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# On complete objects in the category of $T_0$ closure spaces

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Abstract. In this paper we present an example in the setting of closure spaces that fits in the general theory on 'complete objects' as developed by G. C. L. Brümmer and E. Giuli. For $\mathcal{V}$ the class of epimorphic embeddings in the construct $\mathbf{Cl_0}$ of $T_0$ closure spaces we prove that the class of $\mathcal{V}$-injective objects is the unique firmly $\mathcal{V}$-reflective subconstruct of $\mathbf{Cl_0}$. We present an internal characterization of the $\mathcal{V}$-injective objects as 'complete' ones and it turns out that this notion of completeness, when applied to the topological setting is much stronger than sobriety. An external characterization of completeness is obtained making use of the well known natural correspondence of closures with complete lattices. We prove that the construct of complete $T_0$ closure spaces is dually equivalent to the category of complete lattices with maps preserving the top and arbitrary joins.

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## 1. Introduction

A closure space $(X, \mathcal{C})$ is a pair, where $X$ is a set and $\mathcal{C}$ is a subset of the power set $\mathcal{P}(X)$ satisfying the conditions that $X$ and $\varnothing$ belong to $\mathcal{C}$ and that $\mathcal{C}$ is closed for arbitrary unions. The sets in $\mathcal{C}$ are called open sets. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ between closure spaces $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ is said to be continuous if $f^{-1}(D) \in \mathcal{C}$ whenever $D \in \mathcal{D}$. $\mathbf{Cl}$ is the construct of closure spaces as objects and continuous maps as morphisms. Some isomorphic descriptions of $\mathbf{Cl}$ are often used e.g. by giving the collection of all closed sets (the so called Moore family [4]) where, as usual, the closed sets are the complements of the open ones and continuity is defined accordingly. Another isomorphic

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description is obtained by means of a closure operator [4]. The closure operation $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ associated with a closure space $(X, \mathcal{C})$ is defined in the usual way by $x \in \mathrm{cl}\, A \iff (\forall C \in \mathcal{C} : x \in C \Rightarrow C \cap A \neq \varnothing)$ where $A \subset X$ and $x \in X$. This closure need not be finitely additive, but it does satisfy the conditions $\mathrm{cl}\, \varnothing = \varnothing$, $(A \subset B \Rightarrow \mathrm{cl}\, A \subset \mathrm{cl}\, B)$, $A \subset \mathrm{cl}\, A$ and $\mathrm{cl}\,(\mathrm{cl}\, A) = \mathrm{cl}\, A$ whenever $A$ and $B$ are subsets of $X$. Continuity is then characterized in the usual way. Finally closure spaces can also be equivalently described by means of neighborhood collections of the points. These neighborhood collections satisfy the usual axioms, except for the fact that the collections need not be filters. So in a closure space $(X, \mathcal{C})$ the neighborhood collection of a point $x$ is a non empty stack (in the sense that with every $V \in \mathcal{N}(x)$ also every $W$ with $V \subset W$ belongs to $\mathcal{N}(x)$), where every $V \in \mathcal{N}(x)$ contains $x$ and $\mathcal{N}(x)$ satisfies the open kernel condition. In the sequel we will just write $X$ for a closure space and we'll choose the most convenient form for its explicit structure.

Motivations for considering closure spaces can be found in G. Birkhoff's book [4] where he associates closures to binary relations in a natural way. Similar ideas appeared in G. Aumann's work on contact relations with applications to social sciences [3] or in a more recent work of B. Ganter and R. Wille on formal contexts with applications in data analysis and knowledge representation [11]. In recent years closures have also been used in connection with quantum logic and in the representation theory of physical systems, see e.g. [2] or [16]. In these applications the $T_0$ axiom we are dealing with plays a key role [20].

In 1940 G. Birkhoff's motivation for considering closures also came from the observation that the collection of closed sets of a closure space forms a complete lattice. The interrelation between closures and complete lattices has been investigated by many authors and a general discussion of this subject can be found in M. Erné's paper [10]. In the last section of our paper further investigation of the correspondence with complete lattices leads to an external characterization of the complete objects we are studying.

For all categorical terminology we refer the reader to the books [1, 13] or [18].

