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Iterated starcompact topological spaces

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Abstract. Let $\mathcal{P}$ be a topological property. A space X is said to be *k-$\mathcal{P}$-starcompact* if for every open cover $\mathcal{U}$ of X, there is a subspace $A \subseteq X$ with $\mathcal{P}$ such that $\mathrm{st}^k(A,\mathcal{U}) = X$. In this paper, we consider k-$\mathcal{P}$-starcompactness for some special properties $\mathcal{P}$ and discuss relationships among them.

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1. Introduction

Let X be a topological space and $\mathcal{U}$ a collection of subsets of X. For $\emptyset \neq A \subseteq X$, let $\mathrm{st}(A,\mathcal{U}) = \mathrm{st}^1(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$ and $\mathrm{st}^{n+1}(A,\mathcal{U}) = \mathrm{st}(\mathrm{st}^n(A,\mathcal{U}),\mathcal{U})$ for all $n \in \mathbb{N}$. We simply write $\mathrm{st}^n(x,\mathcal{U})$ for $\mathrm{st}^n(\{x\},\mathcal{U})$. A space X is called *n-starcompact* (*$n\frac{1}{2}$-starcompact*) [6] if for every open cover $\mathcal{U}$ of X there is a finite subset F of X (finite subcollection $\mathcal{V}$ of $\mathcal{U}$) such that $\mathrm{st}^n(F,\mathcal{U}) = X$ ($\mathrm{st}^n(\bigcup\mathcal{V},\mathcal{U}) = X$). Let $\widetilde{\mathbb{N}} = \mathbb{N} \cup \{n\frac{1}{2} : n \in \mathbb{N}\}$. By definition, every n-starcompact space is $(n+\frac{1}{2})$-starcompact for $n \in \widetilde{\mathbb{N}}$. It is known that 1-starcompactness is equivalent to countable compactness for Hausdorff spaces. Moreover, every n-starcompact regular space is $2\frac{1}{2}$-starcompact for $n \geq 3, n \in \widetilde{\mathbb{N}}$, and $2\frac{1}{2}$-starcompactness is equivalent to pseudocompactness for Tychonoff spaces [1].

Behaviours of the above mentioned star-covering properties were studied in [1, 6, 7]. By replacing 'finite' with 'countable' in the definition, n-starcompactness was extended to n-star-Lindelöffness in [1]. As we have seen, finiteness plays an important role in the concept of n-starcompactness. In what follows, we may replace finiteness with some topological properties to get some new concepts. Given a topological property $\mathcal{P}$, a space X is called *k-$\mathcal{P}$-starcompact* if for every open cover $\mathcal{U}$ of X, there is a subspace $A \subseteq X$ with $\mathcal{P}$ such that $\mathrm{st}^k(A,\mathcal{U}) = X$. Ikenaga [4] and Song [7] considered 1-$\mathcal{P}$-starcompactness for

$\mathcal{P}$ being compact. We are especially interested in k-$\mathcal{P}$-starcompact spaces for $\mathcal{P}$ being n-starcompact, and call them iterated starcompact spaces in general. More precisely, a space X is said to be (n,k)-*starcompact* if for every open cover $\mathcal{U}$ of X there is an n-starcompact subspace A of X such that $\mathrm{st}^k(A,\mathcal{U}) = X$. For the sake of unification, a compact space is called $\frac{1}{2}$-*starcompact*. In fact, the above definitions appeared in [6] but no further investigation has been done so far. By definition, we have the following lemma.

Lemma 1.1. *(i) Every (n,k)-starcompact space is $(n+k)$-starcompact for $n \in \widetilde{\mathbb{N}}$ and $k \in \mathbb{N}$.*

(ii) Every (n_1,k)-starcompact space is (n_2,k)-starcompact for $n_1, n_2 \in \widetilde{\mathbb{N}}$ with $n_1 \leq n_2$ and $k \in \mathbb{N}$.

(iii) Every (n,k_1)-starcompact space is (n,k_2)-starcompact for $n \in \widetilde{\mathbb{N}}$ and $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$.

By applying known properties, we obtain Diagram 1 in the class of regular spaces. For convenience, (n,k)-starcompactness is abbreviated as $\mathrm{st}^{n,k}$.

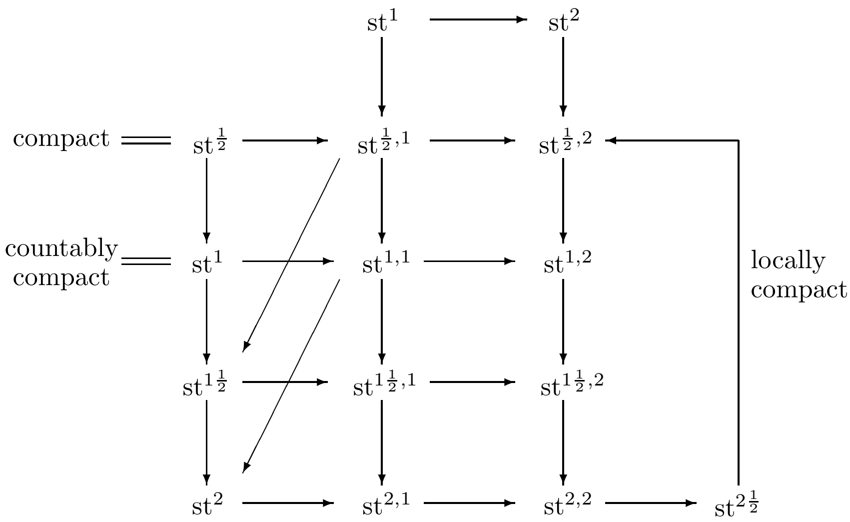

Diagram 1 (In the class of regular spaces)

In Section 2, we provide examples to distinguish iterated starcompact properties around $(1,1)$-starcompactness and consider their relations with other covering properties. Section 3 is devoted to distinguish properties weaker than 2-starcompactness. Throughout this paper, ω (ω_1) is the first infinite (uncountable) cardinal and $\mathfrak{c}$ is the continuum. For any set A, the cardinality of A is denoted by $|A|$. Undefined concepts and symbols can be found in [2].

