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# Resolvability of ball structures

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Abstract. A ball structure is a triple $\mathbb{B} = (X, P, B)$ where $X$, $P$ are nonempty sets and, for any $x \in X$, $\alpha \in P$, $B(x, \alpha)$ is a subset of $X$, which is called a ball of radius $\alpha$ around $x$. It is supposed that $x \in B(x, \alpha)$ for any $x \in X$, $\alpha \in P$. A subset $Y \subseteq X$ is called large if $X = B(Y, \alpha)$ for some $\alpha \in P$ where $B(Y, \alpha) = \bigcup_{y \in Y} B(y, \alpha)$. The set $X$ is called a support of $\mathbb{B}$, $P$ is called a set of radiuses. Given a cardinal $\kappa$, $\mathbb{B}$ is called $\kappa$-resolvable if $X$ can be partitioned to $\kappa$ large subsets. The cardinal $res\ \mathbb{B} = sup\ \{\kappa : \mathbb{B} \text{ is } \kappa\text{-resolvable}\}$ is called a resolvability of $\mathbb{B}$. We determine the resolvability of the ball structures related to metric spaces, groups and filters.

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## 1. Introduction

Let $\mathbb{B}_1 = (X_1, P_1, B_1)$ and $\mathbb{B}_2 = (X_2, P_2, B_2)$ be ball structures, $f\colon X_1 \longrightarrow X_2$. We say that $f$ is a $\prec$-*mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that $f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$ for every $x \in X$. A bijection $f$ is called an *isomorphism* if $f$ and $f^{-1}$ are $\prec$-mappings.

The results from [10], [11], [12] show that the ball structures (with the isomorphisms defined above) are the natural asymptotic counterparts of topological spaces. A good motivation to study ball structures related to metric spaces is in the survey [5].

A topological spaces is called $\kappa$-*resolvable* ($\kappa$ is a cardinal) if it can be partitioned to $\kappa$ dense subsets. For resolvability of topological spaces and topological groups see the surveys [3], [4], [9].

Let $\mathbb{B} = (X, P, B)$ be a ball structure. A subset $Y \subseteq X$ is called *large* if there exists $\alpha \in P$ such that $X = B(Y, \alpha)$. The large subsets of ball structure can be considered as the duplicates of the dense subspaces of topological space. Given a cardinal $\kappa$, we say that $\mathbb{B}$ is $\kappa$-*resolvable* if $X$ can be partitioned to $\kappa$ large subsets. The *resolvability* of $\mathbb{B}$ is the cardinal

$$res\mathbb{B} = \sup\{\kappa : \mathbb{B} \text{ is } \kappa - \text{resolvable}\}.$$

A subset $Y \subseteq X$ is called *small* if $X \setminus B(Y, \alpha)$ is large for every $\alpha \in P$. The small subsets of ball structure can be considered as the duplicates of the nowhere dense subsets of topological space. Assume that every singleton of $X$ is small. Given a cardinal $\kappa$, we say that $\mathbb{B}$ is *$\kappa$-coresolvable* if $X$ can be covered by $\kappa$ small subsets. The *coresolvability* of $\mathbb{B}$ is the cardinal

$$cores\mathbb{B} = min\ \{\kappa : \mathbb{B} \text{ is} \kappa - \text{coresolvable}\}.$$

The referee pointed out that the coresolvability can be considered as the asymptotic duplicate of the Novak number of topological space $X$

$$n(X) = min\{|\mathcal{U}| : \mathcal{U} \text{ is a cover of } X \text{ consisting of nowhere dense subsets}\}.$$

In this paper we determine (or evaluate) the cardinal invariants $res\ \mathbb{B}$ and $cores\ \mathbb{B}$ for a wide spectrum of ball structures $\mathbb{B}$ related to metric spaces, groups and filters. We begin with exposition of results (2), continue with proofs (3) and conclude the paper with comments and open problems (4).