## 2. The construct of $T_0$ spaces

2.1. As is well known [9] **Cl** is a topological construct in the sense of [1]. $\mathbf{Cl_0}$ is the subconstruct consisting of its $T_0$-objects. Applying Marny's definition [15] we say that a closure space $X$ is a $T_0$-*object* in **Cl** if and only if every morphism from the indiscrete object $I_2$ on the two point set $\{0, 1\}$ to $X$ is constant. This equivalently means that for every pair of different points in $X$ there is a neighborhood of one of the points not containing the other one.

$\mathbf{Cl_0}$ is an extremally epireflective subconstruct of **Cl** [15] and as such it is initially structured in the sense of [17, 18]. In particular $\mathbf{Cl_0}$ is complete and cocomplete and well-powered, it is an (epi - extremal mono) category and an (extremal epi - mono) category [13]. Also from the general setting it follows that monomorphisms in $\mathbf{Cl_0}$ are exactly the injective continuous maps and a morphism in $\mathbf{Cl_o}$ is an extremal epimorphism if and only if it is a regular epimorphism if and only if it is surjective and final.

2.2. In order to describe the epimorphisms and the extremal monomorphisms in $\mathbf{Cl_o}$ we need the regular closure operator determined by $\mathbf{Cl_o}$ as introduced in [8, 9]. Given a closure space $X$ and a subset $M \subset X$ one defines the *regular closure* of $M$ in $X$ as follows.

A point $x$ of $X$ is in the closure of $M$ if and only if

(i) for every $T_0$ closure space $Z$ and every pair of morphisms $f, g : X \to Z$,

$$M \subset \{f = g\} \implies f(x) = g(x).$$

Using the fact that $\mathbf{Cl_o}$ is the epireflective hull in $\mathbf{Cl}$ of the two point Sierpinski space $S_2$, we obtain the following equivalent description.

(ii) For every pair of morphisms $f, g : X \to S_2$,

$$M \subset \{f = g\} \implies f(x) = g(x).$$

Quite similar to the topological situation one can prove yet another equivalent formulation.

(iii) For every neighborhood $V$ of $x$:

$$V \cap \mathrm{cl}\{x\} \cap M \neq \varnothing$$

In each of the equivalent cases we'll write $x \in \mathrm{cl}_b^X M$.

It was shown in [9] that the regular closure

$$\mathrm{cl}_b = \{\mathrm{cl}_b^X : \mathcal{P}(X) \to \mathcal{P}(X)\}_{X \in |\mathbf{Cl}|}$$

defines a closure operator on $\mathbf{Cl}$. By the equivalent description (ii) $\mathrm{cl}_b$ coincides with the Zariski closure operator as considered in [7] and [12]. The equivalent formulation (iii) is the formula for the $b$-closure (or front closure) in $\mathbf{Top}$. For this reason we will also call $\mathrm{cl}_b$ the $b$-closure operator on $\mathbf{Cl}$.

It follows from theorem 2.8 in [9] that the epimorphisms in $\mathbf{Cl_o}$ are the $b$-dense continuous maps. So in fact the inclusion functor $\mathbf{Top_o} \hookrightarrow \mathbf{Cl_o}$ preserves epimorphisms. One observes that this is not so for the inclusion functor from $\mathbf{Top_o}$ to the construct $\mathbf{PrTop_o}$ of $T_0$ pretopological spaces.

Using arguments analogous to the ones used in the topological case, one proves that $\mathbf{Cl_o}$ is cowell-powered.

The closure operator $\mathrm{cl}_b$ is idempotent and grounded and is easily seen to be hereditary in the sense that for a closure space $Y$, a subspace $X$ and $M \subset X \subset Y$ we have

$$\mathrm{cl}_b^X M = \mathrm{cl}_b^Y M \cap X$$

Using this fact one can prove that a morphism in $\mathbf{Cl_o}$ is an extremal monomorphism if and only if it is a regular monomorphism if and only if it is a $b$-closed embedding.

Explicit proofs of the previous statements have been worked out in [19].

## 3. Injective objects in $\mathbf{Cl_o}$ and firmness

In this paragraph we consider a particular class of morphisms in $\mathbf{Cl_o}$. Let $\mathcal{V}$ be the class of epimorphic embeddings, i.e. the class of all $b$-dense embeddings. This class $\mathcal{V}$ satisfies the following conditions:

($\alpha$) closedness under composition,

($\beta$) closedness under composition with isomorphisms on both sides.

