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Some problems on selections for hyperspace topologies

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Abstract. The theory of hyperspaces has attracted the attention of many mathematicians who have found a large variety of its applications during the last decades. The theory has taken also its natural course and has yielded lots of problems which, besides their independent inner beauty, provide ties with numerous classical fields of mathematics. In the present note we are concerned with some open problems about selections for hyperspace topologies which have been in the scope of our recent research interests.

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1. The concept of a τ-continuous selection

Let $(X, \mathcal{T})$ be a T_1-space, where $\mathcal{T}$ is the topology of X, and let $\mathcal{F}(X, \mathcal{T})$ be the set of all non-empty closed (with respect to $\mathcal{T}$) subsets of X. Let us stress the reader's attention that $\mathcal{F}(X, \mathcal{T})$ is different for different topologies $\mathcal{T}$ on X, while X is always a subset of $\mathcal{F}(X, \mathcal{T})$ because we may identify each point $x \in X$ with the corresponding singleton $\{x\} \in \mathcal{F}(X, \mathcal{T})$.

Definition 1.1. *A topology τ on $\mathcal{F}(X, \mathcal{T})$ is called* admissible *(see* [18]*) if its restriction on the set of all singletons $\{\{x\} : x \in X\}$ of X coincides with the topology $\mathcal{T}$.*

In the light of Definition 1.1, we may look at $(\mathcal{F}(X, \mathcal{T}), \tau)$ as a *topological extension* of the topological space $(X, \mathcal{T})$ provided τ is an admissible topology. It should be mentioned that the concept of an admissible topology may refer also to some additional structures on X, see [18].

The second basic concept of this paper is related to a selection for a hyperspace topology. Let $\mathcal{D} \subset \mathcal{F}(X, \mathcal{T})$.

Definition 1.2. *A map $f : \mathcal{D} \to X$ is a* selection *for $\mathcal{D}$ if $f(S) \in S$ for every $S \in \mathcal{D}$.*

Definition 1.3. *If τ is a topology on $\mathcal{F}(X, \mathcal{T})$, then a map $f : \mathcal{D} \to X$ is a τ-*continuous *selection for $\mathcal{D}$ if it is a selection for $\mathcal{D}$ which is continuous with respect to the relative topology on $\mathcal{D}$ as a subspace of $(\mathcal{F}(X, \mathcal{T}), \tau)$.*

So far, one of the best known admissible topologies on $\mathcal{F}(X, \mathcal{T})$ is the Vietoris one $\tau_{V(\mathcal{T})}$. Let us recall that all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X, \mathcal{T}) : S \subset \bigcup \mathcal{V} \text{ and } S \cap V \neq \varnothing, \text{ whenever } V \in \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite subsets of $\mathcal{T}$, provide a base for the topology $\tau_{V(\mathcal{T})}$. Any selection has the following property with respect to the Vietoris topology, it appeared in several papers in an explicit or implicit way.

Proposition 1.4. *If $f : \mathcal{F}(X, \mathcal{T}) \to X$ is a selection for $\mathcal{F}(X, \mathcal{T})$, then f is $\tau_{V(\mathcal{T})}$-continuous at $\{x\}$ for every $x \in X$.*

2. Selections and orderability

In what follows, all spaces are assumed to be at least Hausdorff. For a space $(X, \mathcal{T})$ and $0 < n < \omega$, we let

$$\mathcal{F}_n(X) = \{S \subset X : 0 < |S| \leq n\}.$$

Note that $\mathcal{F}_1(X)$ is the set of all singletons of X, and always $\mathcal{F}_n(X) \subset \mathcal{F}(X, \mathcal{T})$.

