Applied General Topology
© Universidad Politécnica de Valencia
Volume 6, No. 2, 2005
pp. 135-142

Compactness properties of bounded subsets of spaces of vector measure integrable functions and factorization of operators

L. M. García-Raffi and E. A. Sánchez-Pérez*

Abstract. Using compactness properties of bounded subsets of spaces of vector measure integrable functions and a representation theorem for q-convex Banach lattices, we prove a domination theorem for operators between Banach lattices. We generalize in this way several classical factorization results for operators between these spaces, as p-summing operators.

2000 AMS Classification: 46G10, 54D30.

Keywords: Compactness, vector measures, integration, factorization.

1. Introduction

Compactness of the unit ball of Banach spaces is a useful tool in the theory of operators between these spaces. One of the basic arguments that provides important applications in this field uses Ky Fan´s Lemma with a family of functions on the unit ball of a Banach space that are continuous with respect to the weak* topology. This argument can be found in the proof of the Pietsch Domination Theorem for p-summing operators, the characterization of the p,q-dominated operators or the Maurey-Rosenthal Theorem for factorization of operators through L_p-spaces (see for instance [12, 15, 5]). Roughly speaking, weak* compactness of the unit ball of a Banach space is one of the keys to relate vector valued norm inequalities and domination/factorization theorems for operators.

*The authors acknowledge the support of the Generalitat Valenciana, Spain, grant GV04B-371, the Spanish Ministry of Science and Technology, Plan Nacional I+D+I , grant BFM2003-02302; and the support of the Universidad Politécnica de Valencia, under grant 2003-4114 for Interdisciplinary Research Projects.

In the context of the spaces $L_q(m)$ of q-integrable functions with respect to the (countably additive) vector measure m, it is possible to obtain more compactness results with respect to topologies that are defined using the properties of the integration map that appears in a natural way in this framework. In particular, we will use the fact that for reflexive sapces $L_q(m)$ the unit ball of $L_q(m)$, $1 < q < \infty$, is compact for the m-weak topology (see Proposition 13 in [13] for the λ-weak topology, assuming $L_q(m)$ is reflexive). A characterization of the compactness of the unit ball of such spaces with respect to other different topology can also be found in [13] (see Theorem 14 for the λ-topology). More compactness results for the integration operator have been recently obtained in [11] (see also [10]).

In this paper we present a domination theorem for operators that satisfy a p-summing type vector norm inequality. For its proof, we use the compactness of bounded sets in one of the topologies quoted above on spaces of q-integrable functions with respect to a vector measure. Every q-convex Banach lattice with order continuous norm and weak order unit can be represented as a space of integrable functions with respect to a vector measure (see Proposition 2.4 in [7]). We use these representations of the Banach lattices and the compactness with respect to the m-weak topology of their unit balls to prove a (representation-depending) general domination theorem for operators on q-convex Banach lattices. Let E be an order continuous q-convex Banach lattice with weak order unit, $1 \leq q < \infty$. We will say that E is *q-represented* by the vector measure $m : \Sigma \to X$, where X is a Banach space, if E is order isomorphic to $L_q(m)$. As a direct consequence of the proposition quoted above, such a representation always exists for every such a Banach lattice E.

We use standard Banach lattice concepts and notation (see [8, 15]). If $1 \leq p \leq \infty$, we write p' for the extended real number that satisfies $1/p + 1/p' = 1$. We will write R for the set of real numbers. Let E be a Banach lattice and $1 \leq r < \infty$. It is said that E is r-convex if there is a constant $c > 0$ such that for every finite sequence $x_1, ..., x_n \in E$,

$$\|(\sum_{k=1}^{n} |x_k|^r)^{\frac{1}{r}}\| \leq c(\sum_{k=1}^{n} \|x_k\|^r)^{\frac{1}{r}}.$$

The real number $M^{(r)}(E)$ defined as the best constant c in the inequality above is called the r-convex constant of E.