2. $(1,1)$-starcompact Spaces

A space X is said to be *$\mathcal{L}$-starcompact* if for every open cover $\mathcal{U}$ of X there exists a Lindelöf subspace L such that $\mathrm{st}(L,\mathcal{U}) = X$. By definition, every $(\frac{1}{2},1)$-starcompact space is $\mathcal{L}$-starcompact, $(1,1)$-starcompact and $1\frac{1}{2}$-starcompact; every $1\frac{1}{2}$-starcompact space is both $(1\frac{1}{2},1)$-starcompact and 2-starcompact; and every $(1,1)$-starcompact space is both 2-starcompact and $(1\frac{1}{2},1)$-starcompact. These relationships can be described in the following diagam.

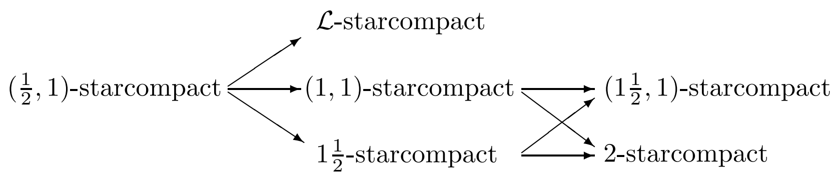

Diagram 2

In this section, we shall first provide some examples to show the difference among concepts in Diagram 2.

Lemma 2.1. [6] *If a regular space X contains a closed discrete subspace Y such that $|Y| = |X| \geq \omega$, then X is not $1\frac{1}{2}$-starcompact.*

Example 2.2. *There is a 2-starcompact, $\mathcal{L}$-starcompact Tychonoff space which is not $(1\frac{1}{2},1)$-starcompact.* Let $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. It is proved that the Isbell-Mrówka space $\Psi = \omega \cup \mathcal{R}$ is 2-starcompact in [1]. Since Ψ is separable, it is $\mathcal{L}$-starcompact. Note that every $1\frac{1}{2}$-starcompact subspace of Ψ is compact. For, if there exists a $1\frac{1}{2}$-starcompact non-compact subspace $X \subseteq \Psi$, then $|X \cap \mathcal{R}| < |X| \leq \omega$ by Lemma 2.1. It follows from $|X \cap \mathcal{R}| = |\{R_1, \cdots, R_n\}| < \omega$ that there exists $A \subseteq X \cap \omega$ such that $|A| = \omega$ and $A \cap \bigcup_{i=1}^{n} R_i = \emptyset$. This implies that X is not pseudocompact, which is a contradiction. Enumerate $\mathcal{R}$ as $\{R_\beta : \beta < \mathfrak{c}\}$. Since the intersection of every compact subspace of Ψ with $\mathcal{R}$ is finite, we can enumerate all compact subsets of Ψ as $\mathcal{K} = \{F_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$, choose $\beta_\alpha > \alpha$ such that $|R_{\beta_\alpha} \cap F_\alpha| < \omega$. In addition, we may requre $\beta_\alpha < \beta_{\alpha'}$ whenever $\alpha < \alpha'$. Choose an open neighborhood $O(R_{\beta_\alpha})$ of R_{β_α} such that $O(R_{\beta_\alpha}) \cap F_\alpha = \emptyset$. Let $I = \{\beta_\alpha : \alpha < \mathfrak{c}\}$. Then

$$\mathcal{U} = \{\{n\} : n \in \omega\} \cup \{\{R_\alpha\} \cup R_\alpha : \alpha \in \mathfrak{c} \setminus I\} \cup \{O(R_{\beta_\alpha}) : \alpha \in \mathfrak{c}\}$$

is an open cover of Ψ. Let K be any compact subspace of Ψ. Then $K = F_\alpha$ for some $\alpha < \mathfrak{c}$. By the construction of $\mathcal{U}$, $R_{\beta_\alpha} \notin \mathrm{st}(K,\mathcal{U})$. Therefore, Ψ is not $(1\frac{1}{2},1)$-starcompact. □

Example 2.3. *There is a $(1,1)$-starcompact Tychonoff space which is neither $1\frac{1}{2}$-starcompact nor $\mathcal{L}$-starcompact.* Let τ be a regular cardinal with $\tau \geq \omega_1$. Let D be the discrete space with $|D| = \tau$ and let $D^* = D \cup \{\infty\}$ be the one-point compactification of D. Consider $X = (D^* \times (\tau+1)) \setminus \{\langle\infty,\tau\rangle\}$ as a subspace of the usual product space $D^* \times (\tau+1)$. Since $D^* \times \tau$ is a countably compact dense subspace of X, X is $(1,1)$-starcompact. But X is not $1\frac{1}{2}$-starcompact, since $|X| = |D|$ and $D \times \{\tau\}$ is a closed discrete subspace of X.