Let τ be a topology on $\mathcal{F}(X, \mathcal{T})$, and let $\mathcal{S}e\ell_\tau(X, \mathcal{T})$ be the set of all *τ-continuous selections* for $\mathcal{F}(X, \mathcal{T})$. Also, let $\mathcal{S}e\ell_{(\tau,n)}(X, \mathcal{T})$, $n > 1$, be the set of all *τ-continuous selections* for $\mathcal{F}_n(X)$, and $\mathcal{S}e\ell_n(X)$ that of all selections (not necessarily τ-continuous) for $\mathcal{F}_n(X)$.

Any selection $f \in \mathcal{S}e\ell_2(X)$ naturally defines an *order-like* relation $\prec_f$ on X [18] by letting for $x \neq y$ that $x \prec_f y$ iff $f(\{x, y\}) = x$. However, in general, $\prec_f$ fails to be a linear order on X. Let us denote by $\mathcal{T}_f$ the topology generated by all possible "open" $\prec_f$-intervals. It is easy to observe that $\mathcal{T}_f$ is also a Hausdorff topology, see [16].

Theorem 2.1 ([18])**.** *Let $(X, \mathcal{T})$ be a space, and let $f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X, \mathcal{T})$. Then,*

(a) *$\mathcal{T}_f \subset \mathcal{T}$.*

If, in addition, $(X, \mathcal{T})$ is connected, then we also have that

(b) *"$\prec_f$" is a proper linear order on X,*
(c) *$(X, \mathcal{T}_f)$ is connected,*
(d) *$f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T}_f)},2)}(X, \mathcal{T}_f)$.*

Theorem 2.2 ([20])**.** *Let $(X, \mathcal{T})$ be a compact space, with $\mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X, \mathcal{T}) \neq \varnothing$. Then,*

(a) *$(X, \mathcal{T})$ is a linear ordered topological space (in particular,* $\operatorname{ind}(X, \mathcal{T}) \leq 1$*),*
(b) *$\mathcal{T}_f = \mathcal{T}$ for every $f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X, \mathcal{T})$,*
(c) *$\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T}) \neq \varnothing$.*

Here, $\operatorname{ind}(X, \mathcal{T})$ means the *small inductive dimension* of $(X, \mathcal{T})$.

In view of Theorem 2.1, it makes some sense to investigate the topology $\mathcal{T}_f$. For instance, the following simple observation was obtained in [16].

Proposition 2.3 ([16])**.** *Let $(X, \mathcal{T}_0)$ be a space, and let $f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T}_0)},2)}(X, \mathcal{T}_0)$. Then, $f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X, \mathcal{T})$ for every topology $\mathcal{T}$ on X which is finer than $\mathcal{T}_0$.*

Consider the natural partial order on all Hausdorff topologies on a set X defined by $\mathcal{T}_1 \ll \mathcal{T}_2$ provided $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, i.e. $\mathcal{T}_1 \subset \mathcal{T}_2$. Then, by Proposition 2.3, $f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T}_0)},2)}(X, \mathcal{T}_0)$ implies $f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X, \mathcal{T})$ for every Hausdorff topology $\mathcal{T}$ on X, with $\mathcal{T}_0 \ll \mathcal{T}$.

Thus, we have the following natural question about a possible $\ll$-minimal topology $\mathcal{T}$ on a set X such that a given selection $f \in \mathcal{S}e\ell_2(X)$ is $\tau_{V(\mathcal{T})}$-continuous. Namely,

Problem 2.4 ([16])**.** *Let X be a set, and let $f \in \mathcal{S}e\ell_2(X)$. Does there exist a topology $\mathcal{T}$ on X which is $\ll$-minimal with respect to the property "$f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X, \mathcal{T})$"?*

Related to this question, let us observe that, by Theorem 2.1, $f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X,\mathcal{T})$ implies $\mathcal{T}_f \ll \mathcal{T}$. So, $\mathcal{T}_f$ is a possible candidate for a $\ll$-minimal topology in that sense. However, we have the following recent example.

Example 2.5 ([16]). There exists a set X and $\sigma \in \mathcal{S}e\ell_2(X)$ such that σ is not $\tau_{V(\mathcal{T}_\sigma)}$-continuous.

