

Continuous representability of interval orders

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ABSTRACT. In the framework of the analysis of orderings whose associated indifference relation is not necessarily transitive, we study the structure of an interval order, and its representability through a pair of continuous real-valued functions. Inspired in recent characterizations of the representability of interval orders, we obtain new results concerning the existence of continuous real-valued representations. Classical results are also restated in a unified framework.

2000 AMS Classification: 54F05, 06A06.

Keywords: Orderings on a set. Interval orders. Numerical representations of orderings. Continuous representations of orderings.

1. INTRODUCTION

Dealing with different classes of *orderings* \prec defined on a nonempty set X , the concept of an *interval order* was introduced by Peter C. Fishburn [17] in contexts of Economic Theory, in order to build *models of preferences whose associated indifference may fail to be transitive*. (See also Fishburn [18], or Bosi and Isler [4]). A very complete and informative study of interval orders appears in Chapter 6 of Bridges and Mehta [6].

An interval order \prec defined on a set X is representable if there exists a pair of real-valued functions $u, v : X \rightarrow \mathbb{R}$ such that $x \prec y \iff v(x) < u(y)$ ($x, y \in X$).

The question of *finding a complete characterization of the representability* of interval orders was solved by Fishburn [20] (see also Fishburn [21], Theorem 5 on p. 135). A different characterization was obtained by Doignon et al. [14]. Furthermore, a new alternative solution using *only one ordinal condition* has been recently obtained by Olóriz et al. [27]. Some further characterizations have appeared in Bosi et al. [3].

¹A substantial part of this work comes from the Ph. D. of M. Zudaire made under the advisement of J. C. Candeal and E. Induráin.

To deal with numerical representations of interval orders *three main techniques* have been used in the literature:

- i) The *first technique* (see e.g. Bridges and Mehta [6], pp. 88 and ff.), associates to each element of the set X where the interval order \prec has been defined, two suitable subsets $A(x)$ and $B(x)$ of natural numbers in a way that $x \prec y \iff B(x) \subsetneq A(y)$.
- ii) The *second technique*, used in Bosi and Isler [4], is based on *measure theory*. The functions u, v that represent the interval order are related to the measures of lower and upper contour sets, in some orderings related to \prec , of the elements of X .
- iii) The *third technique*, introduced in Olóriz et al. [27], is based on the theory of *functional equations*. The idea is to construct a bivariate map $F : X \times X \rightarrow \mathbb{R}$ such that, for instance, $x \prec y \iff F(x, y) > 0$. In the case of an interval order such functions F should also verify a characteristic functional equation, namely the *functional equation of separability*:

$$F(x, y) + F(y, z) = F(x, z) + F(y, y) \quad (x, y, z \in X).$$

Paying attention to *continuity* we will be looking for representations (u, v) of an interval ordered structure (X, \prec) such that u and v are *continuous* (with respect to a given topology τ on X and the usual Euclidean topology on \mathbb{R}). In Chateaufeuf [11] a characterization of the continuous representability of an interval order was given for the particular case of a *connected topological space* X .

The purpose of the present paper is twofold:

Our first task is to *introduce new characterizations* of the *continuous representability of an interval order*, comparing our results with previous ones existing in the literature (see Chateaufeuf [11], Bridges and Mehta [6]), and *looking for a standard and unified notation*.

Our second task is to *adapt the main techniques* that have been considered to get numerical representations of interval orders to obtain characterizations of the *continuous representability*.

2. PREVIOUS CONCEPTS

Let X be a nonempty set. In what follows “ \prec ” will denote an asymmetric binary relation defined on a nonempty set X . Associated to \prec we will also consider the binary relations “ \succsim ” and “ \sim ”, respectively defined as $x \succsim y \iff \neg(y \prec x)$ and $x \sim y \iff x \succsim y, y \succsim x$. The relation \prec is usually called *strict preference*. The relation \succsim is said to be the *weak preference*, and \sim is called the *indifference*, associated to \prec . The *opposite ordering* \prec_{op} of \prec is defined by $x \prec_{op} y \iff y \prec x$. Given $x \in X$, the sets $L(x) = \{y \in X : y \prec x\}$ and $U(x) = \{y \in X : x \prec y\}$ are called, respectively, the *lower contour set* and the *upper contour set* relative to \prec .

Definition 2.1. *The binary relation \prec is said to be an interval order if $(x \prec y, a \prec b) \implies$ either $x \prec b$, or else $a \prec y$ (or both). An interval order \prec is said to be a *semiorder* if in addition $a \prec b \prec c \implies a \prec d$ or else $d \prec c$ ($a, b, c, d \in X$).*

Observe that because \prec is asymmetric, *if it is an interval order then it must be transitive*. Notice also that \prec being an interval order, *the associated relations \succsim and \sim may fail to be transitive*. An *example* is the relation \prec defined on the real line \mathbb{R} as $x \prec y \iff x + 1 < y$. However an interval order \prec is always *pseudotransitive*: That is $x \prec y \succsim z \prec t \implies x \prec t$ ($x, y, z, t \in X$).

It is straightforward to see that:

For an asymmetric binary relation \prec on X , pseudotransitivity is equivalent to the fact of \prec being an interval order.

Moreover:

An asymmetric binary relation \prec defined on X is a semiorder if and only if it satisfies the condition of generalized pseudotransitivity, namely, for every $x, y, z, t \in X$ the following three conditions hold true:

1. $x \prec y \succsim z \prec t \implies x \prec t$,
2. $x \succsim y \prec z \prec t \implies x \prec t$,
3. $x \prec y \prec z \succsim t \implies x \prec t$.

