map associated with a partial semimetric ρ is monotone and strictly monotone with respect to the specialisation orders of its domain and codomain.*

Proof. For any $x, y \in X$, $x \sqsubseteq_{\tau_\rho} y$ is equivalent to $\rho(x,y) = \rho(x,x)$ by Proposition 2.11. But by **(ssd)**, $\rho(y,y) \leq \rho(x,y) = \rho(x,x)$. That is, $\mu_\rho y \leq \mu_\rho x$. Hence the map μ_ρ is monotone.

Here, strict monotonicity is the condition

$$\forall x, y \in X.\ (x \sqsubseteq_\rho y \text{ and } \mu_\rho x = \mu_\rho y) \text{ imply } x = y$$

and one can note that this condition is an equivalent formulation of the T_0 axiom of the space (using Corollary 2.12). □

For the self-distance map μ_ρ associated with a partial symmetric ρ define

$$\mu_\rho(x,\varepsilon) := \{y \in X \mid y \sqsubseteq_{\tau_\rho} x \wedge \mu_\rho y < \mu_\rho x + \varepsilon\}. \tag{5.1}$$

We say that $\mu_\rho(x,\varepsilon)$ is the set of elements of X which are ε-*close to* x.

Lemma 5.2. *Let $\langle X, \rho\rangle$ be a partial semimetric space. Then*

$$\forall\varepsilon > 0.\ \forall x \in P.\ \ \mu_\rho(x, \varepsilon) \subseteq B_\rho(x, \varepsilon).$$

Proof. Suppose $z \in \mu_\rho(x, \varepsilon)$. Since $z \sqsubseteq_\rho x$, we have $\rho(x, z) = \mu_\rho z$ by semimetrizability. Therefore, $\rho(x, z) = \mu_\rho z < \mu_\rho x + \varepsilon$. That is, $z \in B_\rho(x, \varepsilon)$. □

Proposition 5.3. *Let $\langle X, \rho\rangle$ be a partial semimetric space, $x \in X$ and S any subset of X. If $S \sqsubseteq x$ and $\mu_\rho x = \inf\{\mu_\rho s \mid s \in S\}$, then x is the supremum of S.*

Proof. Let $x \in U \in \tau_\rho$ and let u be any upper bound of S. Since U is open, there exists $\varepsilon > 0$ such that $x \in B_\rho(x, \varepsilon) \subseteq U$.

By assumption, there exists $s \in S$ with $\mu_\rho s < \mu_\rho x + \varepsilon$. Since $s \sqsubseteq x$, $s \in \mu_\rho(x, \varepsilon)$. By Lemma 5.2, $s \in B_\rho(x, \varepsilon)$ and so $s \in U$. But the latter set is upper, and therefore $u \in U$. We have shown that for any $U \in \tau_\rho$, $x \in U$ implies $u \in U$ and hence $x \sqsubseteq_{\tau_\rho} u$ follows. This means that $x = \bigsqcup S$. □

5.1. **Stability condition for partial semimetrics.** It happens that there exists a class of partial semimetric spaces, where the distance topology can be recovered from self-distance maps. Continuous domains in their Scott topology (see Section 6.2) are our major example of such spaces. Here, we develop the basics. For more information, consult [13, 25, 24].

Definition 5.4. Let $\langle X, \tau_\rho\rangle$ be a partial symmetric space. We say that the map ρ is *stable* if for all $x, y \in X$ we have

$$\rho(x, y) := \inf\{\mu_\rho z \mid z \sqsubseteq_{\tau_\rho} x, y\}.$$

Lemma 5.5. *Let $\langle X, \rho\rangle$ be a partial semimetric space. Then*

$$\forall\varepsilon > 0.\ \exists\delta > 0.\ \forall x \in P.\ \ \mu_\rho(x, \delta) \subseteq \mathrm{int}(B_\rho(x, \varepsilon)).$$

Proof. Let $x \in X$ and $\varepsilon > 0$. Then by definition, there exists $\delta > 0$ such that $x \in B_\rho(x, \delta) \subseteq \mathrm{int}(B_\rho(x, \varepsilon))$. By Lemma 5.2, $\mu_\rho(x, \delta) \subseteq \mathrm{int}(B_\rho(x, \varepsilon))$. □