Now, we will show that X is not $\mathcal{L}$-starcompact. Enumerate D as $\{d_\alpha : \alpha < \tau\}$. For each $\alpha < \tau$, choose an open set $U_\alpha = \{d_\alpha\} \times (\alpha, \tau]$. Then $\mathcal{U} = \{U_\alpha : \alpha < \tau\} \cup \{D^* \times \tau\}$ is an open cover of X. Let L be any Lindelöf subspace of X. Then $L \cap (D \times \{\tau\})$ must be countable. Let $L' = L \setminus \bigcup\{L \cap U_\alpha : \langle d_\alpha, \tau\rangle \in L \cap (D \times \{\tau\})\}$. Without loss of generality, we may assume $L' \neq \emptyset$. Since L' is closed in L, L' is Lindelöf. Note that $L' \subseteq D^* \times \tau$. Let $\pi : D^* \times \tau \to \tau$ be the projection. Then $\pi(L')$ is a Lindelöf subspace of the countably compact space τ. Therefore there exists $\kappa_0 < \tau$ which is greater than all elements of $\pi(L')$, i.e., $U_\alpha \cap L' = \emptyset$ for all $\alpha \geq \kappa_0$. Since τ is a regular cardinal with $\tau \geq \omega_1$ and $L \cap (D \times \{\tau\})$ is countable, there exists some $\kappa < \tau$ such that $\kappa_0 < \kappa$ and $U_\kappa \cap L = \emptyset$. Because U_κ is the only one element of $\mathcal{U}$ containing d_κ, $d_\kappa \notin \mathrm{st}(L, \mathcal{U})$. Therefore, X is not $\mathcal{L}$-starcompact. □

Example 2.4. *There is an $\mathcal{L}$-starcompact and $(1,1)$-starcompact Tychonoff space X which is not $1\frac{1}{2}$-starcompact.* Let $\Psi = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space, where $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$ and let D be the discrete space such that $|D| = |\mathcal{R}|$ and $D \cap \mathcal{R} = \emptyset$. Let $Y = (D^* \times (\omega_1 + 1)) \setminus \{\langle\infty, \omega_1\rangle\}$, where $D^* = D \cup \{\infty\}$ is the one-point compactification of D. Take a bijection $i : \mathcal{R} \to D \times \{\omega_1\}$. Let X be a quotient space of $\Psi \cup Y$ and $\pi : \Psi \cup Y \to X$ a quotient mapping which identifies r with $i(r)$ for each $r \in \mathcal{R}$. Then $X = \pi(\omega) \cup \pi(Y) = \pi(\Psi) \cup \pi(D^* \times \omega_1)$. Since X is locally compact Hausdorff, it is Tychonoff.

First, we will show that X is $(1,1)$-starcompact. Let $\mathcal{U}$ be an open cover of X. Note that $A = \pi(D^* \times \omega_1)$ is a countably compact dense subset of $\pi(Y)$. Hence $\pi(Y) \subseteq \mathrm{st}(A, \mathcal{U})$. Since $\pi(\omega)$ is relatively countably compact in $\pi(\Psi)$, $B = \omega \setminus \mathrm{st}(A, \mathcal{U})$ is finite. Thus $\mathrm{st}(A \cup B, \mathcal{U}) = X$ and $A \cup B$ is a countably compact subspace of X. Now, we will show that X is $\mathcal{L}$-starcompact. Since A is countably compact, there is a finite subset F of A such that $A \subseteq \mathrm{st}(F, \mathcal{U})$. Moreover, $\pi(\omega)$ is a countable dense subset of $\pi(\Psi)$, thus we have $\mathrm{st}(F \cup \pi(\omega), \mathcal{U}) = X$ and $F \cup \pi(\omega)$ is a Lindelöf subspace of X. But $\pi(\mathcal{R})$ is closed and discrete in X and $|\pi(\mathcal{R})| = |X|$. Therefore, X is not $1\frac{1}{2}$-starcompact. □

Example 2.5. *There is a $1\frac{1}{2}$-starcompact, $\mathcal{L}$-starcompact Hausdorff space which is not $(1,1)$-starcompact.* Let $X = [0,1]$ and let τ_0 be the Euclidean topology on X. Define $\tau_1 = \{U \setminus F : U \in \tau_0, F \text{ is a countable subset of } X\}$. Then (X, τ_1) is Hausdorff. We will show that (X, τ_1) is $1\frac{1}{2}$-starcompact. Let $\mathcal{U}$ be a basic open cover of (X, τ_1). For each $U \in \mathcal{U}$, select an open subset $V(U)$ of (X, τ_0) and a countable subset $F(U)$ of X such that $U = V(U) \setminus F(U)$. Then $\mathcal{V} = \{V(U) : U \in \mathcal{U}\}$ is an open cover of (X, τ_0). Since (X, τ_0) is compact, $\mathcal{V}$ has a finite subcover $\mathcal{V}_0$. Let $\mathcal{U}_0 = \{U \in \mathcal{U} : V(U) \in \mathcal{V}_0\}$. Then $|X \setminus \bigcup\mathcal{U}_0| \leq \omega$. Since every neighborhood of each point of $X \setminus \bigcup\mathcal{U}_0$ meets $\bigcup\mathcal{U}_0$, $\mathrm{st}(\bigcup\mathcal{U}_0, \mathcal{U}) = X$. It is easy to prove that (X, τ_1) is Lindelöf. Note that every countable subset is closed and discrete in (X, τ_1). So every countably compact subspace is finite. Since (X, τ_1) is not countably compact (i.e., not 1-starcompact), it is not $(1,1)$-starcompact. $\square$

A space X is said to be *meta-Lindelöf* (*para-Lindelöf*) if every open cover of X has a point (locally) countable open refinement. It is well-known that every pseudocompact para-Lindelöf Tychonoff space is compact.