By Theorem 2.2, if $(X,\mathcal{T})$ is a compact space, then $\mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X,\mathcal{T}) \neq \varnothing$ if and only if $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X,\mathcal{T}) \neq \varnothing$. On the other hand, $\mathcal{S}e\ell_{(\tau_{V(\mathcal{T}_e)},2)}(\mathbb{R},\mathcal{T}_e) \neq \varnothing$, while $\mathcal{S}e\ell_{\tau_{V(\mathcal{T}_e)}}(\mathbb{R},\mathcal{T}_e) = \varnothing$ (see [8]), where $\mathbb{R}$ are the real numbers and $\mathcal{T}_e$ is the usual Euclidean topology on $\mathbb{R}$. Thus, in view of Theorem 2.1, we get the following natural question.

Problem 2.6 ([16]). *Let $(X,\mathcal{T})$ be a connected space, and let $f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X,\mathcal{T})$. Is it true that $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X,\mathcal{T}) \neq \varnothing$ if and only if $\mathcal{S}e\ell_{\tau_{V(\mathcal{T}_f)}}(X,\mathcal{T}_f) \neq \varnothing$?*

In general, the answer is "No" which was provided by the following example.

Example 2.7 ([16]). There exists a separable, connected and metrizable space $(X,\mathcal{T})$ such that

(i) $\mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X,\mathcal{T}) \neq \varnothing$,
(ii) $\mathcal{S}e\ell_{\tau_{V(\mathcal{T}_f)}}(X,\mathcal{T}_f) \neq \varnothing$, for every $f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X,\mathcal{T})$,
(iii) $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X,\mathcal{T}) = \varnothing$.

In contrast to this, Theorem 2.1 implies that $\mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},n)}(X,\mathcal{T}) = \mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X,\mathcal{T})$, for every $n \geq 2$, provided $(X,\mathcal{T})$ is a connected space. On this base, we have also the following question.

Problem 2.8. *Does there exist a space $(X,\mathcal{T})$ such that $\mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X,\mathcal{T}) \neq \varnothing$ but $\mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},n)}(X,\mathcal{T}) = \varnothing$ for some $n > 2$?*

Suppose that $(X,\mathcal{T})$ is connected, and $f \in \mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X,\mathcal{T})$. Then, by Theorem 2.1, the space $(X,\mathcal{T}_f)$ will be locally compact as a connected linear ordered space. Hence, a possible common point of view to Theorems 2.1 and 2.2 is suggested by the following question.

Problem 2.9. *Let $(X,\mathcal{T})$ be a locally compact space, with $\mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X,\mathcal{T}) \neq \varnothing$. Does there exist a topology $\mathcal{T}_* \ll \mathcal{T}$ on X such that $(X,\mathcal{T}_*)$ is a linear ordered topological space?*

It should be mentioned that all known selection constructions are based on some extreme principle related to "orderability". Hence, it seems natural to expect that some dimension-like function might be bounded. This is, in fact, the motivation for our next question.

Problem 2.10. *Does there exist a space $(X,\mathcal{T})$ such that $\mathcal{S}e\ell_{(\tau_{V(\mathcal{T})},2)}(X,\mathcal{T}) \neq \varnothing$ and $\operatorname{ind}(X,\mathcal{T}) > 1$?*

For some related results and open questions we refer the interested reader to [2, 9, 11].

3. On the cardinality of $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X,\mathcal{T})$

The cardinality of $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X,\mathcal{T})$ may provide some information for $(X,\mathcal{T})$ but mainly when it is finite.

Theorem 3.1. *For a space $(X,\mathcal{T})$, with $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X,\mathcal{T}) \neq \varnothing$, the following holds:*

(a) *If $(X,\mathcal{T})$ is connected, then $|\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X,\mathcal{T})| \leq 2$,* [18].
(b) *$\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X,\mathcal{T})$ is finite if and only if $(X,\mathcal{T})$ has finitely many connected components,* [22].