Let X, Y be a pair of Banach spaces, $1 \leq p < \infty$, and consider an operator $T : X \to Y$. T is p-summing (p-absolutely summing in [12]) if there is a constant $c > 0$ such that for every finite set $x_1, ..., x_n \in X$, the inequality

$$(\sum_{i=1}^{n} \|T(x_i)\|^p)^{\frac{1}{p}} \leq c \sup_{x' \in B_{X'}} (\sum_{i=1}^{n} | < x_i, x' > |^p)^{\frac{1}{p}}$$

holds (see e.g. [12, 5]).

The Pietsch Domination Theorem establishes that an operator $T : X \to Y$ is p-summing if and only if there is a (regular Borel) probability measure μ on the weak* compact set $B_{X'}$ and a positive constant c such that

$$\|T(x)\| \leq c(\int_{B_{X'}} |< x, x' >|^p d\mu)^{\frac{1}{p}}, \qquad x \in X.$$

In this paper we provide a new version of this result. We complete in this way the results of [13] that relates compactness properties of the unit ball of the spaces $L_q(m)$ of a vector measure with domination/factorization theorems (see also [9]). This is the reason we assume through all the paper that the spaces $L_q(m)$ involved are reflexive.

Let X be a Banach space and let (Ω, Σ) be a measurable space. Consider a countably additive vector measure $m : \Sigma \to X$. We say that a measurable function $f : \Omega \to R$ is integrable with respect to m if it is scalarly integrable (i.e. it is integrable with respect to every scalar measure $m_{x'}$, $x' \in X'$, given by $m_{x'}(A) :=< m(A), x' >$, $A \in \Sigma$), and there is an element $\int_\Omega f dm \in X$ such that for every $x' \in X'$, $< \int_\Omega f dm, x' >= \int_\Omega f dm_{x'}$ (see for instance [1]).

A Rybakov measure for m is a measure defined by the variation $|m_{x'}|$ of a measure $m_{x'}$ that controls m (see [6]). The space $L_1(m)$ of integrable functions with respect to m is the Banach space of all the classes of $m_{x'}$-a.e. equal functions, where $m_{x'}$ is a Rybakov measure for m. Endowed with the norm $\|f\|_{L_1(m)} = \sup_{x' \in B_{X'}} \int_\Omega |f| d|m_{x'}|$ and the $|m_{x'}|$-a.e. order, it is a Köthe function space over $|m_{x'}|$ with weak unit. The reader can see [1, 2] for the fundamental facts about these spaces. If $1 < q < \infty$, we say that a measurable function f is q-integrable with respect to m if $|f|^q \in L_1(m)$. The construction of the space $L_q(m)$ follows in the same way that in the case of $L_1(m)$. It is also a Köthe function space over $|m_{x'}|$ and the norm is given by

$$\|f\|_{L_q(m)} = \sup_{x' \in B_{X'}} (\int_\Omega |f|^q d|m_{x'}|)^{\frac{1}{q}} \qquad f \in L_q(m),$$

(see [13, 7]). This space is q-convex when considered as a Banach lattice.

2. Extensions of operators defined on spaces of integrable functions with respect to a vector measure

Let $1 \leq q < \infty$ and consider an element $x' \in X'$ such that the measure $m_{x'}$ is positive. It is easy to see that the operator $i_{x'} : L_q(m) \to L_q(m_{x'})$ defined by $i_{x'}(f) := f$, $f \in L_q(m)$ is well-defined and continuous. Moreover, $\|i_{x'}\| \leq 1$. However, note that we can assure that $i_{x'}$ is an injection only if $m_{x'}$ is a Rybakov measure for m.