Theorem 2.6. *Let X be a meta-Lindelöf T_1 space. If X is $(1,1)$-starcompact, then it is $1\frac{1}{2}$-starcompact.*

Proof. Let $\mathcal{U}$ be an open cover of X. Since X is meta-Lindelöf, we may assume that $\mathcal{U}$ is point countable. Since X is $(1,1)$-starcompact, there exists a countably compact subspace A of X such that $\mathrm{st}(A, \mathcal{U}) = X$. We may assume $A \cap U \neq \emptyset$ for all $U \in \mathcal{U}$. Now, we will show that some finite subcollection $\mathcal{V}$ of $\mathcal{U}$ covers A. Therefore $\mathrm{st}(\bigcup \mathcal{V}, \mathcal{U}) = X$. Suppose that it is not true, and pick an arbitrary point $x_0 \in A$. Denote by $\mathcal{V}_{x_0}$ the subcollection $\{V \in \mathcal{U} : x_0 \in V\}$ of $\mathcal{U}$. Since A is countably compact and $\mathcal{V}_{x_0}$ is countable, $A \setminus \bigcup \mathcal{V}_{x_0} \neq \emptyset$ (Otherwise, we have $A \subseteq \bigcup \mathcal{V}_{x_0}$, and thus there exists a finite subfamily of $\mathcal{V}_{x_0}$ which covers A). Inductively, we can choose an infinite sequence $\{x_n : n \in \omega\}$ such that $x_n \in A \setminus \bigcup_{i<n} \mathcal{V}_{x_i}$ for each $n \in \omega$. But the sequence $\{x_n : n \in \omega\}$ does not have a cluster point in X. This contradicts the countable compactness of A. Hence, there exists a finite subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $A \subseteq \bigcup \mathcal{V}$, which implies $\mathrm{st}(\bigcup \mathcal{V}, \mathcal{U}) = X$. $\square$

A space X is said to be *strongly collectionwise Hausdorff* (*collectionwise Hausdorff*) if for every closed discrete subset D of X, there exists a discrete (pairwise disjoint) open collection $\{U_d : d \in D\}$ such that $U_d \cap D = \{d\}$ for each $d \in D$.

Theorem 2.7. *Let X be a strongly collectionwise Hausdorff space. If X is $(1,1)$-starcompact, then X is countably compact.*

Proof. Suppose that D is a closed discrete subset of X with $|D| = \omega$. Since X is strongly collectionwise Huasdorff, there exists a discrete open collection $\mathcal{U} = \{U_d : d \in D\}$ such that $U_d \cap D = \{d\}$ for every $d \in D$. Then $\mathcal{V} = \{X \setminus D\} \cup \mathcal{U}$ is an open cover of X. If A is a countably compact subspace of X, $\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ is finite. Hence there exists $d \in D$ such that $U_d \cap A = \emptyset$. Since U_d is the only element of $\mathcal{U}$ containing d, $d \notin \mathrm{st}(A, \mathcal{V})$. $\square$

In Theorem 2.7, strongly collectionwise Hausdorffness cannot be replaced by collectionwise Hausdorffness. It is easy to check that the Tychonoff plank is $(\frac{1}{2}, 1)$-starcompact and collectionwise Hausdorff, but not countably compact.

3. More Examples

In this section, we shall provide some examples to distinguish properties weaker than 2-starcompactness.

Lemma 3.1. *If a space X is locally compact and $n\frac{1}{2}$-starcompact for any $n \in \mathbb{N}$, then X is $(\frac{1}{2}, n)$-starcompact.*

Proof. Let $\mathcal{U}$ be an open cover of X. For each $x \in X$, choose an open neighborhood V_x of x such that $\overline{V_x}$ is compact and $\overline{V_x} \subseteq U$ for some $U \in \mathcal{U}$. Then $\mathcal{V} = \{V_x : x \in X\}$ is an open cover of X. Since X is $n\frac{1}{2}$-starcompact, there exists a finite subcollection $\mathcal{V}_0$ of $\mathcal{V}$ such that $\mathrm{st}^n(\bigcup \mathcal{V}_0, \mathcal{V}) = X$. Since $\overline{\bigcup \mathcal{V}_0}$ is compact and $\mathrm{st}^n(\overline{\bigcup \mathcal{V}_0}, \mathcal{U}) = X$, X is $(\frac{1}{2}, n)$-starcompact. □

Example 3.2. *There exists a $(\frac{1}{2}, 2)$-starcompact Tychonoff space which is not 2-starcompact.* Tree [8] constructed a $2\frac{1}{2}$-starcompact locally compact space which is not 2-starcompact. By Lemma 3.1, it is $(\frac{1}{2}, 2)$-starcompact. □

Also, the space in Example 3.2 can be considered as a candidate of a counterexample between $(2, 2)$-starcompactness and $(2, 1)$-starcompactness. But it is not easy to verify that the space is not $(2, 1)$-starcompact directly. So we shall give a detail maximal family to destroy $(2, 1)$-starcompactness. First, we shall outline the construction of Tree in [8].