(c) *If* $(X, \mathcal{T})$ *is infinite and connected, then* $|\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})| = 2$ *if and only if* $(X, \mathcal{T})$ *is compact,* [21].

For some other relations between $|\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})|$ and $(X, \mathcal{T})$, the interested reader is refer to [10, 21, 22].

4. On the variety of $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})$

As it was mentioned above, all known selection constructions are based on some extreme principle, so our knowledge about particular members of $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})$ is mainly related to this. Here are some result about "extreme-like" members of $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})$.

Theorem 4.1 ([17]). *Let* $(X, \mathcal{T})$ *be a space, with* $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T}) \neq \varnothing$. *Then, the set* $\{f(X) : f \in \mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})\}$ *is dense in* $(X, \mathcal{T})$ *provided* $(X, \mathcal{T})$ *is zero-dimensional, while* $(X, \mathcal{T})$ *is totally disconnected provided* $\{f(X) : f \in \mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})\}$ *is dense in* $(X, \mathcal{T})$.

Here, as usual, a space $(X, \mathcal{T})$ is *zero-dimensional* if it has a base of clopen sets, i.e. if $\operatorname{ind}(X, \mathcal{T}) = 0$.

Problem 4.2 ([17]). *Does there exist a space* $(X, \mathcal{T})$ *which is not zero-dimensional but* $\{f(X) : f \in \mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})\}$ *is dense in* $(X, \mathcal{T})$?

Problem 4.3. *Let* $(X, \mathcal{T})$ *be a totally disconnected space, with* $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T}) \neq \varnothing$. *Is the set* $\{f(X) : f \in \mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})\}$ *dense in* $(X, \mathcal{T})$?

Some other results about extreme-like selections are summarized below.

Theorem 4.4. *For a space* $(X, \mathcal{T})$, *with* $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T}) \neq \varnothing$, *the following holds*:

(a) $(X, \mathcal{T})$ *is zero-dimensional provided for every point* $x \in X$ *there exists an* $f_x \in \mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})$, *with* $f_x^{-1}(x) = \{S \in \mathcal{F}(X, \mathcal{T}) : x \in S\}$, [17].

(b) *If* $(X, \mathcal{T})$ *is first countable and zero-dimensional, then for every point* $x \in X$ *there exists an* $f_x \in \mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})$, *with* $f_x^{-1}(x) = \{S \in \mathcal{F}(X, \mathcal{T}) : x \in S\}$, [17].

(c) *If* $(X, \mathcal{T})$ *is separable, then it is zero-dimensional and first countable if and only if for every point* $x \in X$ *there exists an* $f_x \in \mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T})$, *with* $f_x^{-1}(x) = \{S \in \mathcal{F}(X, \mathcal{T}) : x \in S\}$, [10].

5. More about the selection problem for topologically generated hyperspace topologies

Suppose that "R" is a rule by which for any space $(X, \mathcal{T})$ we may assign a topology $\tau_{R(\mathcal{T})}$ on $\mathcal{F}(X, \mathcal{T})$ depending only on the topological structure $\mathcal{T}$ of X. We consider the class $\mathcal{S}e\ell_R$ of those spaces $(X, \mathcal{T})$ which admit a $\tau_{R(\mathcal{T})}$-continuous selection for their hyperspaces $\mathcal{F}(X, \mathcal{T})$ of closed subsets, i.e. $(X, \mathcal{T}) \in \mathcal{S}e\ell_R$ if and only if $\mathcal{F}(X, \mathcal{T})$ has a $\tau_{R(\mathcal{T})}$-continuous selection.

Note that if "V" is the rule by which we assign the Vietoris topology $\tau_{V(\mathcal{T})}$ on $\mathcal{F}(X, \mathcal{T})$, then $(X, \mathcal{T}) \in \mathcal{S}e\ell_V$ if and only if $\mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X, \mathcal{T}) \neq \varnothing$.

