Applied General Topology
© Universidad Politécnica de Valencia
Volume 5, No. 1, 2004
pp. 91- 96

Homeomorphisms of R and the Davey Space

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Abstract. Up to homeomorphism, there are 9 topologies on a three point set $\{a, b, c\}$ [4]. Among the resulting topological spaces we have the so called Davey space, where the only non-trivial open set is, let us say, $\{a\}$. This is an interesting topological space to the extent that every topological space can be embedded in a product of Davey spaces [3]. In this note we will consider the problem of obtaining the Davey space as a quotient R/G, where G is a suitable homeomorphism group. The present work can be regarded as a follow-up to some previous work done by one of the authors and Bernd Wegner [1].

2000 AMS Classification: 54F65.

Keywords: Davey space, homeomorphism group, Cantor set.

1. R/G as the Davey space -necessary conditions

We will take the topological space $(\{a, b, c\}, \tau)$, with $\tau = \{\varnothing, \{a\}, \{a, b, c\}\}$, as a model for the Davey space and the real line will be denoted by R.

Our purpose is to obtain a group G of homeomorphisms of R whose natural action on R gives rise to the Davey space and we start by establishing a number of observations which guided our quest.

Below we assume that the homeomorphism group G is such that R/G is the Davey space and π will stand for the projection from R to R/G.

Proposition 1.1. *G is not finite.*

Proof. If G were finite then, for instance, $\pi^{-1}(b)$ would be finite and, consequently, $\{a, c\}$ would be open.

□

Proposition 1.2. *$\pi^{-1}(a)$ is bounded neither above nor below.*

Proof. Assume that $\pi^{-1}(a)$ is bounded above and let x be its supremum. Then $\pi((x, +\infty))$ is open in R/G and, consequently must contain a.

□

Proposition 1.3. *$\pi^{-1}(\{b, c\})$ is bounded neither above nor below.*

Proof. Assume that x is the supremum of $\pi^{-1}(\{b, c\})$. Since this set is closed in R, x belongs to it. Let us suppose that $\pi(x) = b$. As we will see below it then follows that $|\pi^{-1}(b)| \leq 2$ which, as remarked above, is impossible.

Let y, z be points in $\pi^{-1}(b)$ with $y < z < x$. There is a homeomorphism f in G such that $f(z) = x$. If f were increasing then $f(x) > x$. Therefore f must be decreasing and, since $y < z$, $f(y) > x$ which, again, is impossible.

□

Proposition 1.4. *$\pi^{-1}(b)$, $\pi^{-1}(c)$ are bounded neither above nor below.*

Proof. Assume that $\pi^{-1}(b)$ is bounded above and let x be its supremum. Then $\pi((x, +\infty))$ must be $\{a\}$ and x is an upper bound for $\pi^{-1}(c)$. Consequently $\pi^{-1}(\{b, c\})$ would be bounded above.

□

We are now in a position which allows us to conclude

Theorem 1.5. *The action of G is not free.*

Proof. Let $\pi^{-1}(a) = \bigcup_{i \in I} C_i$, where the C_i's are the connected components. From above it follows that, for each i, $C_i = (a_i, b_i)$.

Fix an i and choose x, y distinct in C_i. There is an $f \in G$ such that $f(x) = y$. Since f maps $[a_i, b_i]$ into itself, it must have a fixed point. □

It is also clear that $\pi^{-1}(\{b, c\})$ is totally disconnected and that every point in it is a limit point of that set.

Proposition 1.6. *$\pi^{-1}(\{b, c\})$ is uncountable.*

Proof. Write $\pi^{-1}(a) = \bigcup_{i \in I} C_i$ and choose x, y in different components, with $x < y$. Then $\pi^{-1}(\{b, c\}) \bigcap [x, y]$ is a compact, Hausdorff space having all its elements as limits points. Therefore it is uncountable [4]. □

2. An example

This section is devoted to the construction of an example of a group G such that R/G is the Davey space. Since they are homeomorphic spaces we will use the open interval $(0, 1)$ instead of R.

Let $\mathbf{C}$ denote the intersection of the Cantor set [2], [5] with $(0, 1)$ and consider the partition $(0, 1) = A \cup B \cup C$, where $A = (0, 1) \setminus \mathbf{C}$ is a union of open intervals, the "middle thirds", B is the set of end-points of the open intervals in A and $C = \mathbf{C} \setminus B$.