All ball structures under consideration are supposed to be uniform. A ball structure $\mathbb{B} = (X, P, B)$ is called *uniform* if $\mathbb{B}$ is symmetric and multiplicative. We say that $\mathbb{B}$ is *symmetric* if, for every $\alpha \in P$, there exists $\beta \in P$ such that $B(x, \alpha) \subseteq B^*(x, \beta)$ for every $x \in X$ and vice versa, where $B^*(x, \beta) = \{y \in X : x \in B(y, \beta)\}$. A ball structure $\mathbb{B}$ is called *multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma(\alpha, \beta) \in P$ such that $B(B(x, \alpha), \beta) \subseteq B(x, \gamma(\alpha, \beta))$ for every $x \in X$. Note that if $\mathbb{B}$ is uniform and $Y \subseteq X$ is large, then there is $\gamma \in P$ such that $B(x, \gamma) \bigcap Y \neq \varnothing$ for all $x \in X$. For more detailed information concerning the uniform ball structures as the asymptotic counterparts of the uniform topological spaces see [11]. Initially, the problem of resolvability of ball structures was motivated by the following question [1]: can every infinite group be partitioned onto two large subsets? For the positive answer to this question see [14] or [15].

## 2. Results

Let $\mathbb{B} = (X, P, B)$ be a ball structure, $\kappa$ be a cardinal. We say that a subset $Y \subseteq X$ is *$\kappa$-crowded* if there exists $\alpha \in P$ such that $|B(y, \alpha) \bigcap Y| \geq \kappa$ for every $y \in Y$. A ball structure $\mathbb{B}$ is called *$\kappa$-crowded* if its support $X$ is $\kappa$-crowded. The *crowdedness* of $\mathbb{B}$ is the cardinal

$$cr\ \mathbb{B} = sup\ \{\kappa : \mathbb{B} \text{ is} \kappa - \text{crowded}\}.$$

Define the preodering $\leq$ on $P$ by the rule $\alpha \leq \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$, there exists $\beta \in P'$ such that $\alpha \leq \beta$. The *cofinality* $cf\ \mathbb{B}$ is the minimal cardinality of the cofinal subsets of $P$.

**Proposition 2.1.** *For every ball structure $\mathbb{B} = (X, P, B)$, the following statements hold*

*(i) if $\mathbb{B}$ is $\kappa$-crowded, then $\mathbb{B}$ is $\kappa$-resolvable;*

*(ii) $cr\ \mathbb{B} \leq\ res\ \mathbb{B} \leq cr\ \mathbb{B} \cdot cf\ \mathbb{B}$;*

*(iii) if $\kappa$ is a finite cardinal and $\mathbb{B}$ is $\kappa$-resolvable, then $\mathbb{B}$ is $\kappa$-crowded.*

By van Douwen-Illanes' theorem [7], if a topological space is $n$-resolvable for every natural number $n$, then it can be partitioned to countably many dense subsets. The referee pointed out that the generalization of van Douwen-Illanes' theorem for the case of countable cofinality lies in [2]. In view of Proposition 2.1 (iii), the following statement can be considered as the analogue of this generalization.

**Proposition 2.2.** *Let $\mathbb{B} = (X, P, B)$ be a ball structure, $< \kappa_n >_{n\in\omega}$ be an increasing sequence of cardinals, $\kappa = sup\ \{\kappa_n : n \in \omega\}$. If $\mathbb{B}$ is $\kappa_n$-crowded for every $n \in \omega$, then $X$ can be partitioned in $\kappa$ large subsets.*

Let $(X, d)$ be a metric space. For any $x \in X$, $r \in \mathbb{R}^+$, put

$$B_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

The ball structure $(X, \mathbb{R}^+, B_d)$ induced by the metric space $(X, d)$ is denoted by $\mathbb{B}(X, d)$. A ball structure $\mathbb{B}$ is called metrizable if $\mathbb{B}$ is isomorphic to $\mathbb{B}(X, d)$ for the appropriate metric space $(X, d)$. By [8], $\mathbb{B}$ is metrizable if and only if $\mathbb{B}$ is uniform, connected and $cf\ \mathbb{B} \leq \aleph_0$. A ball structure $\mathbb{B} = (X, P, B)$ is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $x \in B(y, \alpha)$, $y \in B(x, \alpha)$.