($\alpha$) and ($\beta$) are standing assumptions made in [5] and enable us to apply to $\mathbf{Cl_o}$ the theory developed in that paper. A $T_0$ closure space $B$ is $\mathcal{V}$-*injective* if for each $v : X \to Y$ in $\mathcal{V}$ and $f : X \to B$ there exists $f' : Y \to B$ such that $f' \circ v = f$. In this case $f'$ is called an extension of $f$ along $v$. *Inj* $\mathcal{V}$ denotes the full subcategory of all $\mathcal{V}$-injective objects in $\mathbf{Cl_o}$.

**Proposition 3.1.** *The two point Sierpinski space $S_2$ is $\mathcal{V}$-injective in $\mathbf{Cl_o}$.*

Next consider $\mathcal{R}_{\mathbf{Cl_o}}(\{S_2\})$, the epireflective hull of $S_2$ in $\mathbf{Cl_o}$. In view of the properties of $\mathbf{Cl_o}$ listed in paragraph 2, this hull consists of all $b$-closed subspaces of powers of $S_2$.

Recall that a reflective subcategory is $\mathcal{V}$-reflective if the reflection morphisms all belong to $\mathcal{V}$.

**Proposition 3.2.** *A $T_0$ closure space is $\mathcal{V}$-injective if and only if it is a $b$-closed subspace of some power of $S_2$.*

*Proof.* In view of theorem 37.1 in [13] *Inj* $\mathcal{V}$ is epireflective in $\mathbf{Cl_o}$ and since it contains $S_2$ we immediately have $\mathcal{R}_{\mathbf{Cl_o}}(\{S_2\}) \subset Inj\,\mathcal{V}$. Moreover $\mathcal{R}_{\mathbf{Cl_o}}(\{S_2\})$ clearly is $\mathcal{V}$-reflective, so if $B$ is $\mathcal{V}$-injective, the reflection morphism $v : B \to RB$ belongs to $\mathcal{V}$. We have $f' \circ v = 1_B$ where $f'$ is the extension of $1_B : B \to B$ along $v$. Then clearly $v$ is an isomorphism and therefore $B \in |\mathcal{R}_{\mathbf{Cl_o}}(\{S_2\})|$. $\square$

**Remark 3.3.** The notion of $\mathcal{V}$-injectivity in $\mathbf{Cl_o}$ differs from injectivity related to the class of all embeddings. $\mathcal{V}$-injectivity is a strictly weaker condition as is shown by the example $B = \{(0,0,1), (0,1,1), (0,1,0), (1,1,1)\}$ which is a $b$-closed subspace of $S_2^3$ and hence is $\mathcal{V}$-injective by proposition 3.2. However $B$ is not injective in $\mathbf{Cl_o}$ with respect to embeddings.

We use the terminology of [5] (which slightly differs from [6]). A class $\mathcal{U}$ of morphisms in a category $\mathbf{X}$ (satisfying the standing assumptions ($\alpha$) and ($\beta$)) is said to be

(i) a *subfirm* class: if there exists a $\mathcal{U}$-reflective subcategory with reflector $R$ such that $Rf$ is an isomorphism whenever $f$ is in $\mathcal{U}$.

(ii) a *firm* class: if there exists a $\mathcal{U}$-reflective subcategory with reflector $R$ such that $Rf$ is an isomorphism if and only if $f$ is in $\mathcal{U}$.

In these cases the corresponding subcategory is said to be (sub-)firmly $\mathcal{U}$-reflective and it coincides with *Inj* $\mathcal{U}$ [5]. Again we consider the particular class $\mathcal{V}$ of $b$-dense embeddings in $\mathbf{Cl_o}$. In view of the equivalent descriptions given in 2.2 and the fact that $\mathcal{R}_{\mathbf{Cl_o}}(\{S_2\}) = Inj\,\mathcal{V}$, the class *Inj* $\mathcal{V}$ is $\mathcal{V}$-reflective. So we can apply theorems 1.4 and 1.14 in [5] to formulate the following result.

**Proposition 3.4.** *The class $\mathcal{V}$ of b-dense embeddings is a firm class of morphisms in $\mathbf{Cl_0}$ and Inj $\mathcal{V}$ is the unique firmly $\mathcal{V}$-reflective subcategory of $\mathbf{Cl_0}$.*

In the context of an epireflective subcatgory $\mathbf{X}$ of a topological category, with $\mathcal{S}$ the class of embeddings in $\mathbf{X}$ and $\mathcal{V}$ the class of epimorphic embeddings, the notion of $\mathcal{V}$-injective object can be linked to a few others, as discussed in [6]. An object $X$ in $\mathbf{X}$ is said to be *$\mathcal{S}$-saturated* if an $\mathbf{X}$-morphism $f : X \to Y$ is an isomorphism whenever $f$ is in $\mathcal{V}$. $X$ is said to be *absolutely $\mathcal{S}$-closed* if an $\mathbf{X}$-morphism $f : X \to Y$ is a regular monomorphism whenever $f \in \mathcal{S}$.