The Cantor set can be described in terms of ternary expansions. We then have, for $x \in (0, 1)$, that

$x \in A$ if and only if there is $n \in N$ such that $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$, where, for $i < n$, $x_i = 0$ or $2, x_n = 1$ and $0 < \sum_{i=n+1}^{\infty} \frac{x_i}{3^i} < \frac{1}{3^n}$,

$x \in B$ if and only if there is $n \in N$ such that $x = \sum_{i=1}^{n} \frac{x_i}{3^i}$, where, for $i < n$, $x_i = 0$ or 2, $x_n = 1$ or 2.

$x \in C$ if and only if $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$, with $x_i = 0$ or 2, and there are arbitrarily large i and j for which $x_i = 0$, $x_j = 2$.

Proposition 2.1. *The quotient topological space originated by the partition* $(0,1) = A \cup B \cup C$ *is the Davey space.*

Proof. Let $X = \{a, b, c\}$ be the quotient space obtained by identifying A, B, C to points a, b, c, respectively.

Since A is open in $(0,1)$ it follows that $\{a\}$ is open in X.

Let now $x \in B$ and suppose that $x = \sum_{i=1}^{n-1} \frac{x_i}{3^i} + \frac{1}{3^n}$, where $x_i = 0$ or 2. For $k \geq n+1$, define $y_k \in C$ by $y_k = \sum_{i=1}^{n-1} \frac{x_i}{3^i} + \sum_{i=n+1}^{k} \frac{2}{3^i} + \sum_{j=1}^{\infty} \frac{2}{3^{k+2j}}$. Then the sequence (y_k) converges to $\sum_{i=1}^{n-1} \frac{x_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{2}{3^i}$, which is x.

Similarly if $x = \sum_{i=1}^{n-1} \frac{x_i}{3^i} + \frac{2}{3^n}$, where $x_i = 0$ or 2, for $k \geq n+1$, define $y_k = x + \sum_{j=1}^{\infty} \frac{2}{3^{k+2j}}$. This sequence also converges to x.

Thus every element of B belongs to the closure of C and, since it is an end-point of an open interval in A, it also lies in the closure of A. Hence every open set in X containing b also contains a and c and the only such open set is X itself.

Next consider $x \in C$, say $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$, where $x_i = 0$ or 2. There exists an arbitrarily large i for which $x_i = 2$. Let x_l be the first nonzero term and, for $k \geq l$, define $y_k \in B$ by $y_k = \sum_{i=1}^{k} \frac{x_i}{3^i}$. The sequence (y_k) converges to x.

Thus x lies in the closure of B and, as each y_k is in the closure of A, it also lies in the closure of A. So every open set in X containing c also contains a and b and the only such open set is X itself.

Therefore X is the Davey space. □

Let $G = \{h : (0,1) \to (0,1) \mid h$ is a homeomorphism and $h(A) = A\}$. If $h \in G$ then h takes an open interval in A to an open interval in A and, consequently, the end-points to end-points. So $h(B) = B$ and then $h(C) = C$. If we prove that G acts transitively on A, B and C we may conclude that those subsets are the orbits of the natural action of G on $(0,1)$ and, by Proposition 2.1, $(0,1)/G$ is the Davey space.

Proposition 2.2. *G acts transitively on A.*

Proof. We start by observing that, given any open interval (α, β) in $(0,1)$ and $x, y \in (\alpha, \beta)$, there exists a homeomorphism $h : (0,1) \to (0,1)$ such that $h((\alpha,\beta)) = (\alpha,\beta), h(x) = y$ and $h \mid (0,1) \setminus (\alpha,\beta)$ is the identity function.

To prove transitivity on A it is therefore enough to show that, for any open interval (α,β) in A, with $\alpha, \beta \in B$, there is $h \in G$ such that $h((\alpha,\beta)) = (\frac{1}{3}, \frac{2}{3})$.

Assume $\alpha = \frac{2}{3^{i_1}} + \ldots + \frac{2}{3^{i_k}} + \frac{1}{3^n}$, $1 \leq i_1 < i_2 < \ldots < i_k < n$. So $\beta = \alpha + \frac{1}{3^n}$.