**Theorem 2.3.** *For every metric space $(X, d)$, $res\ \mathbb{B}(X, d) = cr\ \mathbb{B}(X, d)$ and $X$ can be partitioned in $cr\ \mathbb{B}(X, d)$ large subsets.*

Let $G$ be an infinite group with the identity $e$ and let $\kappa$ be an infinite cardinal with $\kappa \leq |G|$. Denote by $\Im(G, \kappa)$ the family of all subsets of $G$ of cardinality $< \kappa$ containing $e$. For all $g \in G$, $F \in \Im(G, \kappa)$, put

$$B_l(g, F) = Fg,\ B_r(g, F) = gF.$$

Denote by $\mathbb{B}_l(G, \kappa)$ and $\mathbb{B}_r(G, \kappa)$ the ball structures $(G, \Im(G, \kappa), B_l)$ and $(G, \Im(G, \kappa), B_r)$. Observe that the mapping $g \longmapsto g^{-1}$ is an isomorphism between $\mathbb{B}_l(G, \kappa)$ and $\mathbb{B}_r(G, \kappa)$. We say that a subset $Y \subseteq X$ is left $\kappa$-large (left $\kappa$-small) if $Y$ is large (small) in the ball structure $\mathbb{B}_l(G, \kappa)$. In other words, $Y$ is called left $\kappa$-large if there exists a subset $F \subseteq G$ such that $|F| < \kappa$ and $G = FY$.

**Theorem 2.4.** *Let $G$ be an infinite group, $\kappa$ be an infinite cardinal and $\kappa \leq |G|$. Then $G$ can be partitioned in $\kappa$ left $\kappa$-large subsets.*

Let $\mathbb{B} = (X, P, B)$ be a ball structure. A subset $Y \subseteq X$ is called *bounded* if there exist $x \in X$, $\alpha \in P$ such that $Y \subseteq B(x, \alpha)$. We say that $\mathbb{B}$ is *bounded* if $X$ is bounded. Assume that $\mathbb{B}$ is unbounded and connected. Then every bounded subset of $X$ is small. Since $X = \bigcup\{B(x, \alpha) : \alpha \in P\}$ for every $x \in X$, we conclude that $cores\ \mathbb{B} \leq cf\ \mathbb{B}$. In particular, $cores\ \mathbb{B}(X,d) \leq \aleph_0$ for every unbounded metric space $(X,d)$. On the other hand, the family of all small subsets of an arbitrary ball structure is an ideal in the Boolean algebra of all subsets of $X$ (see [11]). Thus, $cores\ \mathbb{B} \geq \aleph_0$ for every unbounded ball structure $\mathbb{B}$. Hence, $cores\ \mathbb{B}(X,d) = \aleph_0$ for every unbounded metric space $(X,d)$. The following theorem shows that $cores\ \mathbb{B}$ could be much more less than $cf\ \mathbb{B}$.

**Theorem 2.5.** *Let $G$ be an infinite group, $\kappa$ be an infinite cardinal and $\kappa \leq |G|$. If $\kappa < cf\ (|G|)$, then $cores\ \mathbb{B}_l(G,\kappa) = \aleph_0$.*

Let $G$ be a topological group, $\mathcal{C}(G)$ be a family of all compact subsets of $G$ containing the identity of $G$. A ball structure $(G, \mathcal{C}(G), B_l)$ is denoted by $\mathbb{B}_l(G,\mathcal{C})$. Clearly, $\mathbb{B}_l(G,\mathcal{C}) = \mathbb{B}_l(G,\aleph_0)$ for every discrete group $G$.

