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On the cardinality of indifference classes

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Abstract. Let " $\precsim$ " be a continuous total preorder on some topological space (X, t). Then the cardinality or at least lower and upper bounds of the cardinality of the indifference (equivalence) classes of " $\precsim$ " will be computed. In addition, the relevance of these bounds in mathematical utility theory and the theory of orderable topological spaces will be discussed.

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1. Introduction

Let (X, t) be a topological space that is endowed with a *total preorder* " $\precsim$ ". The reader may recall that a total preorder " $\precsim$ " on X is a reflexive and transitive relation on X that satisfies the additional property that for all pairs $(x, y) \in X^2$ at least one of the relations $x \precsim y$ or $y \precsim x$ holds. The relation $x \sim y \Longleftrightarrow x \precsim y \wedge y \precsim x$ defines an equivalence relation on X. The corresponding eqivalence classes are the *indifference* classes of " $\precsim$ ". The *order topology* $t^{\precsim}$ on X that is induced by " $\precsim$ " is generated by the sets $L(x) := \{y \in X | y \prec x\}$ and $K(x) := \{z \in X | x \prec z\}$ where x runs through X.

In mathematical utility theory usually *continuous* total preorders " $\precsim$ " on X are considered, which means that the order topology $t^{\precsim}$ is coarser than t.

Ideally, the indifference classes of a continuous total preorder " $\precsim$ " on (X, t) only consist of single points. This means from the viewpoint of mathematical utility theory that different alternatives are perfectly distinguishable by preferences (utilities) and from the viewpoint of pure mathematics that the given topology t on X is *orderable*. The reader may notice that in our terminology *orderability* only means that $t^{\precsim} \subset t$. If also $t \subset t^{\precsim}$ then we speak of a *strictly orderable* topology t on X (cf. Kok [12]). Of course, in general, indifference classes are *thick*. This means that, in general, one cannot escape from the fact that one has to deal with a preorder.

In general, a topology t on X is not orderable. Indeed, let $n > 1$ be any natural number. Then the natural topology t_{nat} on $\mathbb{R}^n$ is not orderable (cf. Section 2 and the nice results in Beardon [2], Candeal and Induráin [6] and Candeal, Induráin and Mehta [7]).

Clearly, a topology t on X is not orderable if and only if every continuous total preorder " $\precsim$ " on X has at least one indifference class that contains more than one point. In this case we may select the *greatest* of all these *big* indifference classes. The *smallest* of these greatest classes then represents in some sense the degree up to which t is orderable or not. In this way the concept of an orderable topology t on X seems to be generalizable in a natural way. In addition, by strengthening this idea we are able to measure in a precise sense up to which degree alternatives can or cannot be distinguished by preferences (utilities). Indeed, the fundamental concepts that are introduced and discussed in Section 3 are based upon this idea.

In order to compute or to measure *how big* an indifference class $[x]$ of " $\precsim$ " can be two possibilities seem to be natural.

Indeed, let $\mathcal{A}$ be an appropriate *σ-field* in X that contains t and let μ be an appropriate measure on $\mathcal{A}$. Then for every indifference class $[x]$ of " $\precsim$ " the measure $\mu([x])$ *might* be computed. The reader may recall that the continuity of " $\precsim$ " implies that every indifference class $[x]$ of " $\precsim$ " is a closed subset of X. Since $t \subset \mathcal{A}$, therefore, $\mu([x])$ is defined.

In the Arrow-Hahn or Euclidean distance approach to mathematical utility theory an indifference class $[x]$ of " $\precsim$ " is considered as being *thick* if it contains a non-empty open subset of X, which means that its Lebesgue measure is non-zero. In order to rule out thick indifference classes " $\precsim$ ", thus, is required to be *locally non-satiated*, which means that for every point $x \in X$ and every neighborhood U of x there exists some point $y \in U$ such that $x \prec y$. Besides the original Arrow-Hahn approach [1] the reader may consult, in particular, the book of Bridges and Mehta [5].

On the other hand, one simply could compute the cardinality $|[x]|$ of every indifference class $[x]$ of " $\precsim$ ".

Let $n \geq 1$ be a natural number. Usually on $\mathbb{R}^n$ the Lebesgue measure λ is considered. One verifies immediately that there exist total preorders " $\precsim$ " on $\mathbb{R}^n$ that are continuous with respect to t_{nat} and only have indifference classes $[x]$ the Lebesgue measure $\lambda([x])$ of which is zero. For any two points $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$ one may set, for instance, $x \precsim y \iff x_1 \leq y_1$. With the exception of $n = 1$ in this case the indifference classes are hyperplanes of $\mathbb{R}^n$ which still look *very big*. In addition, although for $n > 1$ the natural topology t_{nat} on $\mathbb{R}^n$ is not orderable (cf. Section 2) this example guarantees, however, the existence of continuous total preorders " $\precsim$ " on $\mathbb{R}^n$ which, following the spirit of the first possibility, only have *thin* or *small* indifference classes. This means that the first possibility is not appropriate if one is interested in measuring or computing the degree of orderability of a topology t or its *potency* of distinguishing between different alternatives by preferences (utilities). Hence, in the remainder of this paper we mainly study the second possibility and, thus, compute or at least estimate the *cardinality* of indifference classes.

After introducing in Section 3 the basic concepts of this paper in Section 4 we, therefore, concentrate on the computation of the cardinality or at least lower and upper bounds of the cardinality of indifference classes.

Meanwhile Hilbert spaces or Banach spaces or even general convex spaces are commonly encountered in mathematical utility theory (cf. [3], [11], [13], [14]). Thus, in particular, the degree of orderability or the potency of distinguishing between different alternatives by preferences (utilities) in these spaces will be computed (Proposition 4.6, Corollary 4.7 and Proposition 4.9). In addition, in Remark 4.10 the (possible) relevance of these results in mathematical utility theory will be discussed.

Finally, also a generalization of well known results on orderable connected topologies will be proved (Proposition 4.11).

2. A first approach

A well known result of Candeal and Induráin [6, Theorem 4] states that for $n \geq 1$ a closed and convex subset C of $\mathbb{R}^n$ can be endowed with a continuous total order if and only if C is homeomorphic and isotonic to a connected subset of $\mathbb{R}$. An immediate and often quoted consequence of this theorem of Candeal and Induráin says that $\mathbb{R}^n$ for $n > 1$ cannot be endowed with a continuous total order. Related but more general results also can be found in Beardon [2] and Candeal, Induráin and Mehta [7].

The quoted consequence of the theorem of Candeal and Induráin also follows from the remarkable fact that for $n > 1$ each indifference class $[x]$ of a continuous total preorder " $\precsim$ " on $\mathbb{R}^n$ such that x is neither a first nor a last element of " $\precsim$ "contains exactly $2^{\aleph_0}$ elements. This result is interesting. Indeed, let $n > 1$ and let S be a countable subset of $\mathbb{R}^n$. Then a well known result of Dugundji [8, Chapter 5, Theorem 2.2] states that $\mathbb{R}^n \backslash S$ is connected. As we shall soon see this result of Dugundji also implies the consequence of the Candeal-Induráin theorem. On the other hand, if $\aleph_1 < 2^{\aleph_0}$ it does not imply that any continuous total preorder on $\mathbb{R}^n$ contains at least one indifference class that consists of $2^{\aleph_0}$ elements.