(See Gensemer [24] for details).

An interval order \prec defined on X is said to be *representable* if there exist two real-valued functions $u, v : X \rightarrow \mathbb{R}$ such that $x \prec y \iff v(x) < u(y)$ ($x, y \in X$). Since \prec is asymmetric, this is equivalent to associate to each element $x \in X$ a real-interval (that eventually may collapse to a single point), $I_x = [u(x), v(x)]$. Thus $x \prec y$ if and only if I_x is located on the left of I_y , and I_x does not meet I_y . This kind of “interval-representation” gave raise to the nomenclature of interval order. However not every interval order is representable (see e.g. Bosi et al. [3] for details).

A semiorder \prec defined on X is said to be *representable* if there exist a real-valued function $u : X \rightarrow \mathbb{R}$ and a non-negative real number α (called *threshold*) such that $x \prec y \iff u(x) + \alpha < u(y)$. Observe that this is a particular case of representation of an interval order, in which $v(x)$ can be defined as $u(x) + \alpha$ ($x \in X$). Observe also that this is equivalent to associate to each element $x \in X$ a real-interval $I_x = [u(x), u(x) + \alpha]$. In this case, *all the intervals have the same length α* . Obviously all them will collapse to a single point if and only if $\alpha = 0$. (See Fishburn [20], Gensemer [22, 23, 24, 25] or Candeal et al. [10] for more details).

Following Fishburn [17, 19] we shall associate to an interval order \prec two new binary relations, respectively denoted by \prec^* and \prec^{**} and defined by $x \prec^* y \iff x \prec z \preceq y$ for some $z \in X$ ($x, y \in X$), and, similarly, $x \prec^{**} y \iff x \preceq z \prec y$ for some $z \in X$ ($x, y \in X$). Define the relation $x \preceq^* y \iff \neg(y \prec^* x)$ and $x \preceq^{**} y \iff \neg(y \prec^{**} x)$ ($x, y \in X$). Then it is straightforward to see that $x \preceq^* y \iff (y \prec z \implies x \prec z \text{ (} z \in X))$ and similarly $x \preceq^{**} y \iff (z \prec x \implies z \prec y \text{ (} z \in X))$. Observe also that in terms of contour sets, it follows that $x \preceq^* y \iff U(y) \subseteq U(x)$ and also $x \preceq^{**} y \iff L(x) \subseteq L(y)$ ($x, y \in X$).

A *preorder* \preceq defined on a nonempty set X is a reflexive and transitive binary relation defined on X . If it is also complete (i.e., either $x \preceq y$ or else $y \preceq x$ for every $x, y \in X$), \preceq is said to be a *total preorder*. A total preorder \preceq is said to be *representable* if there exists a real-valued function, usually called *utility function*, $u : X \rightarrow \mathbb{R}$ such that $x \preceq y \iff u(x) \leq u(y)$. If $x \prec y \iff \neg(y \preceq x)$ and $x \sim y \iff x \preceq y \preceq x$ ($x, y \in X$), then a representable total preorder clearly corresponds to a representation of the weak preference \preceq associated to a semiorder \prec for the special case in which the threshold α equals zero. An antisymmetric total preorder \preceq on a set X (i.e. $x \preceq y \preceq x \implies x = y$ ($x, y \in X$)) is said to be a *total order*.

Let \preceq be a total preorder on a set X , and let $x \prec y$ ($x, y \in X$). We say that the pair (x, y) defines a *jump* if there is no $z \in X$ such that $x \prec z \prec y$. A subset $Y \subseteq X$ is said to be *cofinal* if for every $x \in X$ there exists $y \in Y$ such that $x \prec y$. Similarly Y is said to be *coinitial* if for every $x \in X$ there exists $y \in Y$ such that $y \prec x$. An element $z \in X$ is said to be *minimal* (respectively: *maximal*) with respect to \preceq if $z \preceq x$ (respectively: $x \preceq z$) for every $x \in X$. Consider now the quotient space X/\sim of X through the equivalence relation \sim . The ordering \preceq is compatible with this quotient, so that X/\sim becomes a totally ordered set. The total preorder \preceq is said to be *Dedekind complete* if in X/\sim every subset $C \subseteq X/\sim$ that is bounded above with respect to \prec has a *supremum* (i.e.: smallest upper bound) $\sup C$ in X/\sim .

When dealing with a *total order* \preceq defined on X we can endow X with the *order topology* whose subsbasis is defined by the lower and upper contour sets relative to \prec .

It can be proved that:

Lemma 2.2.

- i) (see Gillman and Jerison [26] p. 3, Birkhoff [2], p. 200, or else Candeal and Induráin [9]) :
*Every total preorder \preceq has a Dedekind complete extension without jumps that has neither minimal nor maximal elements.
This extension is essentially unique.*

- ii) (see e.g. Birkhoff [2], p. 243) :
The order topology relative to a total order \lesssim on a set X is connected if and only if \lesssim is Dedekind complete and has no jumps.

Remark 2.3. The extension corresponding to Lemma 2.2 (i) may produce a set \bar{X} that is much bigger than the given set X . Notice that if we start with a single $X = \{x\}$, with the trivial total ordering, then we arrive to an extension that is isotonic to the real line \mathbb{R} .

Coming back to the study of interval orders and semiorders, it is well-known (see e.g. Proposition 2.1 in Bridges [5]) that:

Lemma 2.4. *Let \prec be an asymmetric binary relation defined on a nonempty set X . Then the following statements are equivalent:*

- i) \prec is an interval order,
- ii) \lesssim^* is a total preorder,
- iii) \lesssim^{**} is a total preorder.