**Theorem 2.6.** *Let $G$ be a non-compact locally compact group, then $cores\ \mathbb{B}_l(G,\mathcal{C}) = \aleph_0$.*

Let $X$ be a set and let $\varphi$ be a filter on $X$ such that $\bigcap \varphi = \varnothing$. For any $x \in X$, $F \in \varphi$, put

$$B(x,F) = \begin{cases} X \backslash F, & \text{if } x \notin F; \\ \{x\}, & \text{if } x \in F; \end{cases}$$

and denote by $\mathbb{B}(X,\varphi)$ the ball structure $(X,\varphi,B)$.

**Theorem 2.7.** *Let $X$ be a set, $\varphi$ be a filter on $X$ such that $\bigcap \varphi = \varnothing$. Then $res\ \mathbb{B} = 1$, $cores\ \mathbb{B} = min\ \{|\psi| : \psi \subseteq \varphi,\ \bigcap \psi = \varnothing\}$.*

## 3. Proofs

*Proof of Proposition 2.1.* (i) Choose $\alpha \in P$ such that $|B(x,\alpha)| \geq \kappa$ for every $x \in X$. By Zorn lemma, there exists a subset $Y \subseteq X$ such that the family $\{B(y,\alpha) : y \in Y\}$ is pairwise disjoint and, for every $x \in X$, there exists $y \in Y$ such that $B(x,\alpha) \bigcap B(y,\alpha) \neq \varnothing$. Since $|B(y,\alpha)| \geq \kappa$ for every $y \in Y$, there exists a family $\Im$ of $\kappa$-many pairwise disjoint subsets of $X$ such that $|F \bigcap B(y,\alpha)| = 1$ for all $y \in Y$ and $F \in \Im$. In view of uniformity of $\mathbb{B}$ and by choice of $Y$, every subset $F \in \Im$ is large. Hence, $\mathbb{B}$ is *$\kappa$-resolvable.*

$(ii)$ The left inequality follows from $(i)$. Let $\Im$ be an arbitrary pairwise disjoint family of large subsets of $X$. Pick a cofinal subset $P' \subseteq P$ with $|P'| = cf\ \mathbb{B}$. For every $\alpha \in P'$, put

$$\Im(\alpha) = \{F \in \Im : B^*(F,\alpha) = X\},$$

where $B^*(F,\alpha) = \bigcup_{x \in F} B^*(x,\alpha)$.

Take any $x \in X$ and $\alpha \in P'$. Since $B(x,\alpha) \bigcap F \neq \varnothing$ for every $F \in \Im(\alpha)$, we have $|\Im(\alpha)| \leq |B(x,\alpha)|$. Hence, $|\Im(\alpha)| \leq cr\ \mathbb{B}$ and the right inequality holds.

$(iii)$ Let $F_1, F_2, ..., F_m$ be pairwise disjoint large subsets of $X$. Choose $\alpha \in P$ such that $B^*(F_i,\alpha) = X$ for every $i \in \{1,2,...,m\}$. Then $B(x,\alpha) \bigcap F_i \neq \varnothing$ for all $x \in X$ and $i \in \{1,2,...,m\}$. It follows that $|B(x,\alpha)| \geq m$ and $\mathbb{B}$ is $m$-crowded. $\square$

*Proof of Proposition 2.2.* It suffices to partition $X = Y \bigcup Z$ so that $Y$ is a disjoint union of $\kappa_0$ large subset and $Z$ is $\kappa_n$-crowded for every $n \in \omega$. We may suppose that $\kappa_0 > 0$. Choose $\alpha \in P$ such that $|B(x, \alpha)| \geq 2\kappa_0$ for every $x \in X$. By Zorn lemma, there exists a subset $A \subseteq X$ such that $\{B(a, \alpha) : a \in A\}$ is a maximal disjoint family. For every $a \in A$, partition $B(a, \alpha) = C(a) \bigcup D(a)$ so that $|C(a)| = \kappa_0$, $|D(a)| \geq \kappa_0$. Put $Y = \bigcup_{a \in A} C(a)$, $Z = X \backslash Y$ and note that $Y$ can be partitioned in $\kappa_0$-many large subsets. Fix $n \in \omega$ and choose $\beta \in P$ such that $|B(x, \beta)| \geq 2\kappa_n$ for every $x \in X$. Since $\mathbb{B}$ is multiplicative, there exists $\gamma \in P$ such that $B(B(x, \beta), \alpha) \subseteq B(x, \gamma)$ for every $x \in X$. Then $|B(z, \gamma) \bigcap Z| \geq \kappa_n$ for every $z \in Z$ and $Z$ is $\kappa_n$-crowded. □