In order to be more precise we are going to outline a proof of the fact that for $n > 1$ each indifference class $[x]$ of a continuous total preorder " $\precsim$ " on $\mathbb{R}^n$ such that x neither is a first nor a last element of " $\precsim$ "contains exactly $2^{\aleph_0}$ elements that relies on standard arguments from topology. Then it follows, in particular, that the result of Dugundji does not necessarily imply that any continuous total preorder on $\mathbb{R}^n$ contains at least one indifference class that consists of $2^{\aleph_0}$ elements.

Let $n > 1$ and a, b two different points of $\mathbb{R}^n$. Then a family $\{f_i\}_{i \in I}$ of paths $f_i : [0,1] \longrightarrow \mathbb{R}^n$ that connect a and b is said to be *separated* if for any pair i, j of different indexes of I the meets of the images Imf_i and Imf_j of f_i

and f_j respectively only consist of the points a and b. Now a straightforward consideration (cf. the general construction in Proposition 4.6) allows us to construct a separated family $\{f_i\}_{i\in I}$ of paths $f_i : [0,1] \longrightarrow \mathbb{R}^n$ that connect a and b and the cardinality of which is $2^{\aleph_0}$. This means that the result of Dugundji can be improved. Indeed, let $n > 1$ and let S be a subset of $\mathbb{R}^n$ such that $|S| < 2^{\aleph_0}$. Then for every pair of different points $a, b \in \mathbb{R}^n \backslash S$ there exists at least one index $i \in I$ such that $Imf_i \cap S = \varnothing$, and we may conclude that $\mathbb{R}^n \backslash S$ is path connected. In case that x is neither a first nor a last element of " $\precsim$ " it follows that $\mathbb{R}^n \backslash [x] = \{y \in \mathbb{R}^n | y \prec x\} \cup \{z \in \mathbb{R}^n | z \succ x\}$ is not connected. Hence, we may conclude that $|[x]| = 2^{\aleph_0}$. Obviously, the original result of Dugundji only implies that $\aleph_0 < |[x]| \leq 2^{\aleph_0}$. In Section 4 the afore-presented observations will be generalized to arbitrary real or complex convex spaces (cf. Proposition 4.6, Corollary 4.7 and Corollary 4.8).

Let S be a non-empty subset of $\mathbb{R}^n$. Then we choose an arbitrary vector $v \in S$ and postulate for the moment the *dimension* of S to be the dimension of the linear subspace of $\mathbb{R}^n$ that is generated by $S - \{v\} := \{w - v | w \in S\}$. The reader may verify that the definition of the dimension of S is independent of any particular chosen vector $v \in S$. Let t_{nat} be the natural topology on $\mathbb{R}^n$. With the help of our definition of the dimension of S the *indifference dimension* of a continuous total preorder " $\precsim$ " on $(\mathbb{R}^n, t_{nat})$ or shortly $\mathbb{R}^n$ is defined as the maximum of all dimensions of all indifference classes $[(x)]$ of " $\precsim$ ". Then the *indifference dimension* of $\mathbb{R}^n$ is defined as the minimum of all dimensions of relations " $\precsim$ " where " $\precsim$ " runs through all continuous total preorders on $\mathbb{R}^n$. Of course, these concepts can be generalized to arbitrary real or complex convex spaces (cf. Remark 4.10). In Proposition 4.9 these types of indifference dimensions on $\mathbb{R}^n$ (at least implicitly) will be computed. In Remark 4.10 (possible) consequences in mathematical utility theory will be discussed. On the other hand, these concepts of measuring the *size* or (*thickness*) of indifference classes in $\mathbb{R}^n$ hardly seem to be generalizable to arbitrary topological spaces. Hence, in general topology we restrict on computing the cardinality of indifference classes.

3. Fundamental concepts and inequalities

Let (X, t) be some arbitrarily but fixed chosen *non-trivial* topological space. Non-trivial means that X contains at least two (different) elements.

According to our considerations in the introduction for every continuous total preorder " $\precsim$ " on X the *weak indifference potency* of " $\precsim$ " is defined by

$$winpot(\precsim) := \begin{cases} 0 & \text{if " $\precsim$ " is an order on } X \\ sup\{|\,[x]\,| \mid [x] \text{ is an indifference class of " $\precsim$ "}, & \text{otherwise} \end{cases}.$$

Then the *weak indifference potency* of X is defined by

$$\text{winpot}(X) := min\{winpot(\precsim) | \text{“} \precsim \text{”} \text{ is a continuous total preorder on } (X,t)\}.$$

In Section 2 the cardinality of an indifference class has been computed with the help of its potency to separate a (connected) component of X. Let, therefore, $\mathbf{C}$ be the set of all (connected) components of X. We consider some continuous total preorder “ $\precsim$ ” on X and choose a component $C \in \mathbf{C}$. Now an indifference class $[x]$ of “ $\precsim$ ” is said to be *calculable with respect to* “ $\precsim_{|C}$ ” if $C \subset [x]$ or there exist points y, $z \in C$ such that $y \prec v \prec z$ for every point $v \in C \cap [x]$. This notation allows us to define the *indifference potency inpot*($\precsim$) of “ $\precsim$ ” by “0” if “ $\precsim$ ” is an order on X and by

$$sup_{C \in \mathbf{C}} min\{|[x]| \, [x] \text{ is a calculable set with respect to “} \precsim_{|C} \text{”}\}, \text{ otherwise.}$$

Then the indifference potency of X is defined by

$$inpot(X) := min\{inpot(\precsim) | \text{“} \precsim \text{”} \text{ is a continuous total preorder on } (X,t)\}.$$

Finally, a subset S of X is said to be *calculable* if there exists a component C of X and some continuous total preorder “ $\precsim$ ” on X such that S is an indifference class of “ $\precsim$ ” and $C \subset S$ or there exist points y, $z \in C$ such that $y \prec x \prec z$ for every point $x \in C \cap S$. Let now for every component $C \in \mathbf{C}$ the set $ca(C)$ consist of all subsets S of X that are indifference classes of continuous total preorders “ $\precsim$ ” on X such that $C \subset S$ or there exist points y, $z \in C$ such that $y \prec x \prec z$ for every point $x \in C \cap S$. Then we still may define the *strong indifference potency of* X by

$$stinpot(X) := \begin{cases} 0, & \text{if } t \text{ is orderable} \\ sup_{C \in \mathbf{C}} \; min \; \{|S| \, | S \in \text{ca}(C)\} & \text{otherwise} \end{cases} .$$

The above definitions imply that the assertions $winpot(X) = inpot(X) = stinpot(X) = 0$ and t is orderable are equivalent.

The following lemma justifies the concept of a calculable set.