Tree's construction: Let C be the lexicographically ordered Cantor square. Then C is a first-countable compact space such that $\dim(C) = 0$ and $\pi w(G) = \mathfrak{c}$ for every non-empty open subset G of C. Let X be the topological sum of ω many copies of C and let $Y = \bigcup\{X_\alpha : \alpha < \mathfrak{c}\}$ be the union of $\mathfrak{c}$ many copies of X. Then Y is a first-countable, locally compact, meta-Lindelöf, non-pseudocompact space such that $\dim(Y) = 0$ and $\pi w(G) = \mathfrak{c}$ for every non-empty open subset G of Y.

Let $\mathcal{B}$ be a base of Y such that every $B \in \mathcal{B}$ is a compact clopen subset of some X_α. Since Y is a locally compact first-countable Hausdorff space, there exists a point-countable π-base $\mathcal{P}$ (see [8] for details) for Y such that

1) $\mathcal{P} \subseteq \mathcal{B}$ and $|\mathcal{P}| = \mathfrak{c}$;

2) for every non-empty set $B \in \mathcal{B}$, $|\{P \in \mathcal{P} : P \subseteq B\}| = \mathfrak{c}$.

Let $D(\mathcal{P})$ be a collection of all sequences $S = \{S_n\}$ of pairwise disjoint open sets from $\mathcal{P}$ that have no cluster point in Y. Enumerate $D(\mathcal{P})$ as $\{T^\alpha : \alpha < \mathfrak{c}\}$ such that $\bigcup T^{\omega\cdot\alpha} \subseteq X_\alpha$ for each $\alpha < \mathfrak{c}$. Inductively we can find, by 1) and 2), $\mathcal{R} = \{S^\alpha : \alpha < \mathfrak{c}, S^\alpha_n \in \mathcal{P}\}$ such that $S^\alpha_n \subseteq T^\alpha_n$ for each α, n, and $S^\alpha_n = S^{\alpha'}_{n'} \Rightarrow \alpha = \alpha'$ and $n = n'$.

Let $\mathcal{A} = \{S^{\omega\cdot\alpha} : \alpha < \mathfrak{c}\}$ and let $\mathcal{R}'$ be a maximal eventually disjoint family of $\mathcal{R}$ with $\mathcal{A} \subseteq \mathcal{R}'$ (that is, if $S \neq S' \in \mathcal{R}'$, there is $n \in \omega$ such that $\bigcup_{i\geq n} S_i \cap \bigcup_{i\geq n} S'_i = \emptyset$). We take a basic neighborhood of $S \in \mathcal{R}'$ as $O_n(S) = \{S\} \cup \bigcup_{i\geq n} S_i$. Then $\mathcal{B}' = \mathcal{B} \cup \{O_n(S) : S \in \mathcal{R}', n \in \omega\}$ is a base for $Y^+ = Y \cup \mathcal{R}'$. Therefore Y^+ is a pseudocompact, locally compact, meta-Lindelöf, Tychonoff space. But Y^+ is not 2-starcompact. Indeed, $\mathcal{U} = \{X_\alpha : \alpha < c\} \cup \{O_1(S) : S \in \mathcal{R}'\}$ is an open cover of Y^+ such that

a) every $y \in Y^+$ is contained in at most countably many members of $\mathcal{U}$

b) if $\mathcal{V} \subseteq \mathcal{U}$ is countable, there exists $S \in \mathcal{A}$ with $O_1(S) \cap \bigcup \mathcal{V} = \emptyset$. □

Lemma 3.3. *Let $Z \subseteq Y^+$ with $|Z \cap \mathcal{A}'| \geq \omega_1$ for some $\mathcal{A}' \subseteq \mathcal{R}'$, and let $\mathcal{U}_0 = \{U_S : S \in \mathcal{A}'\}$ be an open (in Y^+) collection such that $U_S \cap \mathcal{A}' = \{S\}$. If some open (in Y^+) cover $\mathcal{U} \supseteq \mathcal{U}_0$ satisfies a) and b), then Z is not 2-starcompact.*

Proof. Let $\mathcal{O} = \{U \cap Z : U \in \mathcal{U}\}$ be a collection of non-empty sets. Then $\mathcal{O}$ is an open cover of Z and satisfies a) and b). Therefore, Z is not 2-starcompact. □

Example 3.4. *There exists a $(2,2)$-starcompact Tychonoff space which is not $(2,1)$-starcompact.* We will use $\mathcal{A}$ and $\mathcal{R}$ which were consturcted in the above. Let $\{\mathcal{A}_\beta : \beta < \mathfrak{c}\}$ be a partition of $\mathcal{A}$ such that $|\mathcal{A}_\beta| = \omega$ for each $\beta < \mathfrak{c}$ and let $\mathcal{R}_\beta = \{S \in \mathcal{R} : \bigcup S \subseteq \bigcup\{X_\alpha : X_\alpha \cap \bigcup\bigcup \mathcal{A}_\beta \neq \emptyset\}\}$ for each $\beta < \mathfrak{c}$. Then $\mathcal{A}_\beta \subseteq \mathcal{R}_\beta \subseteq \mathcal{R}$. For each $\beta < \mathfrak{c}$, choose a maximal eventually disjoint family $\mathcal{R}'_\beta$ of $\mathcal{R}_\beta$ which contains $\mathcal{A}_\beta$. Finally, we choose a maximal eventually disjoint family $\mathcal{R}'$ of $\mathcal{R}$ which contains $\bigcup\{\mathcal{R}'_\beta : \beta < \mathfrak{c}\}$. Then $\mathcal{A} \subseteq \mathcal{R}' \subseteq \mathcal{R}$ and $Y^+ = Y \cup \mathcal{R}'$ is a locally compact pseudocompact Tychonoff space (and hence $(2,2)$-starcompact).