In the particular situation where moreover in $\mathbf{X}$ extremal monomorphisms coincide with regular monomorphisms and where *Inj* $\mathcal{V}$ is (sub-)firmly $\mathcal{V}$-reflective, it was shown in [6] that for an object $X$ in $\mathbf{X}$ one has

$$\begin{aligned} X \text{ is } \mathcal{V}\text{-injective} &\iff X \text{ is } \mathcal{S}\text{-saturated} \\ &\iff X \text{ is absolutely } \mathcal{S}\text{-closed.} \end{aligned}$$

From the results in paragraph 2 and from Proposition 3.4 we can conclude that the $\mathcal{V}$-injective objects of $\mathbf{Cl_0}$ coincide with the $\mathcal{S}$-saturated or equivalently with the absolutely $\mathcal{S}$-closed $T_0$ objects.

These properties have also been considered by Diers [7] in the setting of $T$-sets and the objects fulfilling the equivalent conditions were called algebraic $T$-sets. Our example also fits in that context.

## 4. Internal characterizations via complete objects

The results displayed so far in paragraph 3 are quite similar to the well known topological situation on $\mathcal{V}$-injective objects in $\mathbf{Top_0}$. In that setting these objects can be internally characterized as $T_0$ topological spaces for which every nonempty irreducible closed set is the closure of a point, i.e. as sober spaces [14, 6].

In this paragraph we give an internal characterization of the $\mathcal{V}$-injective objects in $\mathbf{Cl_0}$. This description for $\mathbf{Cl_0}$, when applied to $\mathbf{Top_0}$ will turn out to deal with a notion much stronger than sobriety.

**Definition 4.1.** Let $X$ be a closure space. For $\mathcal{A} \subset \mathcal{P}(X)$ we write

$$\operatorname{stack} \mathcal{A} = \{B \subset X \mid \exists A \in \mathcal{A} : A \subset B\}$$

and $\mathcal{A}$ is said to be a *stack* if $\mathcal{A} = \operatorname{stack} \mathcal{A}$. A nonempty stack is said to be *open based* if $\mathcal{A} = \operatorname{stack} \{G \in \mathcal{A} \mid G \text{ open}\}$. A proper open based stack $\mathcal{A}$ is said to be *fundamental* if $\mathcal{A}$ contains a member of every open cover of every element of $\mathcal{A}$, i.e. whenever $A \in \mathcal{A}$ and $\mathcal{G}$ is an open cover of $A$ then $\exists\, G \in \mathcal{G} : G \in \mathcal{A}$. More briefly a fundamental nonempty open based stack is called an *$\mathcal{O}$-stack*.

As an easy example we note that in every closure space the neighborhood collection $\mathcal{N}(x)$ of a point $x$ is an $\mathcal{O}$-stack.

**Proposition 4.2.** *On a closure space $X$ and for $\mathcal{A} \subset \mathcal{P}(X)$ we have: $\mathcal{A}$ is an $\mathcal{O}$-stack if and only if there exists a (closed) nonempty set $F \subset X$ such that $\mathcal{A} = \text{stack}\{G \subset X \mid G \text{ open},\ G \cap F \neq \varnothing\}$.*

*Proof.* For $F$ nonempty it is clear that

$$\begin{aligned}\mathcal{A} &= \text{stack}\,\{G \mid G \text{ open},\ G \cap F \neq \varnothing\} \\ &= \text{stack}\,\{G \mid G \text{ open},\ G \cap \operatorname{cl} F \neq \varnothing\}\end{aligned}$$

is an $\mathcal{O}$-stack.