Lemma 3.1. *Let S be a subset of X. Then the following assertions are equivalent:*

(i) *S is calculable.*
(ii) *There exists a continuous total preorder “ $\precsim$ ” on X and some component C of X such that S is an indifference class of “ $\precsim$ ” and $C \subset S$ or $C \backslash S$ is not a connected subset of X with respect to $t^{\precsim}$.*

Proof. (i) $\Longrightarrow$ (ii): Let $S = [x]$ and let there exist points y, $z \in C$ such that $y \prec x \prec z$. It suffices to verify that $C \backslash [x]$ is not connected with respect to $t^{\precsim}$. Let, therefore, $L(x) := \{u \in X | u \prec x\}$ and $K(x) := \{v \in X | x \prec v\}$. Then $L(x)$ and $K(x)$ are two disjoint open subsets of $\left(X, t^{\precsim}\right)$ such that $L(x) \cap C \backslash S \neq \varnothing$

and $K(x)\cap C\backslash S\neq\varnothing$ and $C\backslash S\subset L(x)\cup K(x)$, which implies that $C\backslash S$ is not connected with respect to $t^{\precsim}$.

(ii) $\Longrightarrow$ (i): Let " $\precsim$ " be some continuous total preorder on X such that $S=[x]$ for an indifference class $[x]$ of " $\precsim$ ". Then we assume the existence of a component C of X such that $C\backslash[x]$ is not connected in the order topology $t^{\precsim}$ on X that is induced by " $\precsim$ ". Because of the definition of a calculable set it is sufficient to prove that there exist points $y, z\in C$ such that $y\prec x\prec z$. Since $t^{\precsim}\subset t$ and C is a connected subset of X with respect to t it follows that C is also a connected subset of X with respect to $t^{\precsim}$. In addition, the definition of $t^{\precsim}$ implies that every indifference class $[u]$ of " $\precsim$ " is a connected subset of X with respect to $t^{\precsim}$. Hence, $I:=\{[v]\,C\cap[v]\neq\varnothing\}$ is a connected subset of X with respect to $t^{\precsim}$, which means that I is an interval of $(X,\precsim)$ that is connected with respect to $t^{\precsim}$. Furthermore, the reader may recall that a totally preordered set $(Z,\lesssim)$ is connected with respect to its order topology if and only if it is *order dense* and *Dedekind complete*. Order dense means that for any two points $a<b\in Z$ there exists some point $c\in Z$ such that $a<c<b$. Dedekind complete means that every bounded subset D of Z has a least upper and a greatest lower bound. These results are well known for chains. The reader may consult, for instance, the famous book of Birkhoff [4] on lattice theory. By considering on the set of indifference classes the induced total ordering these results easily can be generalized to total preorders. Since $C\backslash[x]$ is not connected with respect to $t^{\precsim}$ we may summarize our considerations for concluding that x neither can be a first nor a last element of $(I,\precsim)$. Therefore, the definition of I guarantees the existence of points $y, z\in C$ such that $y\prec x\prec z$, and the proof is complete. □

Remark and Example 3.2. Let " $\precsim$ " be a continuous and total preorder on X and let C be a component of X. With the help of the proof of Lemma 3.1 it follows that an indifference class $[x]$ of " $\precsim$ " is calculable with respect to " $\precsim_{|C}$ " if and only if $C\subset[x]$ or $C\backslash[x]$ is not a connected subset of X with respect to $t^{\precsim}$.

In assertion (ii) of Lemma 3.1 the condition that $C\backslash S$ is not a connected subset of X with respect to $t^{\precsim}$ cannot be replaced by the condition that $C\backslash S$ is not a connected subset of X with respect to t. Indeed, let $X:=[0,1]$ be the real unit interval. We set $U:=[0,1)\cap\mathbb{Q}$ and $V:=[0,1)\backslash\mathbb{Q}$ and consider the coarsest topology t on X that contains t_{nat} and both sets U and V. Then $\{1\}$ is an indifference class of the natural total ordering " $\leq=\leq_{|X}$ " on $[0,1]$. In addition, the definition of t imples that $t^{\leq}=t_{nat}\subset t$. Furthermore, since 1 neither is contained in U nor in V it follows that (X,t) is a connected space. Moreover, we may conclude with the help of the definitions of U and V that $X\backslash\{1\}$ is not connected with respect to t. Since $X\backslash\{1\}$ is connected with respect to $t^{\leq}$ it follows that $\{1\}$ cannot be calculable with respect to " $\leq_{|X}$ " = " $\leq$ ". Now the connectedness of t and the particular form of the neighborhoods of 1 also imply that 1 is either a first or a last element of $(X,\precsim)$ for every continuous total preorder " $\precsim$ " on X. Hence, there cannot exist any other continuous total preorder " $\precsim$ " on X such that $X\setminus\{1\}$ is not connected with respect to $t^{\precsim}$, which means that $\{1\}$ cannot be a calculable set.

In order to prepare the main result of this section the reader may recall that (X,t) is said to be *hereditarily Lindelöf* if for every collection $\{O_i\}_{i\in I}$ of open subsets of X there exists a countable subset J of I such that $\bigcup_{i\in I} O_i = \bigcup_{j\in J} O_j$.

In addition, we define for every component $C \in \mathbf{C}$ the *degree of connectedness of C* by

$$dcon(C) := \begin{cases} min\{|Z|\,|Z \in T(C)\}\,, & \text{if } T(C) \neq \varnothing \\ |C|\,, & \text{otherwise} \end{cases}.$$

With the help of this definition we may define the *degree of connectedness* of X by

$$dcon(X) := \begin{cases} 0, & \text{if } t \text{ is orderable} \\ sup_{C\in\mathbf{C}}dcon(C), & \text{otherwise} \end{cases}.$$

The reader may compare the definitions of $dcon(C)$ and $dcon(X)$ with assertion (ii) of Lemma 3.1 and the above given example in order to understand the particular relevance of these concepts within our approach.

Now we still consider the set $S(\mathbf{C})$ of all components C of X such that $dcon(C) < |C|$ and prove the following proposition that in combination with Remark and Example 3.4 will help us to clarify in some degree the interrelations between the afore-introduced concepts.

Proposition 3.3. *The following assertions hold:*

(i) $winpot(X) \geq inpot(X) \geq stinpot(X) \geq dcon(X)$.

(ii) *Let every component C of X be open and closed and let every component C of X be metrizable or regular and hereditarily Lindelöf. Then* $inpot(X) = stinpot(X) = dcon(X)$.

(iii) *Let every component C of X be open and closed and let $S(\mathbf{C}) = \varnothing$. Then* $winpot(X) = inpot(X) = stinpot(X) = dcon(X)$.

Proof. (i): Because of the definitions of $winpot(X)$, $inpot(X)$, $stinpot(X)$ and $dcon(X)$ it suffices to verify that $stinpot(X) \geq dcon(X)$. Let, therefore, some component C of X and a continuous total preorder " $\precsim$ " on X be arbitrarily chosen. Then there exists an indifference class $[x]$ of " $\precsim$ " such that $C\cap[x] \neq \varnothing$. Now we distinguish between the following two cases:

Case 1: $C \subset [x]$. In this case $[x]$ is a calculable subset of X such that $|[x]| \geq dcon(C)$.