Definition 2.1. *Consider two Banach spaces X, Y, a family of Banach spaces $\mathcal{B} = \{X_i : i \in I\}$, and an operator $T : X \to Y$. We say that T can be uniformly extended to $\mathcal{B}$ if the identity map $i_{X_i} : X \to X_i$ is defined, continuous and $\|i_{X_i}\| \leq 1$, for every $i \in I$, and there is a constant $c > 0$ such that all the extensions $T_i : X_i \to Y$ of the operator T (i.e. $T_i \circ i_{X_i}(x) = T(x)$, $x \in X$) are defined, continuous and $\|T_i\| \leq c$.*

Proposition 2.2. *Let $m : \Sigma \to X$ be a countably additive vector measure, Y a Banach space and $1 \leq q < \infty$, and consider an operator $T : L_q(m) \to Y$. Suppose that there is a subset $S \subset X'$ such that for every $x' \in S$, $\|x'\| = 1$ and $m_{x'}$ is a positive measure. Then the following conditions are equivalent.*

(1) *There is a constant $c > 0$ such that for every $x' \in S$,*

$$\|T(f)\| \leq c\Big(\int_\Omega |f|^q dm_{x'}\Big)^{\frac{1}{q}}, \qquad f \in L_q(m).$$

(2) *The operator T can be uniformly extended to all the spaces $L_q(m_{x'})$, $x' \in S$.*

Proof. Let us show (1) $\to$ (2). First note that the inequality of (1) provides the way of extending T to every space $L_q(m_{x'})$, $x' \in S$. Let us write $[f]_{x'}$ for the equivalence class of the function $f \in L_q(m_{x'})$ (only for the aim of this proof, in the rest of the paper we will simply write f). Suppose that $f_1 \neq f_2$ as elements of $L_q(m)$ but $[f_1]_{x'} = [f_2]_{x'}$. Then, (1) gives

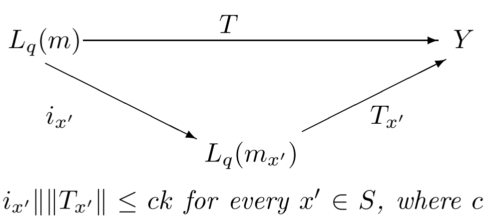
$$\|T(f_1 - f_2)\| \leq c\Big(\int_\Omega |f_1 - f_2|^q dm_{x'}\Big)^{\frac{1}{q}} = 0,$$

and thus $T(f_1) = T(f_2)$.

Now, let us show that the argument above is enough to prove that the operator T is well-defined. The simple functions are dense in the spaces $L_q(m)$ for every countably additive vector measure m (see [13]). Then, for every $x' \in S$ the operator $T_{x'} : L_q(m_{x'}) \to Y$ given by $T_{x'}(f) := T(f)$ for every simple function f and extended to all $L_q(m_{x'})$ by continuity is well-defined. Moreover, we directly obtain $\|T_{x'}\| \leq c$ as a consequence of the inequality (1). Since this argument does not depend on $x' \in S$, we obtain (2). The converse is obvious. □

The theorem above provides a family of factorization theorems through L_q-spaces (indexed by S). Indeed, since the identity map $i_{x'} : L_q(m) \to L_q(m_{x'})$ is continuous, we directly obtain the following

Corollary 2.3. *Let E be a q-convex Banach lattice that can be q-represented by the vector measure m. Consider an operator $T : E \to Y$ that satisfies* (1) *or* (2) *in Proposition 2.2 for a subset $S \subset X'$ satisfying the conditions in this proposition. Then for every $x' \in S$, T can be factorized as follows.*



Moreover, $\|i_{x'}\|\|T_{x'}\| \leq ck$ for every $x' \in S$, where c is the constant given in Proposition 2.2 and k is the corresponding constant of the equivalence of norms between $\|.\|_E$ and $\|.\|_{L_q(m)}$.

A particular straightforward application of this result -that provides also the canonical situation of this extension theorem- is the case when S contains only one element x'. In this case, we obtain directly an extension/factorization theorem through an L_q-space. Using the representation theorem for Banach lattices given by Proposition 2.4 in [7] quoted in Section 1, we obtain a Maurey-Rosenthal type factorization for an operator T whenever it satisfies an inequality as the one given by Theorem 2.2. Moreover, in this case the multiplication operator that defines the factorization is simply the identity.