In addition, \lesssim is transitive if and only if \lesssim , \lesssim^ , and \lesssim^{**} coincide.*

In what concerns semiorders among interval orders, suppose that an interval order \prec has been defined on a set X , and define the following new binary relation, introduced in Fishburn [18]: $x \prec^0 y \iff x \prec^* y$ or else $x \prec^{**} y$ ($x, y \in X$). In Bosi and Isler [4] it is proved the following fact:

Lemma 2.5. *The interval order \prec is actually a semiorder if and only if \prec^0 is asymmetric.*

A key concept, used in Olóriz et al. [27] to get a characterization of the representability of interval orders, is that of *interval order separability* (henceforward *i.o.-separability*). An interval order \prec on a set X is said to be *i.o.-separable* if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exists an element d in D such that $x \prec d \lesssim^{**} y$.

The nub result on the representability of interval orders is in order now:

Lemma 2.6 (Olóriz et al. [27]). *Let X be a nonempty set endowed with an interval order \prec . Then, the following statements are equivalent:*

- i) \prec is *i.o.-separable*,
- ii) *there exists a bivariate map $F : X \times X \rightarrow \mathbb{R}$ such that $x \lesssim y \iff F(x, y) \geq 0$ and $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ for every $x, y, z \in X$,*
- iii) \prec is *representable*.

3. CONTINUOUS REPRESENTATIONS OF INTERVAL ORDERS

Let X be a nonempty set endowed with an interval order \prec and a topology τ . Now we consider the possibility of finding a representation (u, v) for \prec such that both u and v are *continuous* when considering on X the given topology τ and on the real line \mathbb{R} the usual Euclidean topology. To do so, we need to introduce some previous concepts and results.

Definition 3.1. Let (X, τ) be a topological space endowed with an interval order \prec . We say that \prec is τ -continuous if for each $a \in X$ the upper and lower contour sets $L(a)$ and $U(a)$ are τ -open sets.

Let \prec be an interval-order, and \prec^* and \prec^{**} its associated relations as defined in section 2. The τ -continuity of \prec^* (respectively: \prec^{**}) is defined similarly, now considering lower and upper contours relative to \prec^* (respectively: \prec^{**}).

Definition 3.2. Let (X, τ) a topological space endowed with an interval order \prec . The topology τ is said to be natural for the interval order \prec if \prec , \prec^* and \prec^{**} are all τ -continuous.

Observe that a topology τ is natural for the interval order \prec if and only if it is finer than the topology θ , a subbasis of which is given by the family:

$$\begin{aligned} & \{U(a) : a \in X\} \cup \{L(b) : b \in X\} \\ & \bigcup \{U_{\prec^*}(c) : c \in X\} \cup \{L_{\prec^*}(d) : d \in X\} \\ & \bigcup \{U_{\prec^{**}}(e) : e \in X\} \cup \{L_{\prec^{**}}(f) : f \in X\}. \end{aligned}$$

The existence of continuous representations for an interval order will lean, of course, on the continuity of \prec , \prec^* , \prec^{**} with respect to the given topology τ . Also, it will lean on some ordinal condition of *separability* or *density*.

Definition 3.3. Let X be a nonempty set endowed with an interval order \prec . We say that \prec is

- i) *strongly separable* if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$, there exist $a, b \in D$ such that $x \prec a \prec b \prec y$,
- ii) *full* if for every $x, y \in X$ with $x \prec y$, there exist $a, b \in X$ such that $x \prec a \prec b \prec y$.

(It is obvious that strongly separable implies full).

The main well-known result on the continuous representability of an interval order \prec on a topological space (X, τ) was introduced in Chateaufneuf [11] for the case in which (X, τ) is a *connected* topological space.

Theorem 3.4 (Chateaufneuf [11]). *An interval order \prec defined on a connected topological space (X, τ) admits a representation (u, v) with u and v continuous if and only if \prec is strongly separable and in addition \prec^* and \prec^{**} are both τ -continuous. Moreover, in that case there is also a continuous representation (U, V) such that U is a representation for the total preorder \prec^{**} and V is a representation for the total preorder \prec^* .*

We can now improve Chateaufneuf's theorem with more equivalences, having in mind the concept of i.o.-separability, that is weaker than strong separability.

Theorem 3.5. *Let \prec be an interval order defined on a connected topological space (X, τ) . The following conditions are equivalent:*

- i) (X, \prec) has a continuous representation,
- ii) \prec is strongly separable and τ is a natural topology,
- iii) (X, \prec) has a representation (u, v) such that u and v are τ -upper semi-continuous, and \prec^*, \prec^{**} are τ -continuous,
- iv) (X, \prec) has a representation (u, v) such that u and v are τ -lower semi-continuous, and \prec^*, \prec^{**} are τ -continuous,
- v) \prec is strongly separable and \prec^*, \prec^{**} are τ -continuous,
- vi) \prec is i.o.-separable and full, and \prec^*, \prec^{**} are τ -continuous,
- vii) the topology θ is separable (i.e.: there exists a countable subset that meets every nonempty θ -open set) and coarser than τ ,
- viii) \prec is i.o.-separable and τ is a natural topology,
- ix) \prec is i.o.-separable, and \prec^*, \prec^{**} are τ -continuous
- x) (X, \prec) has a continuous representation (u, v) such that u is a representation for \prec^{**} and v is a representation for \prec^* ,
- xi) there exists a bivariate map $F : X \times X \rightarrow \mathbb{R}$ that is continuous with respect to the product $(\tau \times \tau)$ topology on $X \times X$ and the Euclidean topology on \mathbb{R} such that $x \prec y \iff F(x, y) \geq 0$ and $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ for every $x, y, z \in X$,
- xii) the topology θ is second countable (i.e.: there exists a countable basis for such topology) and coarser than τ .