Lemma 5.6. *Let $\langle X, \rho\rangle$ be a stable partial semimetric space. Then for every $x \in X$ and $\varepsilon > 0$ we have*

$$B_\rho(x, \varepsilon) \subseteq \uparrow\mu_\rho(x, \varepsilon).$$

Proof. Let $x \in X$, $\varepsilon > 0$ and $y \in B_\rho(x, \varepsilon)$. Then $\rho(x, y) < \mu_\rho x + \varepsilon$. By stability, there exists $z \sqsubseteq_\rho x, y$ with $\mu_\rho z < \rho(x, y) + \varepsilon$ and hence $z \in \mu_\rho(x, \varepsilon)$. Then, $y \in \uparrow\mu_\rho(x, \varepsilon)$ follows. □

Theorem 5.7. *Let (X, ρ) be a stable partial semimetric space. Then*

$$\{\uparrow\mu_\rho(x, \varepsilon) \mid \varepsilon > 0\}$$

is a neighborhood base at x in τ_ρ.

Proof. Let $x \in X$ and $\varepsilon > 0$. Then $B_\rho(x, \varepsilon) = \uparrow\mu_\rho(x, \varepsilon)$ by Lemma 5.2 and Lemma 5.6. □

6. Distance for continuous domains

As we have mentioned in the introduction, a major motivation for studying general distance spaces was the desire to understand the concept of distance for continuous domains. In previous sections we outlined a general theory, which, we believe, proves especially useful in Domain Theory. In this section we demonstrate that for continuous domains the theory of partial semimetric spaces coincides with the theory of *measurements* introduced by Martin [13] and studied further in the author's PhD thesis [25]. Furthermore, we introduce so called *Lebesgue measurements* (Definition 6.5), which correspond to partial semimetrics, which satisfy condition **(L)**. Finally, we prove a characterization of partial metrizability of algebraic domains using Lebesgue measurements.

First, let us recall some terminology of Domain Theory. See [1] for more information. Let P be a poset. A subset $A \subseteq P$ of P is *directed* if it is nonempty and any pair of elements of A has an upper bound in A. If a directed set A has a supremum, it is denoted $\bigsqcup^{\uparrow} A$. A poset P in which every directed set has a supremum is called a *dcpo*.

Let x and y be elements of a poset P. We say that x *approximates* (*is way-below*) y if for all directed subsets A of P, $y \sqsubseteq \bigsqcup^{\uparrow} A$ implies $x \sqsubseteq a$ for some $a \in A$. We denote this by $x \ll y$. If $x \ll x$ then x is called a *compact* element. The subset of compact elements of a poset P is denoted $K(P)$. Now, $\mathord{\downarrow\!\!\downarrow} x$ is the set of all approximants of x below it. $\mathord{\uparrow\!\!\uparrow} x$ is defined dually. We say that a subset B of a dcpo P is a *basis* for P if for every element x of P, the set $\mathord{\downarrow\!\!\downarrow} x \cap B$ is directed with supremum x. A poset is called *continuous* if it has a basis. It can be shown that a poset P is continuous iff $\mathord{\downarrow\!\!\downarrow} x$ is directed with supremum x, for all $x \in P$. A poset is called a *continuous domain* if it is a continuous dcpo. Note that $K(P) \subseteq B$ for any basis B of P. If $K(P)$ is itself a basis, the domain P is called *algebraic*.

In a continuous domain, every basis is an example of a so called *abstract basis*, which is a set B together with a transitive relation $\prec$ on B, such that

$$\text{(INT)} \qquad M \prec x \text{ implies } \exists y \in B.\, M \prec y \prec x$$

holds for all elements x and finite subsets M of B. For an abstract basis $\langle B, \prec \rangle$ and an element $x \in B$ set $x^* := \{y \in B \mid y \prec x\}$.

For $x \in B$ we also define $x_* := \{y \in B \mid x \prec y\}$. The collection of all sets of the form x_* is a basis for a topology on B called the *pseudoScott topology* [11].

In this paper, an abstract basis such that the relation $\prec$ is reflexive is named a *reflexive abstract basis*. For any algebraic domain P, the set $K(P)$ is an example of a reflexive abstract basis.

For an abstract basis $\langle B, \prec \rangle$ let $\mathcal{I}(B)$ be the set of all ideals (directed, lower subsets) ordered by inclusion. It is called the (*rounded*) *ideal completion* of B. For any algebraic domain P the rounded ideal completion of $K(P)$ is isomorphic to P, in symbols: $\mathcal{I}(K(P)) \cong P$.