*Proof of Theorem 2.3.* Since $cf\ \mathbb{B}(X, d) = \aleph_0$, the first statement follows from Proposition 2.1. The second statement follows from Proposition 2.2. □

To prove the next three theorems we use the filtrations of groups.

Let $G$ be an infinite group with the identity $e$. A *filtration* of $G$ is a family $\{G_\alpha : \alpha < |G|\}$ of subgroups of $G$ such that

(*i*) $G_0 = \{e\}$, $G = \bigcup\{G_\alpha : \alpha < |G|\}$;

(*ii*) $G_\alpha \subset G_\beta$ for all $\alpha < \beta < |G|$;

(*iii*) $\bigcup\{G_\alpha : \alpha < \beta\} = G_\beta$ for every limit ordinal $\beta$;

(*iv*) $G_\alpha < |G|$ for every $\alpha < |G|$.

Using a minimal well-ordering of $G$ it is easy to construct a filtration of $G$ provided that $G$ is not finitely generated. In particular, every uncountable group admits a filtration.

For each $\alpha < |G|$, decompose $G_{\alpha+1} \setminus G_\alpha$ to right cosets by $G_\alpha$ and fix some set $X_\alpha$ of representatives so $G_{\alpha+1} \setminus G_\alpha = G_\alpha X_\alpha$. Take an arbitrary element $g \in G$, $g \neq e$ and choose the smallest subgroup $G_\alpha$ with $g \in G_\alpha$. By (*iii*), $\alpha = \alpha_1 + 1$ for some ordinal $\alpha_1 < |G|$. Hence, $g \in G_{\alpha_1+1} \setminus G_{\alpha_1}$ and there exist $g_1 \in G_{\alpha_1}$, $x_{\alpha_1} \in X_{\alpha_1}$ such that $g = g_1 x_{\alpha_1}$. If $g_1 \neq e$, we choose the ordinal $\alpha_2$, the elements $g_2 \in G_{\alpha_2+1} \setminus G_{\alpha_2}$ and $x_{\alpha_2} \in X_{\alpha_2}$ such that $g_1 = g_2 x_{\alpha_2}$. Since the set of ordinals $< |G|$ is well-ordered, after finite number of steps we get the representation

$$g = x_{\alpha_{s(g)}} x_{\alpha_{s(g)-1}} ... x_{\alpha_2} x_{\alpha_1}, \quad \alpha_{s(g)} < ... < \alpha_1, x_{\alpha_i} \in X_{\alpha_i}.$$

Note that this representation is unique and put

$$\gamma_1(g) = \alpha_1, \quad \gamma_2(g) = \alpha_2, ..., \gamma_{s(g)}(g) = \alpha_{s(g)}, \quad \Gamma(g) = \{\gamma_1(g), ..., \gamma_{s(g)}(g)\}.$$

For every natural number $n$, denote

$$D_n = \{g \in G : s(g) = n\}.$$

*Proof of Theorem 2.4.* First suppose that $|G| = \kappa$. If $G$ is countable, then $\mathbb{B}_l(G, \kappa)$ is metrizable and we can apply Theorem 2.3. Assume that $G$ is uncountable and use the above filtration. For every $\alpha < |G|$, put

$$F_\alpha = \{g \in G : \gamma_{s(g)}(g) = \alpha\}$$

and note that $\{F_\alpha : \alpha < G\}$ is a pairwise disjoint family of left $\kappa$-large subsets.