Case 2: $C\setminus[x] \neq \varnothing$. Since C is connected we may conclude that $I := \{v \in X|C \cap [v] \neq \varnothing\}$ is a connected interval of $(X, \precsim)$ (cf. the corresponding part in the proof of Lemma 3.1). Hence, we may choose some point $z \in I$ that neither is a first nor a last element of I and the indifference class $[z]$ which has minimal cardinality with respect to any other indifference class of an *inner* point of I. It follows from the definition of a calculable set with respect to " $\precsim_{|C}$ " that $[z]$ is a calculable subset of X with respect to " $\precsim_{|C}$ ". Since $C \nsubseteq [z]$ the first part of Remark and Example 3.2 implies that $C \setminus [z]$ is not connected with respect to $t^{\precsim}$, which, in particular, means that $C \setminus [z]$ is not connected with respect to t. Therefore, $[z]$ is a calculable subset of X such that $|[z]| \geq dcon(C)$.

Since C and " $\precsim$ " have been arbitrarily chosen we now may conclude with the help of the definitions of $stinpot(X)$ and $dcon(X)$ that the desired inequality $stinpot(X) \geq dcon(X)$ actually holds.

(ii): Let every component C of X be open and closed and let, in addition, every component C of X be metrizable or regular and hereditarily Lindelöf. Because of assertion (i) it remains to verify that $dcon(C) \geq inpot(X)$. Therefore, we choose for every component $C \in S(\mathbf{C})$ some set $Z \in T(C)$ that has minimal cardinality and construct in three steps a continuous total preorder " $\precsim$ " on X such that $dcon\,(X) \geq inpot\,(\precsim)$. Then the definition of $inpot\,(X)$ implies that the desired inequality $dcon(X) \geq inpot(X)$ actually holds. Let, therefore, some component $C \in S\,(\mathbf{C})$ and a set $Z \in T(C)$ that has minimal cardinality with respect to any other set $Z^{'} \in T(C)$ be arbitrarily but fixed chosen.

1. In the first step we want to show that, without loss of generality, Z may be assumed to be a closed subset of X. Indeed, since $C \setminus Z$ is not connected and C is an open and closed subset of X, there exist open subsets U and V of C such that $U \cap V \cap C \setminus Z = \varnothing$ and $U \cap C \setminus Z \neq \varnothing$ and $V \cap C \setminus Z \neq \varnothing$ and $C \setminus Z \subset U \cup V$. Now we distinguish between the following two cases:

Case 1: $U \cap V = \varnothing$. In this case we set $Z^{'} := C \setminus (U \cup V)$. The inclusion $Z^{'} \subset Z$ allows us to conclude that $U \cap C \setminus Z^{'} \neq \varnothing$ and $V \cap C \setminus Z^{'} \neq \varnothing$. Since $C \setminus Z^{'} = U \cup V$ it, thus, follows that $C \setminus Z^{'}$ is not connected. $Z^{'}$ is a closed subset of X by construction. Hence, in the first case we may assume that Z is closed.

Case 2: $U \cap V \neq \varnothing$. The relation $U \cap V \cap C \setminus Z = \varnothing$ implies that $U \cap V \subset Z$. Let $x \in U \cap V$ be some arbitrarily chosen point. Since the relativized topology $t_{|C}$ on C is metrizable or regular and (hereditarily) Lindelöf it follows that $(C, t_{|C})$ is a normal and, therefore, in particular a completely regular space. Hence, there exists some continuous function f from C into the real interval $[0,1]$ such that $f(x) = 1$ and $f(C \setminus (U \cap V)) = \{0\}$. Let $Z^{'} := f^{-1}\left(\left\{\frac{1}{2}\right\}\right)$. The continuity of f allows us to conclude that $Z^{'}$ is a closed subset of X. In addition, it follows that $Z^{'} \subset U \cap V \subset Z$. Furthermore, the sets $O := \left\{y \in C | f(y) < \frac{1}{2}\right\}$ and $W := \left\{z \in C | f(z) > \frac{1}{2}\right\}$ are two disjoint non-empty open subsets of $C \setminus Z^{'}$ the union of which is $C \setminus Z^{'}$, which means that $C \setminus Z^{'}$ is not connected. Therefore, also in the second case Z may be assumed to be a closed subset of X.

2. In the second step we want to construct a continuous total preorder " $\precsim_C$ " on C such that Z is an indifference class of " $\precsim_C$ ". Because of the first

step we may assume that Z is a closed subset of X. In addition, we already know that $(C, t_{|C})$ is a metrizable or competely regular hereditarily Lindelöf space. Therefore, we distinguish between the following two cases:

Case 1: $(C, t_{|C})$ is metrizable. In this case we choose some metric d on C that induces $t_{|C}$. Then we consider the continuous function $g : C \longrightarrow \mathbb{R}^{\geq 0}$ that is defined for all points $x \in C$ by setting $g(x) := d(Z, x)$. Since $g^{-1}(\{0\}) = Z$ we may conclude that Z is an indifference class of the continuous total preorder " $\precsim_C$ " on C that is defined by $y \precsim_C z \Longleftrightarrow g(y) \leq g(z)$.

Case 2: $(C, t_{|C})$ is a completely regular hereditarily Lindelöf space. Now there exists for every point $x \in C\backslash Z$ a continuous function $f_x : C \longrightarrow [0, 1]$ such that $f_x(x) = 1$ and $f_x(Z) = \{0\}$. Moreover, since $(C, t_{|C})$ is a hereditarily Lindelöf space countably many points $x_1, x_2, ..., x_n, ... \in C\backslash Z$ can be chosen such that $C\backslash Z = \bigcup_{x \in C\backslash Z} f_x^{-1}((0, 1]) = \bigcup_{n \in \mathbb{N}\backslash\{0\}} f_{x_n}^{-1}((0, 1])$, which implies that $Z = \bigcap_{n \in \mathbb{N}\backslash\{0\}} f_{x_n}^{-1}(\{0\})$. Therefore, we set $g := \sum_{n \in \mathbb{N}\backslash\{0\}} \frac{1}{2^n} f_{x_n}$. Then $g : C \longrightarrow [0, 1]$ is a continuous function. Hence, in the same way as in the first case, we may consider the continuous total preorder " $\precsim_C$ " on C that is defined by $y \precsim_C z \Longleftrightarrow g(y) \leq g(z)$. Since the definition of g implies that $g^{-1}(\{0\}) = Z$ it follows that Z is an indifference class of " $\precsim_C$ ".

3. In the third step we define the continuous total preorder " $\precsim$ " on X such that $dcon(X) \geq inpot(\precsim)$. Let, therefore, $x \in X$ be an arbitrarily chosen point. Then there exists a uniquely determined component C of X that contains x, and we set $D(x) := C$. Now we choose an arbitrary total ordering " $\preceq$ " on $\mathbf{C}$ and set

$$y \precsim z \Longleftrightarrow \begin{cases} y \precsim_C z, & \text{if there exists some } C \in S(\mathbf{C}) \\ & \text{that contains both point } y \text{ and } z \\ D(y) \preceq D(z), & \text{otherwise} \end{cases}$$

for every pair of points $y, z \in X$. Since every component of X is open and closed and since every total preorder " $\precsim_C$ " that has been defined is continuous we may conclude that " $\precsim$ " is a continuous total preorder on X. Because of the definition of $S(\mathbf{C})$ and the second step it follows with the help of the definitions of $inpot(\precsim)$ and $dcon(X)$, in addition, that $dcon(X) \geq inpot(\precsim)$ (indeed, the equality $dcon(X) = inpot(\precsim)$ holds), which settles assertion (ii).