Proof. The proof will follow the scheme :

i) \iff ii), i) \iff iii), i) \iff iv), ii) \implies v) \implies i), v) \iff vi), vi) \iff viii), viii) \iff ix), viii) \implies x) \implies i), i) \iff xi), vii) \iff ii), xii) \implies vii) and finally i) \implies xii).

i) \implies ii)

See Proposition 6.2.3, Lemma 6.5.1 and Lemma 6.5.3 in Bridges and Mehta [6].

ii) \implies i)

See Theorem 6.5.5 in Bridges and Mehta [6].

i) \implies iii)

It is obvious that (X, \prec) has a representation (u, v) such that u and v are τ -upper semicontinuous. The τ -continuity of \prec^*, \prec^{**} follows from Lemma 6.5.1 in Bridges and Mehta [6].

iii) \implies i)

By Proposition 6.2.3 in Bridges and Mehta [6], \prec is continuous. Moreover, it is i.o.-separable by Lemma 4, since it is representable. The proof of Theorem 6.5.5 in Bridges and Mehta [6] gives now a construction of a continuous representation for the interval order \prec .

i) \implies iv) \implies i)

This is analogous to i) \implies iii) \implies i). Notice also that an interval order is representable if and only if the opposite interval order \prec_{op} is representable. If (u, v) is a representation for \prec , then $(-v, -u)$ is a representation for \prec_{op} . Finally, a map $f : X \rightarrow \mathbb{R}$ is lower-semicontinuous if and only if $-f$ is upper-semicontinuous.

ii) \implies v)

This is obvious.

v) \implies i)

See the proof of Theorem 6.5.5 in Bridges and Mehta [6].

v) \implies vi) \implies v)

It follows from the equivalence, proved in Bosi et al. [3], that states that strongly separable is the same as i.o.-separable plus full.

vi) \implies viii)

Notice that vi) \iff v) \iff ii) and obviously ii) \implies viii).

viii) \implies vi)

This is obvious.

viii) \implies ix)

This is immediate.

ix) \implies viii)

Following the proof of Theorem 6.5.5 in Bridges and Mehta [6] we first obtain a representation (u, v) for \prec such that u and v are τ -upper semicontinuous. Now Proposition 6.2.3 in Bridges and Mehta [6] proves that \prec is continuous.

viii) \implies x)

The proof of Theorem 6.5.5 in Bridges and Mehta [6] furnishes not only a continuous representation (u, v) for \prec . It also states that u is a continuous representation for \prec^{**} and v is a continuous representation for \prec^* .

x) \implies i)

This is obvious.

i) \implies xi)

Let (u, v) be a continuous representation for \prec , and let $F : X \times X \rightarrow \mathbb{R}$ be defined as follows: $F(x, y) = v(y) - u(x)$ ($x, y \in X$). F is continuous because u and v are. A final checking shows that $x \prec y \iff F(x, y) \geq 0$ and $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ for every $x, y, z \in X$.

xi) \implies i)

Suppose F is given in the conditions of xi). Following Lemma 1 in Olóriz et al. [27], fix an element $x_0 \in X$ and call $u(x) = -F(x, x_0)$; $v(y) = F(y, y) + u(y)$ ($x, y \in X$). It is straightforward to see now that (u, v) is a continuous representation for \prec .

vii) \implies ii)

Because θ is coarser than τ , we have that τ is a natural topology and \prec is θ -connected. The connectedness and separability of θ imply, following the proof of Corollary 6.5.6 in Bridges and Mehta [6], that \prec is strongly separable. Thus we arrive to ii).

ii) \implies vii)

Since τ is a natural topology, it is, by definition, finer than the topology θ . Moreover, the τ -connectedness of X implies the connectedness of X in any topology coarser than τ . Also, \prec is by hypothesis strongly separable. Therefore, as shown in Bosi et al. [3], the following conditions are equivalent:

1. \prec is strongly separable.
2. There exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \prec^* d \prec y$,
3. There exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \prec d \prec^{**} y$.

Let us prove now that θ is separable: To do so, we consider the subbasis

$$\begin{aligned} & \{U(a) : a \in X\} \cup \{L(b) : b \in X\} \\ & \bigcup \{U_{\prec^*}(c) : c \in X\} \cup \{L_{\prec^*}(d) : d \in X\} \\ & \bigcup \{U_{\prec^{**}}(e) : e \in X\} \cup \{L_{\prec^{**}}(f) : f \in X\} \end{aligned}$$

for such topology θ so that our task consists in proving the existence of a countable subset $C \subseteq X$ meeting any nonempty basic θ -open set. A basis for θ appears as the collection of all finite intersections of elements in a subbasis. Also, since condition ii) is equivalent to condition x), it follows that the interval order \prec and the total preorders \preceq^* and \preceq^{**} are representable. In particular, their corresponding order topologies are separable. (See Candeal and Induráin [7] for the case of a total preorder, and Bosi et al. [3] for the case of an interval order). Consequently, it is enough to check the following possibilities:

Case 1. $U_{\prec}(a) \cap U_{\prec^*}(b)$:

If there exists $z \in U_{\prec}(a) \cap U_{\prec^*}(b)$, then $a \prec z$ and $b \prec^* z$. Take a countable subset $D \subseteq X$ corresponding to the strong separability of \prec . Take also a countable subset $R \subseteq X$ corresponding to the Cantor separability of the preorder \prec^* , that is, being $x, y \in X$ with $x \prec^* y$, there exists $r \in R$ such that $x \prec^* r \prec^* y$. (The existence of such R follows from the connectedness of the order topology of the representable preorder \prec^* . For more details, consult

Candeal and Induráin [7].) The family $\{U_{\prec}(d) \cap U_{\prec^*}(r) : d \in D, r \in R\}$ is obviously countable. For any nonempty set S of this family, we choose $s \in S$. This new set of elements, say C , is also countable. Now consider $d \in D$ with $a \prec^* d \prec z$ and $r \in R$ such that $b \prec^* r \prec^* z$. It follows that $z \in U_{\prec}(d) \cap U_{\prec^*}(r)$. Take $s \in C \cap U_{\prec}(d) \cap U_{\prec^*}(r)$. It follows easily that $s \in U_{\prec}(a) \cap U_{\prec^*}(b)$ and we are done.

Case 2. $U_{\prec}(a) \cap U_{\prec^{**}}(b)$:

Since $U_{\prec^{**}}(x) = \bigcup_{\{\eta \in X, x \prec \eta\}} U_{\prec}(\eta)$, this case follows from the separability of the order topology corresponding to \prec , a subbasis for which is the collection $\{U(a) : a \in X\} \cup \{L(b) : b \in X\}$.

Case 3. $U_{\prec^*}(a) \cap U_{\prec^{**}}(b)$:

If there exists $z \in U_{\prec^*}(a) \cap U_{\prec^{**}}(b)$, then $a \prec^* z$ and $b \prec^{**} z$. Take a countable subset $R \subseteq X$ corresponding to the Cantor separability of the preorder \prec^* . Take also a countable subset $S \subseteq X$ corresponding to the Cantor separability of the preorder \prec^{**} . The family $\{U_{\prec^*}(r) \cap U_{\prec^{**}}(s) : r \in R, s \in S\}$ is obviously countable. For any nonempty set P of this family, we choose $p \in P$. This new set of elements, say C , is also countable. Now consider $r \in R$ with $a \prec^* r \prec^* z$ and $s \in S$ such that $b \prec^{**} s \prec^{**} z$. It is plain that $z \in U_{\prec^*}(r) \cap U_{\prec^{**}}(s)$. Take $c \in C \cap U_{\prec^*}(r) \cap U_{\prec^{**}}(s)$. It follows easily that $c \in U_{\prec^*}(a) \cap U_{\prec^{**}}(b)$ and we are done.

Case 4. $L_{\prec}(a) \cap L_{\prec^*}(b)$:

Since $L_{\prec^*}(x) = \bigcup_{\{\xi \in X, \xi \prec x\}} L_{\prec}(\xi)$, this case follows from the separability of the order topology corresponding to \prec .

Case 5. $L_{\prec}(a) \cap L_{\prec^{**}}(b)$:

This case is analogous to Case 1. Actually, if there exists $z \in L_{\prec}(a) \cap L_{\prec^{**}}(b)$, then $z \prec a$ and $z \prec^{**} b$. Take a countable subset $D \subseteq X$ corresponding to the strong separability of \prec . Take also a countable subset $R \subseteq X$ corresponding to the Cantor separability of the preorder \prec^{**} , that is, being $x, y \in X$ with $x \prec^{**} y$, there exists $r \in R$ such that $x \prec^{**} r \prec^{**} y$. The family $\{L_{\prec}(d) \cap L_{\prec^{**}}(r) : d \in D, r \in R\}$ is obviously countable. For any nonempty set S of this family, we choose $s \in S$. This new set of elements, say C , is also countable. Now consider $d \in D$ with $z \prec d \prec^{**} a$ and $r \in R$ such that $z \prec^{**} r \prec^{**} b$. It follows that $z \in L_{\prec}(d) \cap L_{\prec^{**}}(r)$. Take $s \in C \cap L_{\prec}(d) \cap L_{\prec^{**}}(r)$. It follows easily that $s \in L_{\prec}(a) \cap L_{\prec^{**}}(b)$ and we are done.

Case 6. $L_{\prec^*}(a) \cap L_{\prec^{**}}(b)$:

This case is analogous to Case 3.

Case 7. $U_{\prec}(a) \cap L_{\prec^*}(b)$:

Since $L_{\prec^*}(x) = \bigcup_{\{\xi \in X, \xi \prec x\}} L_{\prec}(\xi)$, this case follows from the separability of the order topology corresponding to \prec .

Case 8. $U_{\prec}(a) \cap L_{\prec^{**}}(b)$:

If there exists $z \in U_{\prec}(a) \cap L_{\prec^{**}}(b)$, then $a \prec z$ and $z \prec^{**} b$. Take a countable subset $D \subseteq X$ corresponding to the strong separability of \prec . Take also a countable subset $R \subseteq X$ corresponding to the Cantor separability of the preorder \prec^{**} . The family $\{U_{\prec}(d) \cap L_{\prec^{**}}(r) : d \in D, r \in R\}$ is obviously countable. For any nonempty set S of this family, we choose $s \in S$. This new set of elements, say C , is also countable. Now consider $d \in D$ with $a \prec^* d \prec z$ and $r \in R$ such that $z \prec^{**} r \prec^{**} b$. It follows that $z \in U_{\prec}(d) \cap L_{\prec^{**}}(r)$. Take $s \in C \cap U_{\prec}(d) \cap L_{\prec^{**}}(r)$. It follows easily that $s \in U_{\prec}(a) \cap L_{\prec^{**}}(b)$.