Conversely let $\mathcal{A}$ be an $\mathcal{O}$-stack. Let $F = \{x \in X \mid \mathcal{N}(x) \subset \mathcal{A}\}$. $F$ clearly is nonempty since otherwise there would exist an open cover of $X$ of which all members are not in $\mathcal{A}$. If $G$ is open and $G \cap F \neq \varnothing$ then $\mathcal{N}(x) \subset \mathcal{A}$ for some point $x \in G$. So $G \in \mathcal{A}$. On the other hand, if $G$ is open and belongs to $\mathcal{A}$ then $G$ has to intersect $F$. If not, there would exist an open cover of $G$ of which all members are not in $\mathcal{A}$. So finally we can conclude that

$$\mathcal{A} = \text{stack}\,\{G \mid G \text{ open},\ G \cap F \neq \varnothing\}.$$

Remark that the set $F = \{x \in X \mid \mathcal{N}(x) \subset \mathcal{A}\}$ is in fact closed. □

**Definition 4.3.** A $T_0$ closure space $X$ is called *complete* if every $\mathcal{O}$-stack is a neighborhood collection $\mathcal{N}(x)$ for some (unique) point $x \in X$.

The uniqueness of the point follows from the $T_0$ condition. In view of Proposition 4.2 we get the following equivalent description.

**Proposition 4.4.** *A $T_0$ closure space $X$ is complete if and only if every nonempty closed set is the closure of a (unique) point.*

*Proof.* If $F$ is closed and nonempty then there is a point $x \in X$ such that $\text{stack}\,\{G \mid G \text{ open},\ G \cap F \neq \varnothing\} = \mathcal{N}(x)$. Then clearly $F = \operatorname{cl}\{x\}$.

Conversely if $\mathcal{A}$ is an $\mathcal{O}$-stack, as in the proof of Proposition 4.2, let $F$ be the nonempty closed set

$$F = \{x \in X \mid \mathcal{N}(x) \subset \mathcal{A}\}.$$

Now $F = \operatorname{cl}\{x\}$ implies $\mathcal{A} = \mathcal{N}(x)$. □

Let $\mathbf{CCl_0}$ be the full subconstruct of $\mathbf{Cl_0}$ consisting of the complete objects.

**Proposition 4.5.** *Complete $T_0$ objects are absolutely $\mathcal{S}$-closed.*

*Proof.* Let $X$ be a complete $T_0$ space and let $f : X \to Y$ be an embedding in $\mathbf{Cl_0}$. We prove that $f(X)$ is $b$-closed in $Y$. Let $a \in Y \setminus f(X)$. Either $\operatorname{cl}_Y\{a\} \cap f(X) = \varnothing$ or $\operatorname{cl}_Y\{a\} \cap f(X) = \operatorname{cl}_{f(X)}\{f(z)\}$ for some $z \in X$. In the latter case, let $U = Y \setminus \operatorname{cl}_Y\{f(z)\}$, then by the $T_0$ condition on $Y$ we have $a \in U$. Moreover $U \cap \operatorname{cl}_Y\{a\} \cap f(X) = \varnothing$. So in both cases we can conclude that $a \notin \operatorname{cl}_b^Y f(X)$. □

**Proposition 4.6.** $\mathbf{CCl_0}$ *is $\mathcal{V}$-reflective in* $\mathbf{Cl_0}$.

*Proof.* First we construct the reflector $R : \mathbf{Cl_0} \to \mathbf{CCl_0}$. Let $X$ be a $T_0$ closure space and let $\hat{X} = \{\mathcal{A} \mid \mathcal{A} \text{ is a } \mathcal{O}\text{-stack on } X\}$. On $\hat{X}$ we define a closure space as follows. For $G \subset X$ open let

$$\hat{G} = \{\mathcal{A}\ \mathcal{O}\text{-stack} \mid G \in \mathcal{A}\}$$

then $\{\hat{G} \mid G \subset X \text{ open}\}$ defines a $T_0$ closure structure on $\hat{X}$.

If $\mathcal{A}$ is an $\mathcal{O}$-stack on $X$ then

$$\hat{\mathcal{A}} = \text{stack}\,\{\hat{G} \mid G \subset X,\ G \text{ open},\ G \in \mathcal{A}\}$$

is its neighborhood collection in $\hat{X}$. If $\Psi$ is an $\mathcal{O}$-stack on $\hat{X}$ then

$$\check{\Psi} = \text{stack}\,\{G \mid G \subset X,\ G \text{ open},\ \hat{G} \in \Psi\}$$

is an $\mathcal{O}$-stack on $X$. It follows that $\hat{X}$ is a complete $T_0$ closure space.