(iii): Let $S(\mathbf{C}) = \varnothing$. Then the reader may notice that the definition of " $\precsim$ " in the third step of the proof of assertion (ii) also implies that $winpot(X) = dcon(C)$. This observation finishes the proof of the proposition. □

Remark and Example 3.4. The proof of assertion (ii) of the above proposition motivates the problem of determining all topologies t on X for which every non-empty closed subset Z of X is the indifference class of an appropriate continuous total preorder " $\precsim$ " on X. Meanwhile the first author has obtained some partial results on the characterization of Hausdorff-topologies t

on X that have the property that every non-empty closed subset of X is an indifference class of some continuous total preorder "$\precsim$" on (X,t) (cf. [10]). Clearly, these topologies must be completely regular. If t is countably compact it follows, in addition, that t must satisfy *ccc* (countable chain condition). The authors think that these results somewhat justify the assumptions of assertion (ii) of the proposition.

One may verify that, in general, $winpot(X) > inpot(X) \geq stinpot(X) > dcon(X)$. Indeed, let $[0,1]$ be the real unit interval. Then we substitute 0 by a copy $\mathbb{R}(0)$ of the reals and every real $r \in (0,1]$ by a copy $\mathbb{Q}(r)$ of the rationals in order to set $X := \mathbb{R}(0) \cup \bigcup_{r\in(0,1]} \mathbb{Q}(r)$. Now it follows for every $x \in X$ that $x \in \mathbb{R}(0)$ or $x \in \mathbb{Q}(r)$ for some uniquely determined real $r \in (0,1]$. In this way a canonical function $g : X \longrightarrow [0,1]$ is given, and we may define a total preorder "$\lesssim$" on X by setting $x \lesssim y \Longleftrightarrow g(x) \leq g(y)$. We proceed by considering two arbitrarily but fixed chosen subsets U and V of X such that $U \cap \mathbb{R}(0) \neq \varnothing$ and $V \cap \mathbb{R}(0) \neq \varnothing$ and $U \cap \mathbb{Q}(r) \neq \varnothing$ and $V \cap \mathbb{Q}(r) \neq \varnothing$ for every real $r \in (0,1]$ and $U \cup V = X$ and $U \cap V$ is a non-empty finite set. Then we consider the topology t on X that is generated by the order topology $t^{\lesssim}$ and the sets U and V. Because of the definition of t we may conclude that t is connected and that for every continuous total preorder "$\precsim$" on X either the inclusion $\lesssim\, \subset\, \precsim$ or $\gtrsim\, \subset\, \precsim$ holds. It, thus follows that $winpot(X) = winpot(\lesssim) = winpot(\gtrsim) = 2^{\aleph_0}$ and that $inpot(X) = stinpot(X) = inpot(\lesssim) = inpot(\gtrsim) = \aleph_0$. In addition, since $U \cup V = X$ and $U \cap V$ is a non-empty finite set we may conclude that $dcon(X) = |U \cap V| < \aleph_0$, which settles the example.

On the other hand, it seems to be difficult to construct a topological space (X,t) such that $inpot(X) > stinpot(X)$. The authors conjecture, instead, that $inpot(X) = stinpot(X)$ for every topological space (X,t). Indeed, the set $CTP(X)$ of all continuous total preorders "$\precsim$" on (X,t) is naturally ordered by set inclusion. An application of the Lemma of Zorn now implies that $(CTP(X), \subset)$ contains minimal elements "$\lesssim$". Then with the help of some *uniqueness argument* for "$\lesssim_{|C}$" on every component C of X that is well known for orders (cf., for instance, Eilenberg [9] or any other book or paper on connected orderable spaces) it should be possible to prove that for *any* minimal element "$\lesssim$" of $(CTP(X), \subset)$ the equalities $inpot(\lesssim) = inpot(X) = stinpot(X)$ hold. But at present the authors are not sure about some crucial technical details of the argument.

In order to also generalize the approach that has been sketched in Section 2 we still introduce the following concepts.

Let x, $y \in X$ be different points. Then we denote by $P(x,y)$ the set of all paths $f : [0,1] \longrightarrow X$ that connect x and y and define the *path rank* of the pair (x,y) by

$$prank(x,y) := \begin{cases} min\{|S|\,|S \subset X\setminus\{x,y\} \text{ and } \mathrm{Im}f \cap S \neq \varnothing \\ \text{for every path } f \in P(x,y)\}, & \text{if } P(x,y) \neq \varnothing \\ 0, & \text{otherwise} \end{cases}.$$

This definition allows us to define for an arbitrarily chosen set $D \subset X$ the *path rank* of D by $prank(D) := min\{prank(x,y)|x,y \in D, x \neq y\}$.

Moreover, we define the *separation rank* of the pair (x,y) by

$$srank(x,y) := \begin{cases} min\{|\mathcal{F}|\,|\mathcal{F} \text{ is with respect to set} \\ \text{inclusion a maximal separated} & \text{if } P(x,y) \neq \varnothing \\ \text{subset of } P(x,y)\}, \\ 0, & \text{otherwise} \end{cases},$$

in order to finally denote for every set $D \subset X$ the *separation rank* of D by $srank(D) := min\{srank(x,y)|x,y \in D, x \neq y\}$.

4. Lower and upper bounds and a topological characterization of real intervals

As in the previous section we consider some fixed chosen non-trivial topological space (X,t).

With the help of the proof of assertion (i) of Proposition 3.3 the following proposition follows immediately.

Proposition 4.1. *Let " $\precsim$ " be some continuous total preorder on X. Then for every component C of X there exists at least one indifference class $[x]$ of " $\precsim$ " such that $|[x]| \geq dcon(X)$.*

Let now " $\precsim$ " be an arbitrarily chosen continuous total preorder on X. Then we consider an indifference class $[x]$ of " $\precsim$ " for which the set $\mathbf{C}([x])$ of all components C of X such that $[x]$ is a calculable set with respect to " $\precsim_{|C}$ " is non-empty. Now the following proposition holds.

Proposition 4.2.

$$|[x]| \geq \sum_{C \in \mathbf{C}([x])} dcon(C).$$

Proof. Let some component $C \in \mathbf{C}([x])$ be arbitrarily chosen. Then we may conclude with the help of the first part of Remark and Example 3.2 that $C \subset [x]$ or $C \cap [x] \in T(C)$. Hence, it follows for every component $C \in \mathbf{C}([x])$ that $|C \cap [x]| \geq dcon(C)$. Since the components $C \in \mathbf{C}([x])$ are pairwise disjoint we, thus, may conclude that the desired inequality holds. □

Now we want to generalize the situation that has been considered in Section 2. Let, therefore, two different points $x,y \in X$ be arbitrarily chosen. Then the following proposition holds.