3. A Pietsch type domination theorem for operators on spaces of p-integrable functions with respect to a vector measure

In this section we provide a domination theorem for operators that satisfy a vector valued norm inequality involving strong and weak convergent sequences. We obtain in this way a Pietsch type domination theorem for operators on reflexive q-convex Banach lattices, and complete the research that we started in [13]. In this paper, we obtained a factorization theorem through spaces of Bochner integrable functions and we characterized this situation by means of a vector valued norm inequality, whenever a certain compactness property for the integration operator was fulfilled (Theorem 17 in [13]). The key for the proof of this factorization result is the requirement of compactness of the unit ball of the space $L_q(m)$, where m is a countably additive vector measure, with respect to the m-topology (the λ-topology in [13]). The theorem of this section gives the weak version of this result. However, no compactness requirement is needed in this case, since the unit ball of a reflexive space $L_q(m)$ is always compact with respect to the m-weak topology. These compactness properties of the unit ball of q-convex Banach lattices represented by L_q spaces of a vector measure (Proposition 13 and Theorem 14 in [13]), can be generalized to all bounded subsets under the (obvious) adequate requirements.

First, let us write the definition of the m-weak topology for the space $L_q(m)$, where $m : \Sigma \to X$ is a countably additive vector measure and $q > 1$. This is the topology that has as a basis of neighborhoods of an element $g_0 \in L_q(m)$ the following sets. Let $\epsilon > 0$, $n \in \mathbf{N}$, $x'_1, ..., n'_n \in X'$ and $f_1, f_2, ..., f_n \in L_{q'}(m)$. We define the set

$$\xi_{\epsilon, f_1, \ldots, f_n, x'_1, \ldots, x'_n}(g_0)$$

$$:= \{g \in L_q(m) : | < \int_\Omega f_i (g - g_0) dm, x'_i > | < \epsilon, \forall i = 1, ..., n\}.$$

The m-weak topology is the topology which has as a basis of neighborhoods the family of sets

$$\xi_{\epsilon, f_1, \ldots, f_n, x'_1, \ldots, x'_n}(g_0).$$

It is easy to prove that this topology is a well-defined Hausdorff locally convex topology on $L_q(m)$. The reader can find more information about it in [13].

Theorem 3.1. *Let E be a q-convex Banach lattice that can be q-represented by the vector measure $m : \Sigma \to X$, where X is a Banach space and $1 < q < \infty$. Suppose that $L_{q'}(m)$ is reflexive. Let $1 \leq p < \infty$. Consider an operator*

$T : E \to X$, and suppose that there is a subset $S \subset X'$ such that for every $x' \in S$, $\|x'\| = 1$ and $m_{x'}$ is a positive measure. Then the following conditions are equivalent.

(1) *There is a constant $c > 0$ such that for every pair of finite families $x'_1, ..., x'_n \in S$ and $f_1, ..., f_n \in L_q(m)$*

$$(\sum_{i=1}^{n} \|T(f_i)\|^p)^{\frac{1}{p}} \leq c \sup_{g \in B_{L_{q'}(m)}} (\sum_{i=1}^{n} | < \int_\Omega f_i g dm, x'_i > |^p)^{\frac{1}{p}}.$$

(2) *There is a constant $c > 0$ and a regular Borel probability measure μ over the compact Hausdorff space $B_{L_{q'}(m)}$ endowed with the m-weak topology such that*

$$\|T(f)\| \leq c \inf_{x' \in S} (\int_{B_{L_{q'}(m)}} | < \int_\Omega f g dm, x' > |^p d\mu(g))^{\frac{1}{p}}$$

for every $f \in L_q(m)$.

Moreover, the infimum of all the constants c that satisfy (1) *coincides with the infimum of all the constants c in* (2).

Proof. The conditions on E and m allow us to consider that the operator T is directly defined on $L_q(m)$. For the proof of this result we adapt the argument that proves the Pietsch Domination Theorem for p-summing operators. A direct calculation gives (2) $\to$ (1). For the converse, consider the m-weak compact (convex and Hausdorff) set $B_{L_{q'}(m)}$ and the space $C(B_{L_{q'}(m)})$ of continuous functions on $B_{L_{q'}(m)}$, with respect to the m-weak topology. Consider its dual, the space of regular Borel measures $\mathcal{M}$, and the (compact and convex) subset of probability measures $\mathcal{P}$.