Now, we prove that Y^+ is not $(2,1)$-starcompact. $\mathcal{U} = \{X_\alpha : \alpha < \mathfrak{c}\} \cup \{O_1(S) : S \in \mathcal{R}'\}$ is an open cover of Y^+. Suppose that Z is a 2-starcompact subspace of Y^+ such that $\mathrm{st}(Z,\mathcal{U}) = Y^+$. By Lemma 3.3, $Z \cap \mathcal{A}$ is countable. It follows from $\mathrm{st}(Z,\mathcal{U}) = Y^+$ that $O_1(S) \cap Z \neq \emptyset$ for each $S \in \mathcal{R}'$. Because $Z \cap \mathcal{A}$ is countable, there exists $\beta_0 < \mathfrak{c}$ such that for each $\beta \geq \beta_0$, $O_1(S) \cap Z \neq \emptyset$ and $S \notin Z$ for all $S \in \mathcal{A}_\beta$. Hence for each $S \in \mathcal{A}_\beta$ ($\beta \geq \beta_0$), there exists $n(S) \in \omega$ such that $F_S = S_{n(S)} \cap Z \neq \emptyset$. Note that $\{F_S : S \in \mathcal{A}_\beta\}$ is a sequence of pairwise disjoint open subsets of Z. Since Z is a pseudocompact subspace of Y^+, there exists $S^\beta \in \mathcal{R}' \cap Z$ which is a cluster point of $\{F_S : S \in \mathcal{A}_\beta\}$. Also S^β is a cluster point of $T^\beta = \{S_{n(S)} : S \in \mathcal{A}_\beta\}$. Thus, $S^\beta \in \mathcal{R}'_\beta$ (otherwise, S^β and T^β should be eventually disjoint by the maximalities of $\mathcal{R}'_\beta$ and $\mathcal{R}'$, namely, S^β is not a cluster point of T^β). Let $\mathcal{A}' = \{S^\beta : \beta \geq \beta_0\}$. By Lemma 3.3, Z is not 2-starcompact. This is a contradiction. □

Matveev [5] gave a pseudocompact Tychonoff space in which no infinite subspace is 2-starcompact. This is an example of a pseudocompact Tychonoff space which is not $(2,2)$-starcompact.

Example 3.5. *There exists a (1,2)-starcompact Hausdorff space X which is either (2,1)-starcompact nor $2\frac{1}{2}$-starcompact.* Let $S = \mathbb{R}$ and τ_0 be the Euclidean topology on $\mathbb{R}$. Endow S with a new topology $\tau_1 = \{U \setminus F : U \in \tau_0, |F| \leq \omega\}$. Let $Y_1 = \bigoplus_{\alpha<\omega_1} S_\alpha$, $\mathbb{Q}_\alpha = S_\alpha \cap \mathbb{Q}$, and $\mathbb{P}_\alpha = S_\alpha \setminus \mathbb{Q}_\alpha$, where $S_\alpha = S$ for each $\alpha < \omega_1$ and $\mathbb{Q}$ is the set of rational numbers. Then $E = \bigcup_{\alpha<\omega_1} \mathbb{Q}_\alpha$ is closed and discrete in Y_1. Let D be a discrete space with $|D| = \omega_1$ and $D \cap E = \emptyset$, and $D^* = D \cup \{\infty\}$ the one-point compactification of D. Then $Y_2 = D^* \times (\omega_1 + 1) \setminus \{\langle \infty, \omega_1\rangle\}$ has a dense countably compact subset $A = D^* \times \omega_1$. Hence Y_2 is 2-starcompact. Enumerate D and E such that $D = \{d_\kappa : \kappa < \omega_1\}$ and $E = \{q_\kappa : \kappa < \omega_1\}$. Let X be the quotient space of $Y_1 \cup Y_2$ which identifies $\langle d_\kappa, \omega_1\rangle$ with q_κ for each $\kappa < \omega_1$.

We firstly show that X is (1,2)-starcompact. Let $\mathcal{U}$ be an open cover of X. We will prove $\mathrm{st}^2(A,\mathcal{U}) = X$. Because A is dense in Y_2, $Y_2 \subseteq \mathrm{st}(A,\mathcal{U})$. Note that for every open subset U of X which contains E, $Y_1 \subseteq \mathrm{cl}_X U$. Therefore, $\mathrm{st}^2(A,\mathcal{U}) = X$.