If $\kappa < |G|$, we choose a subgroup $H$ of $G$ with $|H| = \kappa$. By above paragraph, there exists a partition $\mathcal{P}$ of $H$ such that each subset $P \in \mathcal{P}$ is large in $\mathbb{B}_l(H, \kappa)$. Decompose $G$ to right cosets by $H$ and fix some set $X$ or representatives so $G = HX$. Then $\{PX : P \in \mathcal{P}\}$ is a pairwise disjoint family of left $\kappa$-large subsets of $G$. $\square$

*Proof of Theorem 2.5.* If $G$ is countable, then $\mathbb{B}_l(G, \kappa)$ is metrizable and we have *cores* $\mathbb{B}_l(G, \kappa) = \aleph_0$. Suppose that $G$ is uncountable and use the above filtration. Observe that $G \setminus \{e\} = \bigcup_{n=1}^{\infty} D_n$, fix a natural number $n$ and show that $D_n$ is small in $\mathbb{B}_l(G, \kappa)$. Take an arbitrary subset $F \in \Im(G, \kappa)$. By assumption, there exists $\beta \in P$ such that $F \subseteq G_\beta$ so $FD_n \subseteq G_\beta D_n$. Show that $G \setminus G_\beta D_n$ is left $\aleph_0$-large. Choose the elements $a_1, a_2, ..., a_{n+1}$ of $G$ such that

$$\alpha_1 \in G_{\beta+1} \setminus G_\beta, a_2 \in G_{\beta+2} \setminus G_{\beta+1}, ..., a_{n+1} \in G_{\beta+n+1} \setminus G_{\beta+n}.$$

Take an arbitrary element $g \in G_\beta D_n$ and put $g = g_0$. If $\beta + n \in \Gamma(g)$, put $\varepsilon_0 = 0$, otherwise $\varepsilon_0 = 1$. Note that $\beta + n \in \Gamma(a_{n+1}^{\varepsilon_0} g_0)$ and put $g_1 = a_{n+1}^{\varepsilon_0} g_0$. If $\beta + n - 1 \in \Gamma(g_1)$, we put $\varepsilon_1 = 0$, otherwise $\varepsilon_1 = 1$. Note that $\{\beta + n - 1, \beta + n\} \subseteq \Gamma(a_{n+1}^{\varepsilon_1} g_1)$ and put $g_2 = a_n^{\varepsilon_1} g_1$. After $n + 1$ steps we get

$$\{\beta, \beta + 1, ..., \beta + n\} \subseteq \Gamma(a_1^{\varepsilon_n} a_2^{\varepsilon_{n-1}} ... a_{n+1}^{\varepsilon_0} g).$$

It follows that $(a_1^{\varepsilon_n} a_2^{\varepsilon_{n-1}} ... a_{n+1}^{\varepsilon_0} g) \notin G_\beta D_n$. Put $A = \{e, a_1, a_2, ..., a_{n+1}\}$, $K = A^n$. We have shown that $G_\beta D_n \subseteq K^{-1}(G \setminus G_\beta D_n)$. Hence, $G = K^{-1}(G \setminus G_\beta D_n)$ and $G \setminus G_\beta D_n$ is left $\aleph_0$-large. $\square$

*Proof of Theorem 2.6.* If $G$ is $\sigma$-compact, then $cf\ \mathbb{B}_l(G, \mathcal{C}) = \aleph_0$ and $\mathbb{B}_l(G, \mathcal{C})$ is metrizable. Hence, *cores* $\mathbb{B}_l(G, \mathcal{C}) = \aleph_0$. Assume that $G$ is not $\sigma$-compact. Then we can easily construct a filtration $\{G_\alpha : \alpha < |G|\}$ so that every subgroup $G_\alpha$, $\alpha > 0$ is open. Repeat the arguments proving Theorem 2.5. $\square$

*Proof of Theorem 2.7.* Two easy observations. A subset $Y \subseteq X$ is large if and only if $Y \in \varphi$. A subset $Y \subseteq X$ is small if and only if $X \setminus Y$ is large. $\square$