Let $j_1, \ldots, j_l$ be such that $1 \leq j_1 < \ldots j_l < n$, $\{j_1, \ldots, j_l\} \cup \{i_1, i_2, \ldots, i_k\} = \{1, 2, \ldots, n-1\}$, $l + k = n - 1$. Hence
$\alpha + \dfrac{2}{3^{j_1}} + \ldots + \dfrac{2}{3^{j_l}} + \dfrac{1}{3^n} = \displaystyle\sum_{i=1}^{n} \dfrac{2}{3^i} = 1 - \dfrac{1}{3^n}$ and, in the construction of h, we will use $1 - (\dfrac{2}{3^{j_1}} + \ldots + \dfrac{2}{3^{j_l}} + \dfrac{1}{3^n} + \dfrac{s}{3^n}) = \alpha + \dfrac{t}{3^n}$, where $s = 1 - t, t \in [0,1]$.

We define $h : (0,1) \to (0,1)$ as follows:

$(0, \frac{2}{3^{i_1}}]$ is mapped to $(0, \frac{2}{3^2}]$ by $h(\frac{2t}{3^{i_1}}) = \frac{2t}{3^2}$, with $t \in (0,1]$,

for $r = 2, \ldots, k$, $[\frac{2}{3^{i_1}} + \ldots + \frac{2}{3^{i_{r-1}}}, \ \frac{2}{3^{i_1}} + \ldots + \frac{2}{3^{i_r}}]$ is mapped to $[\frac{2}{3^2} + \ldots + \frac{2}{3^r}, \ \frac{2}{3^2} + \ldots + \frac{2}{3^{r+1}}]$ by $h(\frac{2}{3^{i_1}} + \ldots + \frac{2}{3^{i_{r-1}}} + \frac{2t}{3^{i_r}}) = \frac{2}{3^2} + \ldots + \frac{2}{3^r} + \frac{2t}{3^{r+1}}$, with $t \in [0,1]$,

$[\frac{2}{3^{i_1}} + \ldots + \frac{2}{3^{i_k}}, \alpha]$ is mapped to $[\frac{2}{3^2} + \ldots + \frac{2}{3^{k+1}}, \frac{1}{3}]$ by $h(\frac{2}{3^{i_1}} + \ldots + \frac{2}{3^{i_k}} + \frac{t}{3^n}) = \frac{2}{3^2} + \ldots + \frac{2}{3^{k+1}} + \frac{t}{3^{k+1}}$, with $t \in [0,1]$,

$[\alpha, \alpha + \frac{1}{3^n}]$ is mapped to $[\frac{1}{3}, \frac{2}{3}]$ by $h(\alpha + \frac{t}{3^n}) = \frac{1}{3} + \frac{t}{3}$, with $t \in [0,1]$,

$[\alpha + \frac{1}{3^n}, \alpha + \frac{2}{3^n}]$ is mapped to $[\frac{2}{3}, \frac{2}{3} + \frac{1}{3^{l+1}}]$ by $h(1 - (\frac{2}{3^{j_1}} + \ldots + \frac{2}{3^{j_l}} + \frac{s}{3^n})) = 1 - (\frac{2}{3^2} + \ldots + \frac{2}{3^{l+1}} + \frac{s}{3^{l+1}})$, with $s \in [0,1]$,

for $r = 2, \ldots, l$, $[1 - (\frac{2}{3^{j_1}} + \ldots + \frac{2}{3^{j_r}}), 1 - (\frac{2}{3^{j_1}} + \ldots + \frac{2}{3^{j_{r-1}}})]$ is mapped to $[1 - (\frac{2}{3^2} + \ldots + \frac{2}{3^{r+1}}), 1 - (\frac{2}{3^2} + \ldots + \frac{2}{3^r})]$ by $h(1 - (\frac{2}{3^{j_1}} + \ldots + \frac{2}{3^{j_{r-1}}} + \frac{2s}{3^{j_r}})) = 1 - (\frac{2}{3^2} + \ldots + \frac{2}{3^r} + \frac{2s}{3^{r+1}})$, with $s \in [0,1]$,

$[1 - \frac{2}{3^{j_1}}, 1)$ is mapped to $[1 - \frac{2}{3^2}, 1)$ by $h(1 - \frac{2s}{3^{j_1}}) = 1 - \frac{2s}{3^2}$, with $s \in (0,1]$.