To show X is not (2,1)-starcompact, we firstly show that every 2-starcompact subspace of X meets only finitely many $\mathbb{P}_\alpha$. Suppose not. Then there is a 2-starcompact subspace K of X such that $\Gamma = \{\alpha < \omega_1 : K \cap \mathbb{P}_\alpha \neq \emptyset\}$ is infinite. Without loss of generality, we can assume $Y_2 \subseteq K$. For each $\alpha \in \Gamma$, pick a point $p_\alpha \in K \cap \mathbb{P}_\alpha$. Note that for each $q_\kappa \in E$, there is a unique $\alpha(\kappa)$ such that $q_\kappa \in \mathbb{Q}_{\alpha(\kappa)}$. Let $U(q_\kappa) = (\mathbb{P}_{\alpha(\kappa)} \setminus \{p_{\alpha(\kappa)}\}) \cup (\{d_\kappa\} \times (\omega_1 + 1))$ for each $\kappa < \omega_1$. Then $\mathcal{U} = \{K \cap \mathbb{P}_\alpha : \alpha \in \Gamma\} \cup \{K \cap U(q_\kappa) : \kappa < \omega_1\} \cup \{A\}$ is an open cover of K. If $x = p_\alpha$ for some (unique) $\alpha \in \Gamma$, then $\mathrm{st}(x,\mathcal{U}) = K \cap \mathbb{P}_\alpha$, i.e., $\mathrm{st}(x,\mathcal{U}) \cap (K \cap \mathbb{P}_{\alpha'}) = \emptyset$ for all $\alpha' \in \Gamma$ with $\alpha \neq \alpha'$. If $x \in \mathbb{P}_\alpha$ and $x \neq p_\alpha$, then $\mathrm{st}(x,\mathcal{U}) = \bigcup\{U(q_\kappa) : q_\kappa \in \mathbb{Q}_\alpha\}$, i.e., $\mathrm{st}(x,\mathcal{U}) \cap (K \cap \mathbb{P}_{\alpha'}) = \emptyset$ for all $\alpha' \in \Gamma$ with $\alpha \neq \alpha'$. In both cases, since $K \cap \mathbb{P}_{\alpha'}$ is the only element of $\mathcal{U}$ containing $p_{\alpha'}$, $p_{\alpha'} \notin \mathrm{st}^2(x,\mathcal{U})$. If $x \in \mathbb{Q}_\alpha$ and $x = q_\kappa$ for some κ, then $\mathrm{st}(x,\mathcal{U}) = U(q_\kappa)$, i.e., $\mathrm{st}(x,\mathcal{U}) \cap (K \cap \mathbb{P}_{\alpha'}) = \emptyset$ for all $\alpha' \in \Gamma$ with $\alpha \neq \alpha'$. Similarily, we have $p_{\alpha'} \notin \mathrm{st}^2(x,\mathcal{U})$. If $x \in A$, then $|\{\kappa < \omega_1 : x \in U(q_\kappa)\}| \leq 1$. Thus $\mathrm{st}(x,\mathcal{U})$ meets at most one $\mathbb{P}_\alpha$. Hence for any finite subset F of K, $\mathrm{st}^2(F,\mathcal{U}) \neq K$. This is a contradiction. Now we will show that X is not $(2,1)$-starcompact. For each $\alpha < \omega_1$, choose a point $p_\alpha \in \mathbb{P}_\alpha$. For each $q_\kappa \in E$, choose a (unique) $\alpha(\kappa) < \omega_1$ such that $q_\kappa \in \mathbb{Q}_{\alpha(\kappa)}$. Let $U(q_\kappa) = (\mathbb{P}_{\alpha(\kappa)} \setminus \{p_{\alpha(\kappa)}\}) \cup (\{d_\kappa\} \times (\omega_1 + 1))$ for each $\kappa < \omega_1$. Then $\mathcal{U} = \{\mathbb{P}_\alpha : \alpha < \omega_1\} \cup \{U(q_\kappa) : \kappa < \omega_1\} \cup \{A\}$ is an open cover of X. Let K be a 2-starcompact subspace of X. Then $\Gamma = \{\alpha < \omega_1 : K \cap \mathbb{P}_\alpha \neq \emptyset\}$ is finite. So there exists $\alpha < \omega_1$ such that $p_\alpha \notin \mathrm{st}(K,\mathcal{U})$. Therefore, X is not (2,1)-starcompact.

Finally, we show that X is not $2\frac{1}{2}$-starcompact. For each $\alpha < \omega_1$, choose a point $p_\alpha \in \mathbb{P}_\alpha$. For each $q_\kappa \in E$, choose a (unique) $\alpha(\kappa) < \omega_1$ such that $q_\kappa \in \mathbb{Q}_{\alpha(\kappa)}$. Let $U(q_\kappa) = (\mathbb{P}_{\alpha(\kappa)} \setminus \{p_{\alpha(\kappa)}\}) \cup (\{d_\kappa\} \times (\kappa, \omega_1])$ and let $V_\kappa = D^* \times \kappa$ for each $\kappa < \omega_1$. Then $\mathcal{U} = \{\mathbb{P}_\alpha : \alpha < \omega_1\} \cup \{U(q_\kappa) : \kappa < \omega_1\} \cup \{V_\kappa : \kappa < \omega_1\}$ is an open cover of X. Note that $\mathcal{U}$ satisfies the following conditions: *(1)* $\mathbb{P}_\alpha \cap \mathbb{P}_{\alpha'} = \emptyset$ if $\alpha \neq \alpha'$; *(2)* $U(q_\kappa) \cap U(q_{\kappa'}) = \emptyset$ if $\alpha(\kappa) \neq \alpha(\kappa')$; *(3)* each V_κ meets at most countably many $U(q_\kappa)$; and *(4)* $\mathbb{P}_\alpha \cap V_\kappa = \emptyset$ for all $\alpha, \kappa < \omega_1$. It follows that $\mathrm{st}(\mathbb{P}_\alpha,\mathcal{U}) \cap \mathbb{P}_{\alpha'} = \emptyset$ if $\alpha \neq \alpha'$; $\mathrm{st}(U(q_\kappa),\mathcal{U}) \cap \mathbb{P}_{\alpha'} = \emptyset$ if $\alpha(\kappa) \neq \alpha'$ and $\mathrm{st}(V_\kappa,\mathcal{U}) \cap \mathbb{P}_\alpha \neq \emptyset$ for at most countably many α. Hence for every finite subcollection $\mathcal{V}$ of $\mathcal{U}$, $\mathrm{st}(\bigcup\mathcal{V},\mathcal{U})$ meets at most countably many $\mathbb{P}_\alpha$, i.e., there exists $\alpha < \omega_1$ such that $\mathrm{st}(\bigcup\mathcal{V},\mathcal{U}) \cap \mathbb{P}_\alpha = \emptyset$. Since $\mathbb{P}_\alpha$ is the only element of $\mathcal{U}$ containing p_α, $p_\alpha \notin \mathrm{st}^2(\bigcup\mathcal{V},\mathcal{U})$. Therefore, X is not $2\frac{1}{2}$-starcompact. □