For every pair of finite families $x'_1, ..., x'_n \in S$ and $f_1, ..., f_n \in L_q(m)$ we define the function $\phi_{x'_1,...,x'_n,f_1,...,f_n} : \mathcal{P} \to R$,

$$\phi_{x'_1,...,x'_n,f_1,...,f_n}(\mu) := (\sum_{i=1}^{n} \|T(f_i)\|^p) - c^p \int_{B_{L_{p'}}} \sum_{i=1}^{n} | < \int_\Omega f_i g dm, x'_i > |^p d\mu.$$

Note that the inequality given in (1) provides an element $g_0 \in B_{L_{q'}(m)}$ such that

$$\sum_{i=1}^{n} \|T(f_i)\|^p \leq c^p \sum_{i=1}^{n} | < \int_\Omega f_i g_0 dm, x'_i > |^p.$$

Thus, for each function $\phi_{x'_1,...,x'_n,f_1,...,f_n}$, there is a probability measure (the Dirac measure at the point g_0, δ_{g_0}) such that $\phi_{x'_1,...,x'_n,f_1,...,f_n}(\delta_{g_0}) \leq 0$.

It is easy to see that the set of all the functions as $\phi_{x'_1,...,x'_n,f_1,...,f_n}$ is concave. In fact, it is clear that the sum of two such functions gives other function of the family. Moreover, the product of a function like this and a positive scalar is also other function of the family (it is enough to consider the product of the same functions f_i that define the function of the family by the scalar to the power $1/p$ to define the new function). Thus we can apply Ky Fan´s Lemma to obtain an element of $\mathcal{P}$ that satisfies the inequalities of the type of (2) for all functions $\phi_{x'_1,\dots,x'_n,f_1,\dots,f_n}$. Thus, there is a probability measure $\mu \in \mathcal{P}$ such that

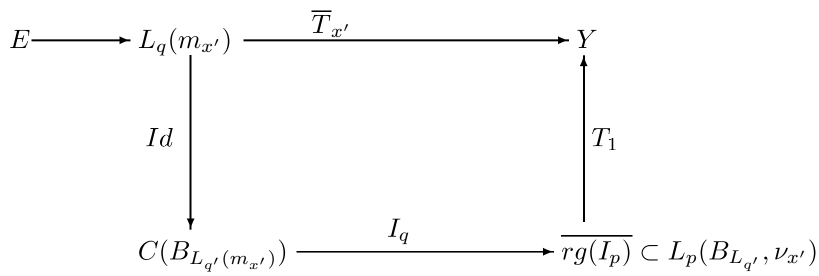
$$\|T(f)\| \leq c\left(\int_{B_{L_{q'}(m)}} |<\int_\Omega fg dm, x'>|^p d\mu(g)\right)^{\frac{1}{p}}$$

for every $f \in L_q(m)$ and each $x' \in S$. This gives the result.

□

The canonical situation that generalizes this theorem is the case of a p-summing operator on $L_q(\nu)$ of a scalar measure ν. In this case, we obtain a factorization through the identity operator $i : C(B_{L_{p'}}) \to L_q(B_{L_{p'}}, \mu)$, as can be obtained as a direct application of the Pietsch Domination Theorem. The set S contains only one element (formally $S = \{1\}$), since the range of ν is a subset of R.

In the general case, an operator T that satisfies the conditions of Theorem 3.1 verifies a family of factorizations indexed by the same set S. Note that the conditions of Proposition 3.1 imply in particular the ones of Theorem 2.2. Moreover, (2) of Theorem 3.1 implies a p-summing inequality for each extension to an $L_q(m_{x'})$-space. Thus, if $x' \in S$ and $T : E \to Y$ satisfies (1) of the theorem, there is a probability measure $\nu_{x'}$ such that T can be factorized as



where $\overline{T}_{x'}$ is the extension of the operator T given in Theorem 2.2, E is included continuously in $L_q(m_{x'})$, Id and I_q are inclusion operators, T_1 is a continuous map and $\overline{rg(I_p)}$ is the (norm) closure of $I_p(C(B_{L_{q'}}))$ in $L_p(B_{L_{q'}})$.