Upper sets inaccessible by directed suprema form a topology called the *Scott topology*. The specialisation order of the Scott topology on a poset coincides with the underlying order. The collection $\{\twoheaduparrow x \mid x \in P\}$ forms a basis for the Scott topology on a continuous domain P. The Scott topology satisfies only weak separation axioms: it is always T_0 on a poset but T_1 only if the order is trivial. The Scott topology on a poset P will be denoted $\sigma(P)$ (or σ for short).

6.1. **Measurements.** We say that a monotone mapping $\mu\colon P \to [0,\infty)^{op}$ *induces the Scott topology* on a poset P if

$$\forall U \in \sigma(P).\ \forall x \in P.\ \exists \varepsilon > 0.\ \mu(x,\varepsilon) \subseteq U,$$

where

$$\mu(x,\varepsilon) := \{y \in P \mid y \sqsubseteq x \text{ and } \mu y < \mu x + \varepsilon\}.$$

We denote this by $\mu \longrightarrow \sigma(P)$.

Definition 6.1. If P is a continuous poset, $\mu\colon P \to [0,\infty)^{op}$ a Scott-continuous map with $\mu \longrightarrow \sigma(P)$, then we will say that μ *measures* P or that μ is a *measurement* on P.

Our definition of a measurement is a special case of the one given by Martin. In the language of [13] our maps are measurements, which induce the Scott topology *everywhere*.

6.2. **Partial semimetrics versus measurements.**

Definition 6.2 (Martin)**.** Let P be a continuous poset with a measurement $\mu\colon P \to [0,\infty)^{op}$. The map $p_\mu\colon P \times P \to [0,\infty)^{op}$ defined by

$$p_\mu(x,y) := \bigsqcup\{\mu z \mid z \ll x,y\} = \inf\{\mu z \mid z \ll x,y\}$$

is the partial semimetric associated with μ (cf. Proposition 6.3 below).

Note that the definition is well-formed if any two elements x,y of P are bounded from below. This condition, however, may be omitted: whenever x,y have no lower bound, we scale the measurement to $\mu^*\colon P \to [0,1)^{op}$ with $\mu^*(x) := \mu x/(1+\mu x)$ for any $x \in P$. Such map is again a measurement (cf. Lemma 5.3.1 of [13], page 135). Now, we define p^*_μ to be:

$$\forall x,y \in P. \quad p^*_\mu(x,y) = \begin{cases} \inf\{\mu^* z \mid z \ll x,y\} & \text{if } \exists z \in P.\ z \sqsubseteq x,y \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 6.3. *Let $\mu\colon P \to [0,\infty)^{op}$ be a measurement on a continuous poset P. Then:*

(1) *p_μ is a Scott-continuous map from $P \times P$ to $[0,\infty)^{op}$.*
(2) *$p_\mu(x,x) = \mu x$ for all $x \in P$.*
(3) *$B_{p_\mu}(x,\varepsilon) = \uparrow\mu(x,\varepsilon)$ for all $x \in P$ and $\varepsilon > 0$. That is, p_μ is a stable partial semimetric, which induces the Scott topology.*
(4) *For a sequence (x_n) and any $x \in P$, $x_n \to x$ in the Scott topology on P iff $\lim p_\mu(x_n,x) = \mu x$.*

Proof. For the proof of statements (1)–(3) consult [13]. Lastly, (4) follows from Proposition 2.10. □

We are now ready to discuss the coincidence of the theory of partial semimetric spaces with the theory of measurements in the framework of continuous domains.

Theorem 6.4. *For a continuous poset P the following are equivalent:*

1. *P admits a stable partial semimetric compatible with the Scott topology;*
2. *P admits a partial semimetric compatible with the Scott topology, which is Scott-continuous as a map from $P \times P$ to $[0,\infty)^{op}$;*
3. *P admits a measurement.*

Proof. The implication (1) ⇒ (2) is trivial. For (2) ⇒ (3), the self-distance mapping of the partial semimetric has the defining measurement property $\mu \to \sigma(P)$ by Lemma 5.5. In addition, it is Scott-continuous since the partial semimetric is. Lastly, (3) ⇒ (1) is a consequence of Proposition 6.3.(3). □

6.3. **Lebesgue measurements.** For the purpose of the next definition we introduce the following notation.

$$\mu(A,\varepsilon) := \bigcup\{\mu(x,\varepsilon) \mid x \in A\}, \tag{6.2}$$

where A is a subset of a continuous domain P, the map $\mu\colon P \to [0,\infty)^{op}$ is monotone and $\varepsilon > 0$.