Proposition 4.3.

$$srank(x,y) \le prank(x,y) \le 2^{\aleph_0} \cdot srank(x,y).$$

Proof. The validity of the inequality $srank(x,y) \le prank(x,y)$ is a straightforward consequence of the definitions of $srank(x,y)$ and $prank(x,y)$ respectively. Hence, only the inequality $prank(x,y) \le 2^{\aleph_0} \cdot srank(x,y)$ has to be verified. The definitions of $srank(x,y)$ and $prank(x,y)$ respectively allow us to assume without loss of generality that $P(x,y) \neq \varnothing$. Let, therefore, $\mathcal{F}$ be some separated subset of $P(x,y)$ that is maximal with respect to set inclusion. Then we set $P(\mathcal{F}) := \bigcup_{f \in \mathcal{F}} (Imf \backslash \{x,y\})$. Now the maximality of $\mathcal{F}$ implies that $P(\mathcal{F}) \cap Imf \neq \varnothing$ for every path $f \in P(x,y)$. Since $P(\mathcal{F}) \subset X \backslash \{x,y\}$, thus, the inequality $prank(x,y) \le |P(\mathcal{F})|$ holds. In addition, it follows from the definition of $P(\mathcal{F})$ that $|P(\mathcal{F})| \le 2^{\aleph_0} \cdot |\mathcal{F}|$. Summarizing these considerations we may conclude that $prank(x,y) \le 2^{\aleph_0} \cdot srank(x,y)$, which still was to be shown. □

Let us now assume, that every component C of X is path connected or locally path connected which means that the components of X with respect to path connectedness coincide with the components C of X. Then the following lemma is an immediate consequence of the definitions of $dcon(C)$ and $prank(C)$ respectively.

Lemma 4.4.

$$dcon(C) = prank(C) \text{ for every component } C \text{ of } X.$$

Because of Proposition 4.3 and Lemma 4.4 also the following proposition holds.

Proposition 4.5.

$$sup_{C \in \mathbf{C}} srank(C) \le \sup_{C \in \mathbf{C}} prank(C) = dcon(X) \le 2^{\aleph_0} \cdot \sup_{C \in \mathbf{C}} srank(C).$$

Let (V,t) be some real or complex convex space the dimension $\dim V$ of which is greater than 1. Then our considerations also imply the following proposition and corollaries (cf. Section 2).

Proposition 4.6.

$$winpot(V) = inpot(V) = stinpot(V) = dcon(V) = |V|.$$

Proof. Let x, $y \in V$ be arbitrarily chosen different points. Because of Proposition 4.3 and Lemma 4.4 and the inequalities $dcon(V) \le stinpot(V) \le inpot(V) \le winpot(V) \le |V|$ it suffices to prove that $|V| \le srank(x,y)$. Therefore, we consider some (closed) hyperplane H of V that separates the points x and y. Then we define for every point $z \in H$ some path $f_z \in P(x,y)$ by setting

$$f_z(\lambda) := \begin{cases} x + 2\lambda(z-x), & \text{if} \quad \lambda \in [0, \frac{1}{2}) \\ z + (2\lambda - 1)(y-z), & \text{if} \quad \lambda \in [\frac{1}{2}, 1] \end{cases}$$

for every real $\lambda \in [0,1]$. A routine argument that uses for different points z,

$z^{'} \in H$ the linear independency of the vectors $z-x$ and $z^{'}-x$, respectively $y-z$ and $y-z^{'}$, implies that the collection $\mathcal{F} := \{f_z\}_{z \in H}$ is a separated family of paths $f_z \in P(x,y)$. Since $|H| = |V|$ the definition of $srank(x,y)$, thus, implies that $|V| = |\mathcal{F}| \leq srank(x,y)$, and the desired inequality follows. □

Corollary 4.7. $|[x]| = |V|$ *for every continuous total preorder "* $\precsim$ *" on* V *and every indifference class* $[x]$ *of "* $\precsim$ *" such that* x *neither is a first nor a last element of "* $\precsim$ *".*

Proof. If x neither is a first nor a last element of " $\precsim$ " then $L(x)$ and $K(x)$ are two non-empty disjoint open subsets of V the union of which is $V \setminus [x]$. Hence, $|V| \geq |[x]| \geq dcon(V)$ and the desired conclusion follows from Proposition 4.6. □

The following corollary strengthens and generalizes Dugundji's result that has been quoted in Section 2 to arbitrary real or complex convex spaces.

Corollary 4.8. *Let* S *be a subset of* V *such that* $|S| < |V|$*. Then* $V \setminus S$ *is path connected.*

Proof. In the proof of Proposition 4.6 it has been shown that $srank(x,y) = |V|$ for every pair of different points $x,\ y \in V$. If $|S| < |V|$ this means, in particular, that for every pair of points $x,\ y \in V \setminus S$ there exists some path $f : [0,1] \longrightarrow V \setminus S$ that connects x and y (cf. the corresponding argument in Section 2). □

Let now $n > 1$ be a natural number. According to the last paragraph of Section 2 we define for every non-empty subset S of $\mathbb{R}^n$ the dimension $dimS$ of S by choosing an arbitrary vector $v \in S$ in order to then identify $dimS$ with the dimension of the linear subspace of $\mathbb{R}^n$ that is generated by $S - \{v\}$. The reader may recall from Section 2 that the definition of $dimS$ is independent of any particular chosen vector $v \in S$. Then the following proposition holds.

Proposition 4.9. *Let "* $\precsim$ *" be a continuous total preorder on* $\mathbb{R}^n$*. Then the following assertions are valid:*

(i) $dim[x] \geq n-1$ *for every indifference class* $[x]$ *of "* $\precsim$ *" such that* x *neither is a first nor a last element of "* $\precsim$ *".*

(ii) $[x]$ *is a hyperplane for every indifference class* $[x]$ *of "* $\precsim$ *" such that* $dim[x] = n-1$ *and* x *neither is a first nor a last element of "* $\precsim$ *".*

Proof. (i): Let $[x]$ be an indifference class of " $\precsim$ " such that x neither is a first nor a last element of " $\precsim$ ". By considering instead of " $\precsim$ " the continuous total preorder " $\precsim_x$ " on $\mathbb{R}^n$ that is defined by $u - x \precsim_x v - x \Longleftrightarrow u \precsim v$ we may assume without loss of generality that $[x]$ contains the zero vector "0". Therefore, we may consider the linear subspace $V_{[x]}$ of $\mathbb{R}^n$ that is generated by $[x]$. Then we assume, in contrast, that $\dim V_{[x]} \leq n-2$. Because of this assumption there exists a linear subspace W of $\mathbb{R}^n$ that is generated by two linearly independent vectors y and z such that $V_{[x]} \cap W = \{0\}$. For every vector $c \in \mathbb{R}^n$ we now set $L^c_y := \{c + \lambda(y-c) | \lambda \in \mathbb{R}\} = \{y + \gamma(c-y) | \gamma \in \mathbb{R}\}$ and