Case 9. $U_{\prec^*}(a) \cap L_{\prec^{**}}(b)$:

If there exists $z \in U_{\prec^*}(a) \cap L_{\prec^{**}}(b)$, then $a \prec^* z$ and $z \prec^{**} b$. Take a countable subset $D \subseteq X$ corresponding to the Cantor separability of the preorder \prec^* . Take also a countable subset $R \subseteq X$ corresponding to the Cantor separability of the preorder \prec^{**} . The family $\{U_{\prec^*}(d) \cap L_{\prec^{**}}(r) : d \in D, r \in R\}$ is obviously countable. For any nonempty set S of this family, we choose $s \in S$. This new set of elements, say C , is also countable. Now consider $d \in D$ with $a \prec^* d \prec^* z$ and $r \in R$ such that $z \prec^{**} r \prec^{**} b$. It follows that $z \in U_{\prec^*}(d) \cap L_{\prec^{**}}(r)$. Take $s \in C \cap U_{\prec^*}(d) \cap L_{\prec^{**}}(r)$. It follows easily that $s \in U_{\prec^*}(a) \cap L_{\prec^{**}}(b)$.

Case 10. $L_{\prec}(a) \cap U_{\prec^*}(b)$:

This case is analogous to Case 8.

Case 11. $L_{\prec}(a) \cap U_{\prec^{**}}(b)$:

This case is analogous to Case 7.

Case 12. $L_{\prec^*}(a) \cap U_{\prec^{**}}(b)$:

Since $U_{\prec^{**}}(x) = \bigcup_{\{\eta \in X, x \prec \eta\}} U_{\prec}(\eta)$ and $L_{\prec^*}(x) = \bigcup_{\{\xi \in X, \xi \prec x\}} L_{\prec}(\xi)$, this case follows from the separability of the order topology corresponding to \prec .

The union of the countable subsets used along the cases considered above is of course countable and meets any nonempty basic θ -open set. Therefore θ is separable.

xii) \implies vii)

Just notice that a second countable topology is always separable. (For aspects concerning General Topology, consult Dugundji [15]).

i) \implies xii)

The topologies θ_1 , θ_2 and θ_3 whose respective subbasis are

$$\begin{aligned}\Sigma_1 &= \{U(a) : a \in X\} \cup \{L(b) : b \in X\} \\ \Sigma_2 &= \{U_{\prec^*}(c) : c \in X\} \cup \{L_{\prec^*}(d) : d \in X\} \\ \Sigma_3 &= \{U_{\prec^{**}}(e) : e \in X\} \cup \{L_{\prec^{**}}(f) : f \in X\}\end{aligned}$$

are second countable because \prec , \prec^* and \prec^{**} are representable. (See Bosi et al. [3] or Candeal and Induráin [7] for more details). Then the topology θ whose subbasis is $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ is also second countable.

This concludes the proof. \square

Remark 3.6.

- i) The version of Chateaufneuf's theorem that appears on pp. 106-107 of Bridges and Mehta [6] corresponds to the equivalence i) \iff v) but actually it is proved x) (a condition that is stronger than i)). Nothing is said about *i.o.-separability*, a condition less restrictive than strong separability, because such concept of interval order separability appeared later in the literature (in Olóriz et al. [27]).
- ii) Several intermediate steps in the proof have appeared in previous works, usually as necessary lemmata to introduce the main Chateaufneuf's theorem. So, for instance, Proposition 6.2.3 in Bridges and Mehta [6] shows that if a structure of interval order (X, \prec) endowed with a topology τ has a representation (u, v) such that u and v are τ -upper semicontinuous then \prec is τ -continuous. Also, Lemma 6.5.1 and Lemma 6.5.3 in the same work state that if (X, \prec) has a continuous representation and X is connected, then \prec^* , \prec^{**} are τ -continuous and \prec is strongly separable. Moreover, Lemma 6.5.4 in Bridges and Mehta [6] states that if \prec^* , \prec^{**} are τ -continuous and \prec is full, then \prec is continuous. Finally, in Bosi et al. [3] it has been proved that strongly separable is the same as i.o.-separable plus full.
- iii) An immediate corollary follows. This is Corollary 6.5.6 in Bridges and Mehta [6] and can be considered as an extension to the context of interval orders of the classical result (see Eilenberg [16]) that states that a continuous total preorder defined on a connected and separable topological space has a continuous numerical representation by means of a utility function. In this new context of interval orders, from the equivalent condition vii) in the statement of Theorem 3.5, it follows that:

\ll An interval order \prec on a connected and separable topological space (X, τ) has a continuous representation if and only if τ is a natural topology (in other words: if and only if \prec , \prec^* and \prec^{**} are all τ -continuous). \gg
- iv) Some method of proof of Chateaufneuf's theorem, whose main ideas have been used to state some equivalences of Theorem 3.5 above, reminds us the classical *Debreu's open gap lemma* (see Debreu [12, 13] or

else Bridges and Mehta [6], Ch. 3) that states the possibility of getting a *continuous* utility function representing a *total preorder*, if it is the case in which a utility function (continuous or not) is available.