In what follows, let us recall that, for a space $(X, \mathcal{T})$, the *Fell topology* $\tau_{F(\mathcal{T})}$ on $\mathcal{F}(X, \mathcal{T})$ is defined by all basic Vietoris neighbourhoods $\langle \mathcal{V} \rangle$ such that $X \setminus \bigcup \mathcal{V}$ is compact. As it becomes clear, we will use "F" to denote the rule that assigns the Fell topology.

Under this terminology, some of the known results can be summarized as follows.

Theorem 5.1. *Let* $(X, \mathcal{T})$ *be a strongly zero-dimensional metrizable space. Then,*

(a) $(X, \mathcal{T}) \in \mathcal{S}e\ell_V$ *if and only if* $(X, \mathcal{T})$ *is completely metrizable,* [6, 8, 19].

(b) $(X,\mathcal{T}) \in \mathcal{S}e\ell_F$ *if and only if* $(X,\mathcal{T})$ *is locally compact and separable,* [15].

The statement (b) of Theorem 5.1 is not surprising since the Fell topology $\tau_{F(\mathcal{T})}$ on $\mathcal{F}(X,\mathcal{T})$ is, in general, not admissible. Related to this, let us recall that a space $(X,\mathcal{T})$ is *topologically well-orderable* [8] if there exists a linear order $\prec$ on X such that $(X,\mathcal{T})$ is a linear ordered space with respect to "$\prec$", and every non-empty closed subset of $(X,\mathcal{T})$ has a "$\prec$"-minimal element. For instance, a strongly zero-dimensional metrizable space $(X,\mathcal{T})$ is topologically well-orderable if and only if it is locally compact and separable, [8].

Theorem 5.2 ([14])**.** *A space* $(X,\mathcal{T})$ *is topologically well-orderable if and only if* $(X,\mathcal{T}) \in \mathcal{S}e\ell_F$.

A further generalization of Theorem 5.2 based on its proof was obtained in [1, 13].

6. Selections in metrizable spaces

Theorem 6.1 ([6, 8])**.** *Let* $(X,\mathcal{T})$ *be a completely metrizable space such that* $\dim(X,\mathcal{T}) = 0$*. Then, there exists a* $\tau_{V(\mathcal{T})}$*-continuous selection for* $\mathcal{F}(X,\mathcal{T})$*.*

Here, $\dim(X,\mathcal{T})$ means the *covering dimension* of $(X,\mathcal{T})$.

Most of the hypotheses in Theorem 6.1 are the best possible. A metrizable space $(X,\mathcal{T})$ is completely metrizable provided there exists a $\tau_{V(\mathcal{T})}$-continuous selection for $\mathcal{F}(X,\mathcal{T})$ [19] (see Theorem 5.1); The assumption $\dim(X,\mathcal{T}) = 0$ cannot be dropped or even weakened to $\dim(X,\mathcal{T}) \le 1$ [8, 20]. Related to this, the following question seems to be open.

Problem 6.2. *Does there exist a zero-dimensional metrizable space* $(X,\mathcal{T})$ *such that* $\mathcal{F}(X,\mathcal{T})$ *has a* $\tau_{V(\mathcal{T})}$*-continuous selection but* $\dim(X,\mathcal{T}) > 0$*?*

7. More continuous selections for metric-generated hyperspace topologies

The continuity of a selection $f \in \mathcal{S}e\ell_{\tau_{V(\mathcal{T})}}(X,\mathcal{T})$ can be improved in several directions involving hyperspace topologies weaker than the Vietoris one. Towards this end, let us briefly recall some of the most important admissible hyperspace topologies on a metric space (X,d). In what follows, we use $\mathcal{T}_d$ to denote the topology on X generated by a metric d on X.