Let $r_X : X \to \hat{X}$ be the natural injective map sending $x \in X$ to $\mathcal{N}(x) \in \hat{X}$. Clearly for $G \subset X$ open, we have $r_X^{-1}(\hat{G}) = G$ and hence $r_X$ is an embedding (in $\mathbf{Cl_0}$). This embedding is $b$-dense since for an $\mathcal{O}$-stack $\mathcal{A}$ and $G \in \mathcal{A}$ there exists $x \in G$ such that $\mathcal{N}(x) \subset \mathcal{A}$ and therefore

$$\hat{G} \cap \text{cl}_{\hat{X}}\,\{\mathcal{A}\}\ \cap r_X(X) \neq \varnothing.$$

Now let $B$ be a complete $T_0$ closure space and $f : X \to B$ a continuous map. By Proposition 3.4 *Inj* $\mathcal{V}$ is $\mathcal{V}$-reflective, so that the reflection map, say $s_B$, belongs to $\mathcal{V}$. Hence $s_B$ is a $b$-dense embedding and by Proposition 4.5 $s_B$ is also $b$-closed, therefore it is an isomorphism. Thus $B$ is $\mathcal{V}$-injective. So there is an extension $f'$ of $f$ along $r_X : X \to \hat{X}$. Since $r_X$ is an epimorphism in $\mathbf{Cl_0}$ this extension moreover is unique. □

**Corollary 4.7.** $\mathbf{CCl_0}$ *is* $\mathcal{V}$ *firmly reflective in* $\mathbf{Cl_0}$ *and therefore coincides with the class of all* $\mathcal{V}$*-injective* $T_0$ *objects.*

**Remark 4.8.** The previous conclusion combined with the characterization of $\mathbf{CCl_0}$ given in Proposition 4.4 and the remarks at the end of paragraph 3, imply the result stated in example 9.7 in [7] that algebraic $T_0$ closure spaces are those for which every nonempty closed set is the closure of a point.

## 5. An external characterization via the natural correspondence with complete lattices

In the topological counterpart on complete $T_0$ objects the duality between sober topological spaces and spatial frames leads to an external characterization of 'completeness'. In this paragraph we base our external characterization on the correspondence between closure spaces and complete lattices.

Let $\mathbf{CLat}_{\vee,\mathbf{1}}$ be the category whose objects are complete lattices and whose morphisms are maps preserving arbitrary joins and the top element. The dual category will be denoted $\mathbf{CLat}^{\mathbf{op}}_{\vee,\mathbf{1}}$.

To every closure space $X$ we associate the lattice $O(X)$ of its open subsets. With $f : X \to Y$ we associate the map

$$O(Y) \to O(X) : G \mapsto f^{-1}(G).$$

This correspondence defines a functor

$$\Omega^c : \mathbf{Cl} \to \mathbf{CLat}^{\mathbf{op}}_{\vee,\mathbf{1}}.$$

In order to define an adjoint for $\Omega^c$, let $L$ be a complete lattice. A point of $L$ is a surjective $\mathbf{CLat}_{\vee,\mathbf{1}}$ morphism $L \to 2$ where $2 = \{0, 1\}$ is the two point complete lattice. In the sequel we'll use $\mathrm{pts}(L)$ to denote the set of points of $L$ and for $u \in L$ we'll write $\Sigma_u = \{\xi \in \mathrm{pts}(L) \mid \xi(u) = 1\}$. Observe that in contrast to the topological and frame counterpart, for objects $u$ and $v$ in $L$, we always have

$$u \neq v \Rightarrow \Sigma_u \neq \Sigma_v.$$

With this notation we can describe the functor

$$\Sigma^c : \mathbf{CLat}^{\mathbf{op}}_{\vee,\mathbf{1}} \to \mathbf{Cl}$$

sending a lattice $L$ to the set $\mathrm{pts}(L)$, endowed with the closure structure $\{\Sigma_u \mid u \in L\}$, and $f : M \to L$ to $\Sigma^c M \to \Sigma^c L : \xi \mapsto \xi \circ f$.

**Proposition 5.1.** *For a complete lattice $L$, $\Sigma^c L$ is a complete $T_0$ closure space.*

*Proof.* Consider two distinct $\xi_1, \xi_2 \in \mathrm{pts}(L)$ of $\Sigma^c L$. There exist a $u \in L$ such that $\xi_1(u) \neq \xi_2(u)$. Hence either $\xi_1 \in \Sigma_u$ and $\xi_2 \notin \Sigma_u$ or $\xi_1 \notin \Sigma_u$ and $\xi_2 \in \Sigma_u$. So $\Sigma^c$ is $T_0$.