In the context of an interval order \prec on a connected topological space (X, τ) such that \prec^* and \prec^{**} are continuous, a technique to get a continuous representation (U, V) starts by taking a utility function v for the total preorder \lesssim^* . This utility function may or may not be continuous, but using Debreu's open gap lemma the continuity of v can be assumed without loss of generality. From v and following the proof of Chateauneuf's theorem that appears in Bridges and Mehta [6], p. 107, we can construct a real-valued function u that is upper semicontinuous and such that (u, v) represents \prec . From Proposition 6.2.3 in Bridges and Mehta [6] it follows now that \prec is continuous. Then Lemma 6.5.3 in Bridges and Mehta [6] states that \prec is strongly separable, and finally the proof of Theorem 6.5.5 in Bridges and Mehta [6] furnishes a *continuous* representation (U, V) for \prec . A final glance to this process tells us that from just a utility function for \lesssim^* , even discontinuous, we are able to get a *continuous* representation (U, V) for the interval order \prec . This is the kind of ideas underlying in the classical Debreu's open gap lemma.

- v) The equation $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ is known as the *functional equation of separability* because it corresponds to maps F defined in $X \times X$ such that F can be *separated* as the sum of two functions of only one variable each, that is $F(x, y) = G(x) + H(y)$; $G, H : X \rightarrow \mathbb{R}$. (For more details, consult Aczél and Dhombres [1]).
- vi) In Theorem 3.5 connectedness cannot be ruled out, nor even in the particular case of a total preorder \lesssim on a set X on which we consider the order topology. For instance, it is not difficult to see that the order topology given by the lexicographic ordering on $\mathbb{R} \times \{0, 1\}$ is separable, but this ordering is not representable.
- vii) Similarly to Eilenberg's theorem, the classical Debreu's theorem on second countability (see Debreu [12, 13] or else Ch. 3 in Bridges and Mehta [6]) states that a continuous total preorder always admit a continuous utility function that represents it. Looking for an extension of this result to the context of *interval orders*, we must observe that the equivalent condition xii) that appears in the statement of Theorem 3.5 proves such a fact for the particular case of *connected* topological spaces. Also, the continuity here is considered with respect to a natural topology. It is an *open question* to prove or disprove the existence of continuous representations for interval orders defined on a nonempty set X on which we consider either the topology θ , or more generally a natural topology, for which X is second countable. That is: the task would consist in proving the equivalence $i) \iff xii)$ in Theorem 3.5

making no use of connectedness, or else disprove it by an adequate counterexample.

- viii) Some steps in the proof of Theorem 3.5 could be shortened if \prec is a *semiorder* instead of just an interval order. For instance, if X is a nonempty set endowed with a semiorder \prec , *even without assuming connectedness* then, it is not difficult to prove that the strong separability of \prec carries the topological separability of θ_1 , the topology on X whose subbasis is $\Sigma_1 = \{U(a) : a \in X\} \cup \{L(b) : b \in X\}$. Indeed, if x, y in X are such that $x \prec y$ and $U_{\prec}(x) \cap L_{\prec}(y) \neq \emptyset$, we have that there exists $z \in X$ such that $x \prec z \prec y$. Let $D \subseteq X$ be a countable subset that furnishes the strong separability of \prec . Then there exist $d_1, d_2 \in D$ such that $x \prec d_1 \prec^{**} z \prec d_2 \prec^{**} y$. By definition of \prec^{**} , there exist $a, b \in X$ such that $x \prec d_1 \prec a \prec z \prec d_2 \prec b \prec y$. By the generalized pseudotransitivity of the semiorder \prec , where in particular the fact “ $\prec \prec$ ” implies “ \prec ”, it follows now that $x \prec d_1 \prec d_2 \prec b \prec y \implies x \prec d_1 \prec y$, so that θ_1 is separable.

A remarkable question that we must consider now concerns the *existence of continuous representations for interval orders in the general case* (i.e.: not necessarily connected). In the case of a total preorder, the crucial Debreu’s open gap theorem provides continuous representations once we have a utility representation. But, as far as we know, the validity or not of a similar result for interval orders is still an *open problem*. Anyways, we can study some cases different from the connected case.

Definition 3.7. *An interval order \prec on a nonempty set X is said to be dense-in-itself if for every $x, y \in X$ with $x \prec y$ there exists $z \in X$ such that $x \prec z \prec y$.*

Proposition 3.8. *Let X be a nonempty set endowed with an interval order \prec and a natural topology τ for which X is connected. Then it holds that either the interval order \prec is dense-in-itself or there exist elements $x, y, z \in X$ such that $x \prec y \prec z \prec x$. In particular, if \prec is a total preorder, \prec is dense-in-itself.*

Proof. Actually, if $x, y \in X$ are such that $x \prec y$ and there is no $z \in X$ such that $y \prec z \prec x$ then it must exist an element $a \in X$ with $x \prec a \prec y$ since otherwise the open and nonempty subsets $L_{\prec}(y)$ and $U_{\prec}(x)$ would give raise to a disconnection of the set X , in contradiction with the hypothesis of connectedness.

In the particular case of a total preorder, the situation $x \prec y \prec z \prec x$ would imply $x \prec x$, against the irreflexivity of the binary relation \prec . \square

An important fact that we can prove at this point, states that on suitable topologies, representable interval-orders that are dense-in-itself admit a *continuous* representation.