We have then a homeomorphism $h : (0,1) \to (0,1)$ such that $h((\alpha,\beta)) = (\frac{1}{3}, \frac{2}{3})$. On each interval of its definition, h is of the form $h(x) = \lambda x + \mu$, for some $\lambda, \mu \in R$. Hence it takes middle thirds in $(0, \frac{1}{3^{i_1}})$ to middle terms in

$(0, \frac{1}{3^2})$, middle thirds in $(\frac{2}{3^{i_1}} + \ldots + \frac{2}{3^{i_{r-1}}},\ \frac{2}{3^{i_1}} + \ldots + \frac{2}{3^{i_{r-1}}} + \frac{1}{3^r})$ to middle thirds in $(\frac{2}{3^2} + \ldots + \frac{2}{3^r},\ \frac{2}{3^2} + \ldots + \frac{2}{3^r} + \frac{1}{3^{r+1}})$, for $r = 2, \ldots, k$, and so on for the other intervals. So $h(A) = A$ and $h \in G$ as required.

□

Proposition 2.3. *G acts transitively on B.*

Proof. The homeomorphism h constructed above maps $[\alpha, \beta]$ to $[\frac{1}{3}, \frac{2}{3}]$, with $h(\alpha) = \frac{1}{3}, h(\beta) = \frac{2}{3}$ and every element in B is such an α or β.

Composing h with the reflection of $(0,1)$ that sends x to $1 - x$ gives an element of G that takes β to $\frac{1}{3}$. Hence, for $\alpha \in B$, there exists $g \in G$ with $g(\alpha) = \frac{1}{3}$. Therefore G acts transitively on B.

□

Proposition 2.4. *G acts transitively on C.*

Proof. Since $\frac{1}{4} \in C$, it suffices to show that, for $\gamma \in C$, there is $h \in G$ such that $h(\gamma) = \displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{2n}} = \frac{1}{4}$.

Let $\gamma = \displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{i_n}}$, $I = \{i_1, i_2, \ldots\}$, $J = N \setminus I = \{j_1, j_2, \ldots\}$. Define $h : (0,1) \to (0,1)$ as follows:

$(0, \frac{2}{3^{i_1}}]$ is mapped to $(0, \frac{2}{3^2}]$ by $h(\frac{2t}{3^{i_1}}) = \frac{2t}{3^2}$, with $t \in (0,1]$,

for $n = 2, \ldots$, $[\frac{2}{3^{i_1}} + \ldots + \frac{2}{3^{i_{n-1}}},\ \frac{2}{3^{i_1}} + \ldots + \frac{2}{3^{i_n}}]$ is mapped to $[\frac{2}{3^2} + \frac{2}{3^4} + \ldots + \frac{2}{3^{2n-2}},\ \frac{2}{3^2} + \ldots + \frac{2}{3^{2n}}]$ by $h(\frac{2}{3^{i_1}} + \ldots + \frac{2}{3^{i_{n-1}}} + \frac{2t}{3^{i_n}}) = \frac{2}{3^2} + \frac{2}{3^4} + \ldots + \frac{2}{3^{2n-2}} + \frac{2t}{3^{2n}}$, with $t \in [0,1]$,

$$h\Big(\sum_{n=1}^{\infty} \frac{2}{3^{i_n}}\Big) = \sum_{n=1}^{\infty} \frac{2}{3^{2n}},$$

for $n = 2, \ldots$, $[1 - (\frac{2}{3^{j_1}} + \ldots + \frac{2}{3^{j_n}}), 1 - (\frac{2}{3^{j_1}} + \ldots + \frac{2}{3^{j_{n-1}}})]$ is mapped to $[1 - (\frac{2}{3} + \frac{2}{3^3} + \ldots + \frac{2}{3^{2n-1}}), 1 - (\frac{2}{3} + \frac{2}{3^3} + \ldots + \frac{2}{3^{2n-3}})]$ by $h(1 - (\frac{2}{3^{j_1}} + \ldots + \frac{2}{3^{j_{n-1}}} + \frac{2s}{3^{j_n}})) = 1 - (\frac{2}{3} + \frac{2}{3^3} + \ldots + \frac{2}{3^{2n-3}} + \frac{2s}{3^{2n-1}})$, with $s \in [0,1]$,

$[1 - \frac{2}{3^{j_1}}, 1)$ is mapped to $[\frac{1}{3}, 1)$ by $h(1 - \frac{2s}{3^{j_1}}) = 1 - \frac{2s}{3}$, with $s \in (0,1]$.

We have therefore defined an $h \in G$ with $h(\gamma) = \frac{1}{4}$ as required.

□

We can now conclude with our main result.

Theorem 2.5. *There is a group G of homeomorphisms of R such that the quotient space R/G is the Davey space.*

Acknowledgements. The authors are very grateful to Alan West for discussions on the topic of this paper which have led to a complete proof of Theorem 2.5.

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Received December 2002
Accepted April 2003

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