$L_z^c := \{c+\lambda(z-c)|\lambda \in \mathbb{R}\} = \{z+\gamma(c-z)|\gamma \in \mathbb{R}\}$. Then we choose two arbitrary vectors a, $b \in \mathbb{R}^n \backslash V_{[x]}$. Since $V_{[x]} \cap W = \{0\}$ it follows by direct computation that is based upon contraposition that $L_y^a \cap V_{[x]} = \varnothing$ or $L_z^a \cap V_{[x]} = \varnothing$ and $L_y^b \cap V_{[x]} = \varnothing$ or $L_z^b \cap V_{[x]} = \varnothing$. We may assume without loss of generality that $L_y^a \cap V_{[x]} = \varnothing$ and $L_z^b \cap V_{[x]} = \varnothing$. The remaining possibilities can be settled by analogous arguments. The equations $L_y^a \cap V_{[x]} = \varnothing$ and $V_{[x]} \cap W = \{0\}$ and $L_z^b \cap V_{[x]} = \varnothing$ allow us to define a path $f : [0,1] \longrightarrow \mathbb{R}^n \backslash V_{[x]}$ by setting

$$f(\lambda) := \begin{cases} a + 3\lambda(y-a), & \text{if } \lambda \in [0, \frac{1}{3}) \\ y + (3\lambda - 1)(z-y), & \text{if } \lambda \in [\frac{1}{3}, \frac{2}{3}) \\ z + (3\lambda - 2)(b-z), & \text{if } \lambda \in [\frac{2}{3}, 1] \end{cases}$$

for every real $\lambda \in [0,1]$. Since the vectors a, $b \in \mathbb{R}^n \backslash V_{[x]}$ have been arbitrarily chosen we may conclude that $\mathbb{R}^n \backslash V_{[x]}$ is path connected and, therefore, connected. Now the connectedness of $\mathbb{R}^n \backslash V_{[x]}$ implies with the help of a straightforward indirect argument that is based upon the fact that every non-empty open subset of $\mathbb{R}^n$ contains a base of linearly independent vectors of $\mathbb{R}^n$ that also $\mathbb{R}^n \backslash [x]$ is connected, which contradicts the proof of Corollary 4.7 and, thus, finishes the proof of assertion (i).

(ii): Let $x \in \mathbb{R}^n$ be any point that neither is a first nor a last element of " $\precsim$ " such that $\dim [x] = n-1$. In order to prove that $[x]$ is a hyperplane of $\mathbb{R}^n$ we may assume, as in the proof of assertion (i), that $0 \in [x]$. Using the notation of the proof of assertion (i) it suffices to verify that $V_{[x]} \subset [x]$. We assume, in contrast, that there exists some non-empty subset S of $V_{[x]}$ that is not contained in $[x]$. We may assume without loss of generality that $|S| = 1$. Indeed, the last argument in the proof of assertion (i) shows in which way the case that $|S| > 1$ can be reduced to the case that $|S| = 1$. Let, therefore, $S = \{v\}$ for some vector $v \in V_{[x]}$. Then we set $V_{[x]}^- := V_{[x]} \backslash \{v\}$ and prove that $\mathbb{R}^n \backslash V_{[x]}^-$ is connected. As in the proof of assertion (i) it, thus, follows that also $\mathbb{R}^n \backslash [x]$ is connected, which contradicts the proof of Corollary 4.7. Hence, if we are able to verify that $\mathbb{R}^n \backslash V_{[x]}^-$ is connected nothing remains to be shown. In order to prove that $\mathbb{R}^n \backslash V_{[x]}^-$ is connected we choose two arbitrary points a, $b \in \mathbb{R}^n \backslash V_{[x]}^-$ and distinguish between the cases that $a = v$ or $b = v$ and that neither $a = v$ nor $b = v$.

Case 1: $a = v$ or $b = v$. In this case, we may assume without loss of generality, that $a = v$ and that $b \neq v$. Since $a = v$ and $b \notin V_{[x]}$ it follows that $\{a + \lambda(b-a) | \lambda \in \mathbb{R}\} \cap V_{[x]}^- = L_b^a \cap V_{[x]}^- = \varnothing$. This means that $f : [0,1] \longrightarrow \mathbb{R}^n \backslash V_{[x]}^-$ defined by $f(\lambda) := a + \lambda(b-a)$ for every real $\lambda \in [0,1]$ is a path that connects a and b.

Case 2: $a \neq v$ and $b \neq v$. Now it follows that $\{a + \lambda(v-a) | \lambda \in \mathbb{R}\} \cap V_{[x]}^- = L_v^a \cap V_{[x]}^- = \varnothing$ and that $\{v + \gamma(b-v) | \gamma \in \mathbb{R}\} \cap V_{[x]}^- = L_b^v \cap V_{[x]}^- = \varnothing$. Hence, $f : [0,1] \longrightarrow \mathbb{R}^n \backslash V_{[x]}^-$ defined by

$$f(\lambda) := \begin{cases} a + 2\lambda(v - a), & \text{if} \quad \lambda \in [0, \frac{1}{2}) \\ v + (2\lambda - 1)(b - v), & \text{if} \quad \lambda \in [\frac{1}{2}, 1] \end{cases}$$

for every real $\lambda \in [0, 1]$ is a path that connects a and b (cf. the definition of f_z in the proof of Proposition 4.6). Since a, $b \in \mathbb{R}^n \setminus V_{[x]}^-$ have been arbitrarily chosen we may summarize our considerations for concluding that $\mathbb{R}^n \setminus V_{[x]}^-$ is path connected and, therefore, also connected. Thus, the proof of assertion (ii) is complete. □

Remark 4.10. An analysis of the proof of Proposition 4.9 implies that Proposition 4.9 can be generalized to arbitrary real or complex convex spaces V the dimension $\dim V$ of which is greater than 1. But then the conclusion $\dim [x] \geq n - 1$ has to be replaced by the condition that in case that $0 \notin [x]$ the linear subspace of V that is generated by $[x]$ coincides with V and that in case that $0 \in [x]$ the linear subspace of V that is generated by $[x]$ coincides with V or is a maximal linear subspace of V. In combination with Proposition 4.6 and Corollary 4.7 this additional result, thus, provides a rather complete survey on the *size* of indifference classes of a continuous total preorder " $\precsim$ " on a real or complex convex space. In this sense Proposition 4.9 completes Proposition 4.6 and Corollary 4.7. Indeed, in mathematical utility theory a particular chosen set of coordinates or a particular chosen base of linearly independent vectors of a real or complex convex space V can be interpreted as the collection of those (latent) factors or dimensions that influence preferences between alternatives.

Let, for the moment, an indifference class of a continuous total preference relation " $\precsim$ " on V for which x neither is a first nor a last element of " $\precsim$ " said to be an *inner* indifference class of " $\precsim$ ". Then we learn from Proposition 4.6, Corollary 4.7 and Proposition 4.9 that inner indifference classes not only have maximal cardinality but that, in addition, preferences between alternatives y and z, the corresponding indifference classes of which are inner, can be assumed to be determined by different expressions of exactly one coordinate or different expressions of the same coordinates or different expressions of *nearly the same* coordinates, which means that one of the given alternatives y or z can be influenced by one more coordinate than the other alternative. Indeed, if $[y]$ and $[z]$ are hyperplanes then the disjointness of $[y]$ and $[z]$ implies that the preference between $[y]$ and $[z]$ can be assumed to be influenced by exactly one coordinate (cf. the corresponding example in the introduction). Otherwise, at least one of the indifference classes $[y]$ or $[z]$ is influenced by a maximal set of coordinates, which means that $[y]$ or $[z]$ can be assumed to be influenced by at most one more coordinate than any other inner indifference class of " $\precsim$ ".