Definition 6.5. Let P be a continuous domain. A Scott-continuous map $\mu\colon P \to [0,\infty)^{op}$ is a *Lebesgue measurement* on P if for all Scott-compact (we may assume saturated) subsets $K \subseteq P$ and for all Scott-open subsets $U \subseteq P$,

$$K \subseteq U \;\Rightarrow\; \exists\, \varepsilon > 0.\ \mu(K,\varepsilon) \subseteq U.$$

One can immediately see from the definitions that Lebesgue measurements are measurements.

Proposition 6.6. *Let P be a continuous domain equipped with a measurement $\mu\colon P \to [0,\infty)^{op}$. The following are equivalent:*

1. *μ is a Lebesgue measurement.*
2. *If $\uparrow x \subseteq U$ for some x in P and a Scott-open subset U of P, then there exists $\varepsilon > 0$ such that $\mu(\uparrow x, \varepsilon) \subseteq U$.*

Proof. For the nontrivial direction, suppose that $K \subseteq U$, where K is a Scott-compact saturated subset of P and U is Scott-open in P. Then for all $k \in K$ choose an element $l \ll k$ with $l \in U$. The collection $\{\twoheaduparrow l \mid l \ll k\}$ is an open cover of K. Hence a finite subcollection $\twoheaduparrow l_1, \twoheaduparrow l_2, \ldots, \twoheaduparrow l_n$ covers K already. By assumption, for every l_i, $i = 1,\ldots,n$ there is $\varepsilon_i > 0$ with $\mu(\uparrow l_i, \varepsilon_i) \subseteq U$. Set $\varepsilon := \min\{\varepsilon_i \mid i = 1,2,\ldots,n\}$. Then $K \subseteq \mu(K,\varepsilon) \subseteq U$. □

Next, we demonstrate that whenever μ is a Lebesgue measurement on a continuous domain, its induced partial semimetric p_μ satisfies condition **(L)**.

Proposition 6.7. *Let P be a continuous domain measured by μ. The following are equivalent:*

(1) *μ is a Lebesgue measurement;*
(2) *For all $x \in P$ and for all sequences $(x_n), (y_n)$ of P,*

$$p_\mu(x, x_n) \to \mu x \text{ and } p_\mu(x_n, y_n) \to \mu x_n \text{ imply } p_\mu(x, y_n) \to \mu x.$$

Proof. Let $x \in P$ and take a Scott-open set U with $x \in U$. The assumption $p_\mu(x, x_n) \to \mu x$ is equivalent to saying that $(x_n) \to_\sigma x$ by Proposition 6.3. That is,

$$\exists\, N_1 \in \omega.\, \forall n \geq N_1.\ \ x_n \in U.$$

By definition, $p_\mu(x_n, y_n) = \inf\{\mu z \mid z \ll x_n, y_n\}$, for each $n \in \omega$. Therefore, there exists a sequence (z_n) with $z_n \ll x_n, y_n$ and $\lim \mu z_n = \lim p_\mu(x_n, y_n)$, for all $n \in \omega$. Since by assumption, $\lim p_\mu(x_n, y_n) = \lim \mu x_n$, we have $\lim \mu z_n = \lim \mu x_n$. This means

$$\forall\, \varepsilon > 0.\, \exists\, N_2 \in \omega.\, \forall n \geq N_2.\ \ \mu z_n - \mu x_n < \varepsilon.$$

Note that the set $K := \{x_n \mid n \geq N_1\} \cup \{x\}$ is a compact subset of U, and since μ is a Lebesgue measurement, the condition specialises to:

$$\exists \lambda > 0.\, \forall x_n \in K.\, \forall z \in P.\ \ [z \sqsubseteq x_n \text{ and } \mu z < \mu x_n + \lambda] \Rightarrow z \in U.$$

Hence $z_n \in U$ for all $n \geq \max\{N_1, N_2(\lambda)\}$. Since $z_n \ll y_n$ for all $n \in \omega$, (y_n) is cofinally in U. That is, $(y_n) \to_\sigma x$, or equivalently, $p_\mu(x, y_n) \to \mu x$.