The *Hausdorff topology* $\tau_{H(d)}$ on $\mathcal{F}(X,\mathcal{T}_d)$ depends essentially on the metric d on X. It is the topology on $\mathcal{F}(X,\mathcal{T}_d)$ generated by the *Hausdorff distance* $H(d)$ associated to d. Let us recall that $H(d)$ is defined by

$$H(d)(S,T) = \sup\{d(S,x) + d(x,T) : x \in S \cup T\}, \quad S,\, T \in \mathcal{F}(X,\mathcal{T}_d).$$

It is well-known that $\tau_{V(\mathcal{T}_d)}$ coincides with $\tau_{H(d)}$ if and only if X is compact [18] while, in general, these two topologies are not comparable. In view of that, we need also some hyperspace topologies which are coarser than both $\tau_{V(\mathcal{T}_d)}$ and $\tau_{H(d)}$. A very interesting such topology is the *d-proximal topology* $\tau_{\delta(d)}$ on $\mathcal{F}(X,\mathcal{T}_d)$ [4]. A base for $\tau_{\delta(d)}$ is defined by all collections of the form

$$\langle\langle \mathcal{V} \rangle\rangle_d = \left\{S \in \langle \mathcal{V} \rangle : d\left(S, X \setminus \bigcup \mathcal{V}\right) > 0\right\},$$

where $\mathcal{V}$ is again a finite family of open subsets of $(X,\mathcal{T}_d)$. Here, and in the sequel, we assume that $d(S,\varnothing) = +\infty$ for every $S \in \mathcal{F}(X,\mathcal{T}_d)$.

Another topology of this type is the *d-ball proximal topology* $\tau_{\delta B(d)}$ on $\mathcal{F}(X,\mathcal{T}_d)$. A base for $\tau_{\delta B(d)}$ is defined by all collections of the form $\langle\langle \mathcal{V} \rangle\rangle_d$, where $\mathcal{V}$ is a finite family of open subsets of $(X,\mathcal{T}_d)$ such that $X \setminus \bigcup \mathcal{V}$ is a finite union of *closed balls* of (X,d). A very similar to the d-ball proximal topology is the *d-ball topology* $\tau_{B(d)}$ on $\mathcal{F}(X,\mathcal{T}_d)$ generated by all collections of the form $\langle \mathcal{V} \rangle$, where $\mathcal{V}$ runs over the finite families of open subsets of $(X, \mathcal{T}_d)$ such that $X \setminus \bigcup \mathcal{V}$ is a finite union of closed balls of (X, d).

Finally, we need also the *Wijsman topology* $\tau_{W(d)}$ which is the weakest topology on $\mathcal{F}(X, \mathcal{T}_d)$ such that all distance functionals $d(x, \cdot) : \mathcal{F}(X, \mathcal{T}_d) \to \mathbb{R},\ x \in X$, are continuous.

It should be mentioned that $\tau_{\delta(d)}$, $\tau_{\delta B(d)}$, $\tau_{B(d)}$ and $\tau_{W(d)}$ also depend on the metric d on X. However, they are metrizable only under additional conditions on the metric space (X, d). On the other hand, we always have the following (usually strong) inclusions

$$\tau_{W(d)} \subset \tau_{\delta B(d)} \subset \tau_{\delta(d)} \subset \tau_{V(\mathcal{T}_d)} \bigcap \tau_{H(d)},$$

and

$$\tau_{\delta B(d)} \subset \tau_{B(d)} \subset \tau_{V(\mathcal{T}_d)}.$$

For these and other properties of the above hyperspace topologies, we refer the interested reader to [3] and [4].