Theorem 3.9. *A dense-in-itself and representable interval order \prec defined on a nonempty set X endowed with a natural topology τ has a continuous representation.*

Proof. It suffices to prove the existence of a continuous representations when the topology considered on X is θ . Let $a, b \in X$ with $a \prec^* b$. By definition of \prec^* there is an element $c \in X$ such that $a \prec c \preceq b$. Since \prec is dense-in-itself, there also exists an element $d \in X$ such that $a \prec d \prec c \preceq b$. Thus $a \prec d \prec^* b$ which implies $a \prec^* d \prec^* b$. Therefore \prec^* is also dense-in-itself. In a completely analogous way we can prove that \prec^{**} is dense-in-itself, too. Since \prec is representable, the associated total preorders \preceq^* and \preceq^{**} are representable. (See Bosi et al. [3]). Since \prec^* is dense-in-itself, standard techniques of construction of utility functions for total preorders allow us to consider a utility function v for \prec^* with the additional property that $(0, 1) \cap \mathbb{Q} \subseteq v(X) \subseteq [0, 1]$, so that $v(X)$ is dense in $[0, 1]$ with respect to the Euclidean topology on \mathbb{R} . (See, e.g., Birkhoff [2], p. 200 or else Candeal and Induráin [7, 8, 9]). The proof concludes as in Bridges and Mehta [6], p.107, proof of the continuity of the representation in Chateauneuf’s theorem. \square

Definition 3.10. *An interval order \prec on a nonempty set X endowed with a topology τ is said to be τ -locally non-satiated (see, e.g. Bridges and Mehta [6], pp. 33 and 93) if for every $x \in X$ it holds that $U_{\prec}(x)$ meets every τ -neighbourhood of x .*

If τ is a natural topology, \prec is τ -locally non-satiated, and $a, b \in X$ are such that $a \prec b$, then since $L_{\prec}(b)$ is a neighbourhood of a there must exist an element $c \in U_{\prec}(a) \cap L_{\prec}(b)$. In particular, \prec is dense-in-itself.

Corollary 3.11. *A representable and τ -locally non-satiated interval order \prec defined on a nonempty set X endowed with a natural topology τ has a continuous representation.*

A deep analysis of the proof of Theorem 6.5.5 in Bridges and Mehta [6] tell us that the key to finally get a continuous representation (U, V) for an interval order \prec defined on a nonempty set X which we shall consider endowed with a natural topology τ is nothing else but the fact that *the associated total preorder \prec^* is dense-in-itself* (dually, it is also true that under the same conditions for X , when the associated total preorder \prec^{**} is dense-in-itself, the interval order \prec admit a continuous representation (U, V)). This fact implies important consequences:

Theorem 3.12. *Let X be a nonempty set endowed with an interval order \prec and a natural topology τ . Then if \prec is strongly separable it admits a continuous representation (U, V) .*

Proof. By the previous comments, it is enough to prove that \prec^* is dense-in-itself. Let $x, y \in X$ be such that $x \prec^* y$. It follows, by definition of \prec^* , that there exists an element $z \in X$ such that $x \prec z \preceq y$. Since \prec is strongly

separable, there exists now an element $a \in X$ such that $x \prec^* a \prec z \succ y$. Therefore $x \prec^* a \prec^* y$, and we are done. \square

Remark 3.13.

- i) It is an *open problem* to prove whether or not Theorem 3.12 remains valid when the condition of strong separability is substituted for the weaker one of i.o.-separability.
- ii) *The converse of Theorem 3.12 is not true:* A clear example is the set of natural numbers \mathbb{N} endowed with the usual Euclidean total ordering and the discrete topology. Considered as a particular case of interval order, such ordering is i.o.-separable, but not strongly separable. And it is plain that it admits a continuous representation.

In what concerns the *techniques* used to obtain *continuous representations of structures of interval order*, the first impression is that they are different from the main techniques used to just get representations (not necessarily continuous). Let us explain why:

The *first technique*, based on constructions using the auxiliar sets $A(x)$, $B(x)$, leans on numerical *series*. (See e.g. Bridges and Mehta [6], pp. 88 and ff. , or else Bosi et al. [3]). However, it is well known that functions defined through series fail to be continuous in a countable set of points (for instance, if $(q_n)_{n=1}^{\infty}$ is an enumeration of the set \mathbb{Q} of rational numbers, the function $F : \mathbb{R} \rightarrow \mathbb{R}$ given by $F(x) = \sum_{\{k \in \mathbb{N} : q_k \leq x\}} 2^{-k}$ fails to be continuous at each rational number). Consequently, the first technique is so good to obtain representations of interval orders, but it is not so good to get *continuous* representations.

The *third technique*, introduced in Olóriz et al. [27] and based on the theory of *functional equations* presents similar problems than the first technique, because the typical constructions of the bivariate map $F : X \times X \rightarrow \mathbb{R}$ that satisfies $x \prec y \iff F(x, y) > 0$ and is a solution of the *functional equation of separability* $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ ($x, y \in X$), are also based on suitable numerical *series* that could lead to the *discontinuity* of F .

Fortunately, the *second technique*, based on *measure theoretical constructions* may lead to a *continuous* representation, as proved in Bosi and Isler [4].

On the other hand, it is clear that the techniques used to prove Theorem 3.5 above, understood as a generalization of Chateauneuf's theorem (as proved e.g. in Bridges and Mehta [6], pp. 106-107) are different from the three techniques, just cited, used to get representations (continuous or not) of interval-orders. The key now is obtaining a continuous representation v for \prec^* (making use of Debreu's open gap lemma, eventually) and then use v to get a continuous representation u for \prec^{**} in such a way that (u, v) is a continuous representation for \prec .

Acknowledgements. This work has been partially supported by the research project PB98-551 “Estructuras ordenadas y aplicaciones”. Ministerio de Educación y Cultura. España. (Dec. 1999).

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RECEIVED MAY 2003

ACCEPTED AUGUST 2003

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