To prove the completeness we choose an $\mathcal{O}$-stack $\mathcal{A}$ in $\Sigma^c L$ and consider $v = \bigvee\{u \in L \mid \Sigma_u \notin \mathcal{A}\}$. Next we define the point $\xi : L \to 2 : u \mapsto \begin{cases} 1 & u \not\leq v \\ 0 & u \leq v \end{cases}$. We have that $\xi(v) = 0$, hence $(\Sigma_u \notin \mathcal{A} \Rightarrow \xi(u) = 0)$. Conversely, if $\Sigma_u \in \mathcal{A}$ then $u \not\leq v$ since $\mathcal{A}$ is an $\mathcal{O}$-stack. Thus $(\Sigma_u \in \mathcal{A} \Rightarrow \xi(u) = 1)$. Finally $\Sigma_u \in \mathcal{A} \iff \xi(u) = 1$. Therefore $\mathcal{A} = \mathcal{N}(\xi)$ in $\Sigma^c L$. □

**Proposition 5.2.** *The restrictions*

$$\Omega^c : \mathbf{CCl_0} \to \mathbf{CLat}^{\mathbf{op}}_{\vee,\mathbf{1}}$$

$$\Sigma^c : \mathbf{CLat}^{\mathbf{op}}_{\vee,\mathbf{1}} \to \mathbf{CCl_0}$$

*define an equivalence of categories.*

*Proof.* The proof consists of three parts.

(1) Let $L$ be a complete lattice then $L \simeq \Omega^c\Sigma^c L$. Choose the isomorphism as follows:

$$\epsilon_L : \Omega^c\Sigma^c L \to L : \Sigma_u \mapsto u$$

This is a well defined $\mathbf{CLat}^{\mathbf{op}}_{\vee,\mathbf{1}}$-isomorphism.

(2) Let $X$ be a complete $T_0$ closure space then $X \simeq \Sigma^c\Omega^c X$. Define the following map

$$\eta_X : X \to \mathrm{pts}(\Omega^c X) : x \mapsto \xi_x$$

where $\xi_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$, for all open sets $A$.

$\eta_X$ is injective since by the $T_0$ property we have for $x \neq y$ an open subset $A$ such that $\xi_x(A) \neq \xi_y(A)$. Therefore $\xi_x \neq \xi_y$.

To show that $\eta_X$ is surjective we choose a point $\xi$, and consider stack $\xi^{-1}(1)$. This is obviously a stack with an open basis, so that if

$\bigcup_{i\in I} A_i \in \operatorname{stack}\xi^{-1}(1)$ then there exists $B \in \xi^{-1}(1) : B \subset \bigcup_{i\in I} A_i$. So we get $1 = \xi(B) \leq \xi(\bigcup_{i\in I} A_i) = \bigvee_{i\in I} \xi(A_i)$. Hence there exists an $i \in I$ with $A_i \in \xi^{-1}(1)$ and so $\operatorname{stack}\xi^{-1}(1)$ is an $\mathcal{O}$-stack. Therefore there is an $x \in X$ such that $\mathcal{N}(x) = \operatorname{stack}\xi^{-1}(1)$. We now have

$$\xi_x(A) = 1 \Leftrightarrow A \in \mathcal{N}(x) \Leftrightarrow \exists B \in \xi^{-1}(1) : B \subset A \Leftrightarrow \xi(A) = 1.$$

Hence $\xi = \xi_x$.

Moreover $\eta_X$ and $\eta_X^{-1}$ are both continuous. This follows from

$$\begin{aligned}&\eta_X^{-1}(\Sigma_A) = \{x \in X \mid \xi_x(A) = 1\} = A,\\&\eta_X(A) = \{\xi_x \mid x \in A\} = \{\xi_x \mid \xi_x(A) = 1\} = \Sigma_A,\end{aligned}$$

where $A$ is open.

(3) To see the naturality of $\eta$, consider continuous $f : X \to Y$ where $X$ and $Y$ are complete $T_0$ closure spaces. We have the following compositions: $(\Sigma^c(\Omega^c(f))\circ\eta_X)(x) = (\Sigma^c(f^{-1}))(\xi_x) = \xi_x \circ f^{-1} = \xi_{f(x)}$ and $\eta_Y \circ f(x) = \xi_{f(x)}$. Hence $\eta = (\eta_X)_{X\in|\mathbf{CCl_0}|}$ is a natural isomorphism $\eta : 1_{\mathbf{CCl_0}} \simeq \Sigma^c\Omega^c$.