For the converse, observe that by Proposition 2.19 for any Scott-open set $U \subseteq P$ and $x \in U$ we have $q_{p_\mu}(\uparrow x, P \setminus U) > 0$. This is however equivalent to saying that there exists $\varepsilon > 0$ such that

$$B_{p_\mu}(\uparrow x, \varepsilon) := \bigcup_{x \sqsubseteq y} B_{p_\mu}(y, \varepsilon) \subseteq U.$$

But the map p_μ is stable and hence $B_{p_\mu}(y, \varepsilon) = \uparrow\mu(y, \varepsilon)$ by the proof of Theorem 5.7. We conclude that $\mu(\uparrow x, \varepsilon) \subseteq U$. Therefore, the map μ is a Lebesgue measurement by Proposition 6.6. □

The last result can be easily extended and stated in a form analogous to Theorem 6.4.

Theorem 6.8. *For a continuous domain P the following are equivalent:*

(1) *P admits a stable partial semimetric with* **(L)** *for the Scott topology;*
(2) *P admits a partial semimetric with* **(L)** *for the Scott topology, which is Scott-continuous as a mapping from $P \times P$ to $[0, \infty)^{op}$;*
(3) *P admits a Lebesgue measurement.*

Proof. It is enough to show (2) ⇒ (3). Let $p\colon P \times P \to [0, \infty)$ be a partial semimetric for the Scott topology, which satisfies **(L)**. Then for any Scott-compact subset K and for any Scott-open subset U of P with $K \subseteq U$, we have $q_\rho(K, P \setminus U) > 0$, by Proposition 2.19. It is equivalent to say that for some $\varepsilon > 0$ we have $K \subseteq B_p(K, \varepsilon) \subseteq U$. Denote the self-distance map for p by μ_p.

Now, by Lemma 5.2, we have $\mu_p(K,\varepsilon) \subseteq B_p(K,\varepsilon) \subseteq U$ and so the mapping μ_p is a Lebesgue measurement on P. □

6.4. **Partial metrization of algebraic domains.** In this section we apply the knowledge about Lebesgue measurements to obtain a complete characterization of partial metrizability of the Scott topology on an algebraic domain. We start from a similar result obtained by Künzi and Vajner in Proposition 3 p.73 of [10].

Theorem 6.9 (Künzi and Vajner)**.** *A poset X admits a partial metric for its Alexandrov topology iff there is a function $|\cdot|\colon X \to [0,\infty)$ such that*

(∗) $\forall x \in X.\ \exists \varepsilon > 0.\ \forall y \in \uparrow x.\ \forall z \in \downarrow y \setminus \uparrow x.\ \ |z| - |y| \geq \varepsilon.$

For the following crucial lemma, recall the notation from the beginning of Section 6.

Lemma 6.10. *Let $(B,\prec)$ be a reflexive abstract basis equipped with a mapping $\mu\colon B \to [0,\infty)^{op}$, which satisfies (∗). Then there exists a partial metric compatible with the Scott topology on the rounded ideal completion $\mathcal{I}(B)$ of B.*

Proof. Note that by the discussion in [10] p.74 we can always assume that μ is bounded by 1. Therefore, applying Theorem 6.9 we conclude that there exists a partial metric $p\colon B \times B \to [0,\infty)$ which is bounded by 3 and such that $\mu_p = \mu$. The partial metric p satisfies

(+) $\forall x \in X.\ \exists \varepsilon_x > 0.\ \uparrow x = \{y \mid p(x,y) = p(x,x)\} = \{y \mid p(x,y) < p(x,x) + \varepsilon_x\}$,

Now, extend the function p to $\mathcal{I}(B)$ in the following way: for each $I, J \in \mathcal{I}(B)$ define $\hat{p}\colon \mathcal{I}(B) \times \mathcal{I}(B) \to [0,\infty)$ by

$$\hat{p}(I,J) := \inf\{p(x,y) \mid x \in I, y \in J\}.$$

Since p is continuous as a map from B equipped with the pseudoScott topology to $[0,\infty)^{op}$ in its Scott topology, the mapping $\hat{p}\colon \mathcal{I}(B) \times \mathcal{I}(B) \to [0,\infty)^{op}$ is Scott continuous. Note that for every $x, y \in B$ we have $\hat{p}(x^*, y^*) = p(x,y)$. This means that p induces the subspace Scott topology on the image of B in $\mathcal{I}(B)$ under the canonical embedding.