## 4. Comments and open problems

**Problem 4.1.** *Let $\mathbb{B} = (X, P, B)$ be a ball structure, $\kappa$ be a cardinal such that $\mathbb{B}$ is $\kappa'$-crowded for every $\kappa' < \kappa$. Can $X$ be partitioned in $\kappa$ large subsets? By Proposition 2.2, this is so if $cf \kappa = \aleph_0$.*

**Problem 4.2.** *Let $G$ be an infinite group, $\kappa$ be an infinite cardinal, $\kappa \leq |G|$. Can $G$ be $\kappa$-partitioned so that each cell of the partition is left and right $\kappa$-large? If $\kappa = \aleph_0$, this is so [14].*

Let $G$ be an infinite amenable (in particular, Abelian) group, $\mu$ be a Banach measure on $G$. Clearly, $\mu(A) > 0$ for every left $\aleph_0$-large subset $A$ of $G$. It follows, that *res* $\mathbb{B}_l(G, \aleph_0) = \aleph_0$. On the other hand, every free group of infinite rank $\kappa$ can be partitioned in $\kappa$ left $\aleph_0$-large subsets [11].

**Problem 4.3.** *Let $G$ be a free Abelian group of rank $\aleph_2$. Can $G$ be partitioned in $\aleph_2$ $\aleph_1$-large subsets.*

**Problem 4.4.** *Let $G$ be an infinite group, $\kappa$ be an infinite cardinal, $\kappa \leq |G|$. Can $G$ be partitioned in $\aleph_0$ left $\kappa$-small subsets? By Theorem 2.5, this is so if $|G|$ is a regular cardinal.*

A topological space is called *irresolvable* if it can not be partitioned in two dense subsets. Let us say that a ball structure $\mathbb{B}$ is *irresolvable* if $res\mathbb{B} = 1$. By Proposition 2.1, $\mathbb{B}$ is irresolvable if and only if $cr\mathbb{B} = 1$.

A topological space $X$ is called *$\kappa$-extraresolvable* if $X$ admits a family $\Im$, $|\Im| = \kappa$ of dense subsets such that $F_1 \bigcap F_2$ is nowhere dense for all distinct subsets $F_1, F_2 \in \Im$. It is important to remark that if $1 < \kappa < \omega$ then $\kappa$-extraresolvability is equivalent to $\kappa$-resolvability. The concept of $\kappa$-extraresolvability was introduced by V. I. Malykhin [8]. As the referee pointed out, the published paper where this concept appears for the first time in the literature is [6].

Let us say that a ball structure $\mathbb{B} = (X, P, B)$ is *$\kappa$-extraresolvable* if $X$ admits a family $\Im$, $|\Im| = \kappa$ of large subsets such that $F_1 \bigcap F_2$ is small for all distinct subsets $F_1, F_2 \in \Im$.

If $\mathbb{B}$ is unbounded and $\kappa$-crowded, then there exists a family $\Im$, $|\Im| = \kappa^{\aleph_0}$ of large subset of $X$ such that $F_1 \bigcap F_2$ is finite for all distinct subsets $F_1, F_2 \in \Im$, so $\mathbb{B}$ is $\kappa^{\aleph_0}$-extraresolvable. The extraresolvability of $\mathbb{B}$ is the cardinal $sup\ \{\kappa : \mathbb{B}$ is $\kappa$-extraresolvable$\}$.

**Problem 4.5.** *Determine (or evaluate) extraresolvability of ball structures of metric spaces and groups.*

The referee asked "what about the infinite case in (iii) of Proposition 2.1?" Let $k$ be an uncountable ordinal and let $G$ be a free group of rank $k$. By [13], $\mathbb{B}_l(G, \aleph_0)$ can be partitioned in $k$-many large subsets, but $cr\ \mathbb{B}_l(G, \aleph_0) = \aleph_0$. On the other hand, the statement (iii) remains true for $k = \aleph_0$.

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