The naturality of $\epsilon$ follows since if $h : L \to M$ is a $\mathbf{CLat}_{\vee,\mathbf{1}}$-morphism, we have the compositions

$$\begin{aligned}(\epsilon_M \circ \Omega^c(\Sigma^c(h)))(\Sigma_u) &= \epsilon_M((\Sigma^c(h))^{-1}(\Sigma_u))\\&= \epsilon_M(\{\xi \in \operatorname{pts}(M) \mid \xi \circ h \in \Sigma_u\})\\&= \epsilon_M(\{\xi \in \operatorname{pts}(M) \mid \xi \circ h(u) = 1\})\\&= \epsilon_M(\Sigma_{h(u)}) = h(u)\end{aligned}$$

and $h \circ \epsilon_L(\Sigma_u) = h(u)$. Therefore $\epsilon = (\epsilon_L)_{L\in|\mathbf{CLat}_{\vee,\mathbf{1}}|}$ is a natural isomorphism $\epsilon : \Omega^c\Sigma^c \simeq 1_{\mathbf{CLat}_{\vee,\mathbf{1}}}$.

Hence we have proven the above equivalence.

□

## References


[1] J. Adámek, H. Herrlich and G. Strecker, *Abstract and Concrete Categories* (Wiley, New York, 1990).

[2] D. Aerts, *Foundations of quantum physics: a general realistic and operational approach*, Internat. J. Theoret. Phys. **38 (1)** (1999), 289–358.

[3] G. Aumann, *Kontaktrelationen*, Bayer. Akad. Wiss. Math.-Nat. Kl. Sitzungsber (1970), 67–77.

[4] G. Birkhoff, *Lattice Theory* (American Mathematical Society, Providence, Rhode Island, 1940).

[5] G. C. L. Brümmer and E. Giuli, *A categorical concept of completion of objects*, Comment. Math. Univ. Carolin. **33 (1)** (1992), 131–147.

[6] G. C. L. Brümmer, E. Giuli and H. Herrlich, *Epireflections which are completions*, Cahiers Topologie Géom. Diff. Catég. **33 (1)** (1992), 71–93.

[7] Y. Diers, *Categories of algebraic sets*, Appl. Categ. Structures **4 (2-3)** (1996), 329–341.

[8] D. Dikranjan and E. Giuli, *Closure operators. I.* Topology Appl. **27** (1987), 129–143.

[9] D. Dikranjan, E. Giuli and A. Tozzi, *Topological categories and closure operators*, Quaestiones Math. **11 (3)** (1988), 323–337.

[10] M. Erné, *Lattice representations for categories of closure spaces*, Categorical Topology (Heldermann Verlag, Berlin 1983), 197–222.
[11] B. Ganter and R. Wille, *Formal Concept Analysis* (Springer Verlag, Berlin, 1998).
[12] E. Giuli, *Zariski closure, completeness and compactness*, Mathematik - Arbeitspapiere **54** (2000), Universität Bremen, Proceedings of Catmat 2000, H. Herrlich and H. Porst editors, 207–216.
[13] H. Herrlich and G. Strecker, *Category theory*, Sigma Series in Pure Mathematics (Heldermann Verlag, Berlin, 1979).
[14] R.-E. Hoffmann, *Topological functors admitting generalized Cauchy-completions*, in Categorical Topology, Lecture Notes in Math. **540** (1976), 286–344.
[15] Th. Marny, *On epireflective subcategories of topological categories*, Gen. Topology Appl. **10 (2)** (1979), 175–181.
[16] D. J. Moore, *Categories of representations of physical systems*, Helv. Phys. Acta **68** (1995), 658–678.
[17] L. D. Nel, *Initially structured categories and Cartesian closedness*, Canad. J. Math. **27 (6)** (1975), 1361–1377.
[18] G. Preuss, *Theory of Topological Structures* (D. Reidel Publishing Company, Dordrecht, 1988).
[19] A. Van der Voorde, *Separation axioms in extensiontheory for closure spaces and their relevance to state property systems*, PhD Thesis Vrije Universiteit Brussel, July 2001.
[20] B. Van Steirteghem, $T_0$ *separation in axiomatic quantum mechanics*, Internat. J. Theoret. Phys. **39 (3)** (2000), 955–962.

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