Step 1: First we will prove that the map so defined satisfies all the partial metric axioms except $(\mathbf{T_0})$. For the sharp triangle inequality, note that for all $x \in I$, $y \in J$ and $z \in K$, where $I, J, K \in \mathcal{I}(B)$, we have $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ and this inequality extends to infima. Similarly we prove the **(ssd)** axiom. Symmetry is trivial.

Step 2: If $I \subseteq J$ in $\mathcal{I}(B)$, then $\{p(x,y) \mid x \in I, y \in I\} \subseteq \{p(x,y) \mid x \in I,\ y \in J\}$. Therefore, $\hat{p}(I,I) \geq \hat{p}(I,J)$. Hence $\hat{p}(I,I) = \hat{p}(I,J)$ by **(ssd)**.

Step 3: To prove that $\hat{p}$ induces the Scott topology on $\mathcal{I}(B)$ we will show that $\twoheaduparrow x^* = B_{\hat{p}}(x^*,\varepsilon)$ for some $x \in B$ and $\varepsilon > 0$. Suppose $x^* \ll K$ for some $K \in \mathcal{I}(B)$. Then $x^* \subseteq K$ and so $\hat{p}(x^*,K) = \hat{p}(x^*,x^*)$ by Step 2. Hence, $\twoheaduparrow x^* \subseteq B_{\hat{p}}(x^*,\varepsilon)$ for any $\varepsilon > 0$. Conversely, if for some $L \in \mathcal{I}(B)$ we have that

$\hat{p}(x^*, L) < \hat{p}(x^*, x^*) + \varepsilon_x$, then $\hat{p}(x^*, w^*) < \hat{p}(x^*, x^*) + \varepsilon_x$ for some $w \in L$. This is however equivalent to $p(x, w) < p(x, x) + \varepsilon_x$ and so $w \in B_p(x, \varepsilon_x) = \uparrow x$ by (+). That is, $x \prec w$. This means that $x \in L$ and so $x^* \ll L$. We have shown that $B_{\hat{p}}(x^*, \varepsilon_x) \subseteq \Uparrow x^*$.

Step 4: By Step 3, the order induced by $\hat{p}$ agrees with subset inclusion, which is the specialisation order for the Scott-topology on $\mathcal{I}(B)$. Hence if $\hat{p}(I, J) = \hat{p}(I, I) = \hat{p}(J, J)$ for some $I, J \in \mathcal{I}(B)$, then $I \subseteq J$ and $J \subseteq I$. Therefore, $I = J$ and so $\hat{p}$ is a partial metric on $\mathcal{I}(B)$. □

Lemma 6.11. *Let P be an algebraic domain with a monotone map $\mu\colon P \to [0, \infty)^{op}$. Then the local triangle axiom for μ implies $(*)$ for the restriction of μ to $K(P)$.*

Proof. Take any $x \in K(P)$. The set $\uparrow x$ is Scott open and Scott-compact in P. By the local triangle axiom,

$$q_\mu(\uparrow x, P \setminus \uparrow x) > 0.$$

Therefore, there exists an $\varepsilon > 0$ such that

$$q_\mu(\uparrow x, P \setminus \uparrow x) > \varepsilon.$$

This implies in particular that for every $y \in \uparrow x$ and $z \in \downarrow y \setminus \uparrow x$ we have $p_\mu(y, z) \geq \mu y + \varepsilon$. Hence $\mu z = p_\mu(y, z) \geq \mu y + \varepsilon$. This shows $(*)$. □

Therefore, we have obtained a complete characterization of partial metrizability on algebraic domains:

Theorem 6.12. *Let P be an algebraic dcpo. The following are equivalent:*

(1) *P admits a Lebesgue measurement $\mu\colon P \to [0, \infty)^{op}$.*
(2) *P admits a partial metric compatible with the Scott topology on P.*

Proof. By Lemma 6.11 and 6.10, every Lebesgue measurement on P extends to a partial metric for the Scott topology on $\mathcal{I}(P) \cong P$. Conversely, by Corollary 2.18, the self-distance map for a partial metric for the Scott topology satisfies **(L)** and is Scott-continuous by a result from [9]. Hence it is a Lebesgue measurement on P by Theorem 6.8. □

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