For a metrizable space $(X, \mathcal{T})$, let $\mathcal{M}(X, \mathcal{T})$ denote the set of all metrics d on X compatible with the topology of X, i.e. for which $\mathcal{T}_d = \mathcal{T}$. Concerning hyperspace topologies which are "mixed" – where the definition includes a topological part from the topological space $(X, \mathcal{T})$ and a metric part from a compatible metric on X, there arise at least three different points of view given by how useful selections for these hyperspace topologies are. Let $\tau_{\mathcal{R}}$ be such a class of hyperspace topologies which are generated by the compatible metrics on X, i.e. for every $d \in \mathcal{M}(X, \mathcal{T})$ we have a corresponding topology $\tau_{\mathcal{R}(d)}$ on $\mathcal{F}(X, \mathcal{T})$. For convenience, we will restrict our attention only to strongly zero-dimensional metrizable spaces considering the following:

- $(\mathrm{S})_w$ The class w-$\mathcal{S}e\ell_R$ of those strongly zero-dimensional metrizable spaces $(X, \mathcal{T})$ which have the *Weak $\tau_{\mathcal{R}}$-Selection Property* defined by $(X, \mathcal{T}) \in w$-$\mathcal{S}e\ell_R$ if and only if there exists a $\tau_{\mathcal{R}(d)}$-continuous selection for $\mathcal{F}(X, \mathcal{T})$ for some $d \in \mathcal{M}(X, \mathcal{T})$.
- (S) The class $\mathcal{S}e\ell_R$ of those strongly zero-dimensional metrizable spaces $(X, \mathcal{T})$ which have the *$\tau_{\mathcal{R}}$-Selection Property* defined by $(X, \mathcal{T}) \in \mathcal{S}e\ell_R$ if and only if $\mathcal{F}(X, \mathcal{T})$ has a $\tau_{\mathcal{R}(d)}$-continuous selection for every $d \in \mathcal{M}(X, \mathcal{T})$.
- $(\mathrm{S})_s$ The class s-$\mathcal{S}e\ell_R$ of those strongly zero-dimensional metrizable spaces $(X, \mathcal{T})$ which have the *Strong $\tau_{\mathcal{R}}$-Selection Property* defined by $(X, \mathcal{T}) \in s$-$\mathcal{S}e\ell_R$ if and only if $\mathcal{F}(X, \mathcal{T})$ has a selection which is $\tau_{\mathcal{R}(d)}$-continuous for every $d \in \mathcal{M}(X, \mathcal{T})$.

Obviously, we always have s-$\mathcal{S}e\ell_R \subset \mathcal{S}e\ell_R \subset w$-$\mathcal{S}e\ell_R$. However, in general, no one of these inclusions is invertible, see [15]. To become more specific, we will use $\mathcal{R} = W$ for the *Wijsman* topology; $\mathcal{R} = \delta B$ for the *ball proximal* topology; $\mathcal{R} = B$ for the *ball* topology; and $\mathcal{R} = \delta$ for the *proximal* topology.

Theorem 7.1 ([15]). *In the class of strongly zero-dimensional metrizable spaces, the following holds*:

- *(a)* $\mathcal{S}e\ell_F = s$-$\mathcal{S}e\ell_W = \mathcal{S}e\ell_W \subsetneqq w$-$\mathcal{S}e\ell_W$.
- (b) $\mathcal{S}e\ell_F = s$-$\mathcal{S}e\ell_{\delta B} = \mathcal{S}e\ell_{\delta B} \subsetneqq w$-$\mathcal{S}e\ell_{\delta B}$.
- (c) $\mathcal{S}e\ell_F = s$-$\mathcal{S}e\ell_B \subsetneqq \mathcal{S}e\ell_B \subset w$-$\mathcal{S}e\ell_B$.
- (d) s-$\mathcal{S}e\ell_\delta \subsetneqq \mathcal{S}e\ell_\delta \subsetneqq w$-$\mathcal{S}e\ell_\delta$.

Related the the above theorem, the following two questions are of interest.