Every $(2,2)$-starcompact regular space is $2\frac{1}{2}$-starcompact, but the implication is not true for Hausdorff spaces. Example 3.5 is a counterexample.

Example 3.6. *There exists a $(1\frac{1}{2}, 2)$-starcompact Hausdorff space which is not $(1,2)$-starcompact.* Let $X_1 = \mathbb{R}$ and τ_0 be the Euclidean topology on $\mathbb{R}$. Endow X_1 with a new topology $\tau_1 = \{U \setminus F : U \in \tau_0, |F| \leq \omega\}$. Suppose X_2 is a homeomorphic copy of the subspace $[0,1]$ of X_1 and $X_1 \cap X_2 = \emptyset$. For convenience, we assume $X_2 = [0,1]$. Let $h : X_1 \to (0,1)$ be a homeomorphism. Let D and E be closed discrete subsets of X_1 and $(0,1)$ respectively satisfying the following conditions: *(1)* $h(D) = E$; *(2)* D is dense in the Euclidean topology on $\mathbb{R}$; and *(3)* E is dense in the Euclidean topology on $(0,1)$. Let X be the quotient space of $X_1 \cup X_2$ which identifies d with $h(d)$ for each $d \in D$. Then X is Hausdorff. From Example 2.4, X_2 is $1\frac{1}{2}$-starcompact and every open subset U of X with $\widetilde{D} \subset U$ is dense in X ($\widetilde{D}$ is the equivalence class on X). Let $\mathcal{U}$ be an open cover of X. Since $\mathrm{st}(X_2,\mathcal{U})$ is open in X and $\widetilde{D} \subseteq \mathrm{st}(X_2,\mathcal{U})$, $\mathrm{st}^2(X_2,\mathcal{U}) = X$. Thus, X is $(1\frac{1}{2},2)$-starcompact.

Next, we show X is not $(1,2)$-starcompact. Every countable subset of X is closed and discrete in X. So every countably compact subspace is finite. Hence it is enough to show that X is not 2-starcompact. For each $n \in \mathbb{Z}$, let $I_n = (n, n+1)$ and $J_n = (n, n+2)$. We denote $J'_n = h(J_n)$ and choose a point $p_n \in I_n \setminus D$ for each $n \in \mathbb{Z}$. Define $U_n = J'_n \cup (J_n \setminus \{p_n, p_{n+1}\})$ for each $n \in \mathbb{Z}$. Then $\mathcal{U} = \{I_n \setminus \widetilde{D} : n \in \mathbb{Z}\} \cup \{U_n : n \in \mathbb{Z}\} \cup \{X_2 \setminus \widetilde{D}\}$ is an open cover of X. One can prove easily that $\{n : \mathrm{st}(x,\mathcal{U}) \cap (I_n \setminus \widetilde{D}) \neq \emptyset\}$ is finite for each $x \in X_1$. We will prove it only for $x \in X_2 \setminus \widetilde{D}$. If $x = \widetilde{n+1}$, then $\mathrm{st}(x,\mathcal{U}) = (X_2 \setminus \widetilde{D}) \cup U_n$. If $\widetilde{n} < x < \widetilde{n+1}$, then $\mathrm{st}(x,\mathcal{U}) = (X_2 \setminus \widetilde{D}) \cup U_{n-1} \cup U_n$. Hence $\{n : \mathrm{st}(x,\mathcal{U}) \cap (I_n \setminus \widetilde{D}) \neq \emptyset\}$ is finite for each $x \in X_2 \setminus \widetilde{D}$. Thus, for every finite subset F of X, there exists $n \in \mathbb{Z}$ such that $\mathrm{st}(F,\mathcal{U}) \cap (I_n \setminus \widetilde{D}) = \emptyset$. Since $I_n \setminus \widetilde{D}$ is the only element of $\mathcal{U}$ containing p_n, $p_n \notin \mathrm{st}^2(F,\mathcal{U})$. □

Remark 3.7. *Most of existing examples in the theory of star-covering properties are regular and locally compact (see* [1], [6], [7], *and* [8]*). By Lemma 3.1, every locally compact* $2\frac{1}{2}$*-starcompact space is* $(\frac{1}{2},2)$*-starcompact. It seems not easy to construct regular spaces distinguishing properties between* $(\frac{1}{2},2)$*-starcompactness and* $2\frac{1}{2}$*-starcompactness in Diagram 1.*

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