In order to complete these considerations we still consider a continuous total preference relation " $\precsim$ " on a real or complex convex space V that is locally non-satiated (cf. the introduction) and has the additional property that all its inner indifference classes are path connected. *In this case the authors conjecture that there exists a homeomorphism $h : V \longrightarrow V$ such that every inner* *indifference class of the continuous total preorder* " $\precsim_h$ " *on* X *that is defined by* $h(x) \precsim_h h(y) \Longleftrightarrow x \precsim y$ *is a hyperplane of* V.

If, however, an arbitrary abstract topological space is given an appropriate generalization of Proposition 4.9 hardly seems to be possible, and the cardinality approach gains additional importance.

We conclude by proving the following proposition that is a slight generalization of well known results in general topology (cf. Eilenberg [9] or Willard [15, Section 28] or Kok [12, Chapter II]). It provides a new proof of these classical results and includes for every natural number $n \geq 1$ the family of all connected and convex subspaces of the Euclidean space $(\mathbb{R}^n, t_{nat})$. Therefore, it also generalizes Theorem 4 of Candeal and Induráin [6] and is closely related to generalizations of this theorem by Beardon [2] and Candeal, Induráin and Mehta [7] (cf., in particular, [7, Theorem 4]), where necessary and sufficient conditions for the agreement of the Euclidean and the order topology on totally ordered subsets of $\mathbb{R}^n$ are discussed.

Proposition 4.11. *Let* (X, t) *be a separable, connected and locally connected Hausdorff space. Then the following assertions are equivalent:*

(i) *Every non-trivial connected subspace* $\left(Y, t_{|Y}\right)$ *of* (X, t) *contains some point* y *such that* $Y \setminus \{y\}$ *splits into two components.*

(ii) *There exists some continuous total order* " $\preceq$ " *on* X.

(iii) *There exists some real interval* I *such that* (X, t) *is homeomorphic to* (I, t_{nat}).

Proof. Since the proof of the implication "(iii)$\Longrightarrow$(i)" is straightforward it suffices to verify the validity of the implications "(i)$\Longrightarrow$(ii)" and "(ii)$\Longrightarrow$(iii)".

(i)$\Longrightarrow$(ii): We construct by transfinite induction a continuous total order " $\preceq$ " on X.

$\alpha = 0$: We set $\precsim_0 := X \times X$.

$0 < \alpha$ is not a limit ordinal: Let $I_{\alpha-1} := \{[x] \mid [x]\ \textit{is a non-trivial indifference class of}\ "\precsim_{\alpha-1}"\}$. In case that $I_{\alpha-1} = \varnothing$ we set $\preceq := \precsim_{\alpha-1}$ in order to then finish the induction process. Otherwise, we choose an arbitrary indifference class $[x] \in I_{\alpha-1}$. Because of the induction hypothesis we may assume that $\left(([x], t_{|[x]}\right)$ is a connected (and locally connected) Hausdorff space and that $[x] = X$ or there exists a single point $s \in X$ such that

$$s = \max\{u \in X | u \prec_{\alpha-1} x\}$$

or there exists a single point $t \in X$ such that

$$t = \min\{v \in X | x \prec_{\alpha-1} v\}.$$

We abbreviate these assumptions on $[x]$ by (*). In addition, the induction hypothesis allows us to assume that the sets $d_{\alpha-1}(s) := \{u \in X | u \precsim_{\alpha-1} s\}$ and $i_{\alpha-1}(s) := \{u \in X | s \precsim_{\alpha-1} u\}$ or $d_{\alpha-1}(t) := \{v \in X | v \precsim_{\alpha-1} t\}$ and $i_{\alpha-1}(t) := \{v \in X | t \precsim_{\alpha-1} v\}$ are closed subsets of X. Let us abbreviate these last assumptions by (**). Since the afore-presented cases can be settled by analogous arguments we may concentrate without loss of generality on the case that there exist single points s, $t \in X$ such that $s = \max\{u \in X | u \prec_{\alpha-1} x\}$ and $t = \min\{v \in X | x \prec_{\alpha-1} v\}$. Assertion (i) implies the existence of some point $z \in [x]$ such that $[x] \setminus \{z\}$ splits into two (non-empty) connected and then, obviously, locally connected components C_1^α and C_2^α. The assumptions (**) imply that $[x]$ is an open subset of X. Since (X, t) is a locally connected Hausdorff space we, thus, may conclude that also C_1^α and C_2^α are open subsets of X. Furthermore, these assumptions on (X, t) allows us to assume without loss of generality that the (topological) closure $\overline{C_1^\alpha}$ of C_1^α is $C_1^\alpha \cup \{s, z\}$ and that the (topological) closure $\overline{C_2^\alpha}$ of C_2^α is $C_1^\alpha \cup \{z, t\}$. Hence, we set $\precsim_\alpha := \precsim_{\alpha-1} \setminus \{(u, v) \in X \times X | u \in C_2^\alpha \cup \{z\}$ *and* $v \in C_1^\alpha$ *or* $u \in C_2^\alpha$ *and* $v \in C_1^\alpha \cup \{z\}\}$. Summarizing the afore-considered arguments it follows that both sets C_1^α and C_2^α are indifference classes of " $\precsim_\alpha$ " and that the assumptions (*) and (**) are also satisfied with respect to " $\precsim_\alpha$ ".

$0 < \alpha$ is a limit ordinal: In this case we set $\precsim_\alpha := \bigcap_{\beta < \alpha} \precsim_\beta$.

The reader may verify that in this way, actually, a total order " $\preceq$ " on X has been constructed. With the help of the assumptions (**) we may conclude, in addition, that " $\preceq$ " is a continuous total order on X.

(ii)$\Longrightarrow$(iii): Let " $\preceq$ " be a continuous total order on X. The general assumptions of the proposition imply with the help of Eilenberg's well known re-presentation theorem [9] that there exists some real interval I such that $(X, t^{\preceq})$ is homeomorphic to (I, t_{nat}). It, thus, suffices to show that $t^{\preceq} = t$. The continuity of " $\preceq$ " allows us to conclude that $t^{\preceq} \subset t$ and it remains to verify that also $t \subset t^{\preceq}$. Since (X, t) is locally connected we may choose some connected open subset O of X. Then we must prove that $O \in t^{\preceq}$. Because of the assumption that " $\preceq$ " is a continuous total order on X it follows from the connectedness of O that O is an interval of $(X, \preceq)$. Since O is an open subset of X the connectedness of (X, t) and the continuity of " $\preceq$ " imply, moreover, that O must be an open interval of $(X, \preceq)$. Hence, $O \in t^{\preceq}$ and nothing remains to be shown. □

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