Therefore, Theorem 3.1 provides a family of mixed factorization schemes. We can obtain a factorization of the operator T through an L_q-space, a $C(K)$-space and an L_p-space for each $x' \in S$. The general theory of operator ideals and its applications in the theory of Banach spaces can then be used to relate this result with well-known properties of operators and Banach spaces (see [4]). For instance, we directly obtain that the conditions of our theorem imply that it is (p, q')-factorable (see Theorem 19.4.6 in [12]).

The results of this section complete in this way the domination/factorization results given in [13] (see also [9]); all of them can be obtained using the compactness properties of the unit ball of $L_q(m)$-spaces with respect to different topologies defined by means of the integration operator associated to m.

References

[1] G. P. Curbera, *Operators into L_1 of a vector measure and applications to Banach lattices*, Math. Ann. **293** (1992), 317-330.

[2] G. P. Curbera, *Banach space properties of L_1 of a vector measure*, Proc. Amer. Math. Soc. **123** (1995), 3797-3806.

[3] A. Defant, *Variants of the Maurey-Rosenthal theorem for quasi Köthe function spaces*, Positivity **5** (2001), 153-175.

[4] A. Defant and K. Floret, "Tensor Norms and Operator Ideals", North Holland, Amsterdam (1993).

[5] J. Diestel, H. Jarchow and A. Tonge, "Absolutely Summing Operators", Cambridge studies in advanced mathematics **43**, Cambridge (1995).

[6] J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys, **15**, Amer. Math. Soc., Providence, RI. 1977.

[7] A. Fernández, F. Mayoral, F. Naranjo, F. Sáez and E. A. Sánchez-Pérez, *Spaces of p-integrable functions with respect to a vector measure*, Positivity, to appear.

[8] J. Lindenstrauss and L. Tzafriri, "Classical Banach Spaces I and II", Springer, Berlin (1996).

[9] F. Martínez-Giménez and E. A. Sánchez-Pérez, *Vector measure range duality and factorizations of (D, p)-summing operators from Banach function spaces*, Bull. Braz. Math. Soc., New Series **35**(1)(2004), 51-69.

[10] S. Okada and W. J. Ricker, *The range of the integration map of a vector measure*, Arch. Math. **64**(1995), 512-522.

[11] S. Okada, W. J. Ricker and L. Rodríguez-Piazza, *Compactness of the integration operator associated with a vector measure*, Studia Math. **150**(2) (2002), 133-149.

[12] A. Pietsch, "Operator Ideals", North-Holland, Amsterdam (1980).

[13] E. A. Sánchez-Pérez, *Compactness arguments for spaces of p-integrable functions with respect to a vector measure and factorization of operators through Lebesgue-Bochner spaces*, Illinois J. Math. **45**(3) (2001), 907-923.

[14] E. A. Sánchez-Pérez, *Spaces of integrable functions with respect to vector measures of convex range and factorization of operators from L_p-spaces*, Pacific J. Math. **207** (2) (2002), 489-495.

[15] P. Wojtaszczyk, "Banach Spaces for Analysts", Cambridge University Press. Cambridge. 1991.

Received September 2004

Accepted September 2004

Luis M. García-Raffi (lmgarcia@mat.upv.es)
E.T.S.I. Caminos, Canales y Puertos. Departamento de Matemática Aplicada.
Universidad Politécnica de Valencia. 46071 Valencia, Spain.

E. A. Sánchez-Pérez (easancpe@mat.upv.es)
E.T.S.I. Caminos, Canales y Puertos. Departamento de Matemática Aplicada.
Universidad Politécnica de Valencia. 46071 Valencia, Spain.