Problem 7.2 ([15]). *Does there exist a strongly zero-dimensional non-separable metrizable space $(X, \mathcal{T})$ such that $X \in w$-$\mathcal{S}e\ell_R$ for some $\mathcal{R} \in \{W, \delta B, B\}$?*

Problem 7.3 ([15])**.** *Does there exist a strongly zero-dimensional metrizable space* $(X, \mathcal{T})$ *such that* $X \in w\text{-}\mathcal{S}e\ell_B \backslash \mathcal{S}e\ell_B$*?*

Finally, we have also the following two general questions:

Problem 7.4 ([7, 15])**.** *Let* $\mathcal{R} \in \{W, \delta B, B, \delta\}$*,* $(X, \mathcal{T})$ *be a strongly zero-dimensional metrizable space, and let* $d \in \mathcal{M}(X, \mathcal{T})$ *be a compatible metric. Does there exist a topological property* $\mathcal{P}$ *such that* $\mathcal{F}(X, \mathcal{T})$ *has a* $\tau_{\mathcal{R}(d)}$*-continuous selection if and only if* $(\mathcal{F}(X, \mathcal{T}), \tau_{\mathcal{R}(d)}) \in \mathcal{P}$*?*

Problem 7.5 ([7, 15])**.** *Let* $\mathcal{R} \in \{W, \delta B, B, \delta\}$*,* $(X, \mathcal{T})$ *be a strongly zero-dimensional metrizable space, and let* $d \in \mathcal{M}(X, \mathcal{T})$ *be a compatible metric. Does there exist a metric property* $\mathcal{D}$ *such that* $\mathcal{F}(X, \mathcal{T})$ *has a* $\tau_{\mathcal{R}(d)}$*-continuous selection if and only if* $d \in \mathcal{D}$*?*

The interested reader is referred to [5, 7, 12, 15] for some additional discussion on the topic.

8. Special metrics and selections

Let (X, d) be a metric space. A subset $A \subset X$ is called *d-clopen* if $d(A, X \backslash A) > 0$, [7, 15]. Every d-clopen set is clopen but the converse fails. For more information about this concept, see [7, 15].

We shall say that a metric space (X, d) is *totally disconnected* with respect to d, or *totally d-disconnected*, if every singleton of X is an intersection of d-clopen subsets of (X, d), [7].

Example 8.1 ([7])**.** There exists a metric space (X, d) with only two non-isolated points which is not totally d-disconnected.

In view of this example, the following questions about the selection problem for the d-proximal topology are still open.

Problem 8.2 ([7])**.** *Let* $(X, \mathcal{T})$ *be a* (*strongly zero-dimensional*) *completely metrizable space, and let* $d \in \mathcal{M}(X, \mathcal{T})$ *be such that* (X, d) *is totally d-disconnected. Does there exist a* $\tau_{\delta(d)}$*-continuous selection for* $\mathcal{F}(X, \mathcal{T})$*?*

Problem 8.3 ([7])**.** *Let X be a metrizable scattered space, and let* $d \in \mathcal{M}(X, \mathcal{T})$ *be such that* (X, d) *is totally d-disconnected. Does there exist a* $\tau_{\delta(d)}$*-continuous selection for* $\mathcal{F}(X, \mathcal{T})$*?*

Problem 8.4 ([7])**.** *Let* $(X, \mathcal{T})$ *be a metrizable scattered space, and* $d \in \mathcal{M}(X, \mathcal{T})$*. Does there exist a* $\tau_{\delta(d)}$*-continuous selection for* $\mathcal{F}(X, \mathcal{T})$*?*

The above question is open even in the special case when $(X, \mathcal{T})$ has only two non-isolated points, The answer is "Yes" if $(X, \mathcal{T})$ has only one non-isolated point [7, Theorem 5.5].

Finally, the following further question seems to be also interesting.

Problem 8.5. *Let* $(X, \mathcal{T})$ *be a metrizable space which is scattered with respect to compact subsets, i.e. every non-empty closed subset of* $(X, \mathcal{T})$ *contains a non-empty compact and relatively open subset. Also, let* $d \in \mathcal{M}(X, \mathcal{T})$*. Does there exist a* $\tau_{\delta(d)}$*-continuous selection for* $\mathcal{F}(X, \mathcal{T})$*?*

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