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Tightness of function spaces

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Abstract. The purpose of this paper is to give higher cardinality versions of countable fan tightness of function spaces obtained by A. Arhangel'skiĭ. Let $\mathrm{vet}(X), \omega\mathrm{H}(X)$ and $\mathrm{H}(X)$ denote respectively the fan tightness, ω-Hurewicz number and Hurewicz number of a space X, then $\mathrm{vet}(\mathrm{C}_p(X)) = \omega\mathrm{H}(X) = \sup\{\mathrm{H}(X^n) : n \in \mathbb{N}\}$.

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The general question in the theory of function spaces is to characterize topological properties of the space, $\mathrm{C}(X)$, of continuous real-valued functions on a topological space X. A study of some convergence properties in function spaces is an important task of general topology. It have been obtained interested results on some higher cardinal properties of first-countability, Fréchet properties, tightness[2, 4, 6, 9]. Arhangel'skiĭ-Pytkeev theorem[2] is a nice result about tightness of function spaces: $\mathrm{t}(\mathrm{C}_p(X)) = \sup\{\mathrm{L}(X^n) : n \in \mathbb{N}\}$ for any Tychonoff space X. The following result on countable fan tightness of function spaces is shown by A. Arhangel'skiĭ[1]: $\mathrm{C}_p(X)$ has countable fan tightness if and only if X^n is a Hurewicz space for each $n \in \mathbb{N}$ for an arbitrary space X. In this paper the higher cardinality versions of countable fan tightness of $\mathrm{C}_p(X)$ are obtained.

In this paper all spaces will be Tychonoff spaces. Let α be a network of compact subsets of a space X, which is closed under finite unions and closed subsets. Then the space $\mathrm{C}_\alpha(X)$ is the set $\mathrm{C}(X)$ with the set-open topology as follows[9]: The subbasic open sets of the form $[A, V] = \{f \in \mathrm{C}(X) : f(A) \subset V\}$, where $A \in \alpha$ and V is open in $\mathbb{R}$. Then $\mathrm{C}_\alpha(X)$ is a topological vector space[9]. The family of all compact subsets of X generates the compact-open topology,

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denoted by $C_k(X)$. Also the family of all finite subsets of X generates the topology of pointwise convergence, denoted by $C_p(X)$. For each $f \in C(X)$, a basic neighborhood of f in $C_p(X)$ can be expressed as $W(f, K, \varepsilon)$ for each finite subset K of X and $\varepsilon > 0$, here $W(f, K, \varepsilon) = \{g \in C(X) : |f(x) - g(x)| < \varepsilon$ for each $x \in K\}$. In this paper the alphabet λ is an infinite cardinal number, γ is an ordinal number, and i, m, n, j, k are natural numbers.

The fan tightness of a space X is defined by $\mathrm{vet}(X) = \sup\{\mathrm{vet}(X, x) : x \in X\}$, here $\mathrm{vet}(X, x) = \omega + \min\{\lambda :$ for each family $\{A_\gamma\}_{\gamma<\lambda}$ of subsets of X with $x \in \bigcap_{\gamma<\lambda} \overline{A}_\gamma$ there is a subset $B_\gamma \subset A_\gamma$ with $|B_\gamma| < \lambda$ for each $\gamma < \lambda$ such that $x \in \overline{\bigcup_{\gamma<\lambda} B_\gamma}\}$. A space X has countable fan tightness[1] if and only if $\mathrm{vet}(X) = \omega$. An α-cover of a space X is a family of subsets of X such that every member of α is contained in some member of this family. An α-cover is called a k-cover if α is the set of all compact subsets of X. Also an α-cover is called an ω-cover if α is the set of all finite subsets of X. The α-Hurewicz number of X is defined by $\alpha\mathrm{H}(X) = \omega + \min\{\lambda :$ for each family $\{\mathcal{U}_\gamma\}_{\gamma<\lambda}$ of open α-covers of X there is a subset $\mathcal{B}_\gamma \subset \mathcal{U}_\gamma$ with $|B_\gamma| < \lambda$ for each $\gamma < \lambda$ such that $\bigcup_{\gamma<\lambda} B_\gamma$ is an α-cover of $X\}$. The α-Hurewicz number of X is called the Hurewicz number of X and written $\mathrm{H}(X)$ if α consists of the singleton of X. A space X is Hurewicz space[5] if and only if $\mathrm{H}(X) = \omega$.

Theorem 1. *$vet(C_\alpha(X)) = \alpha H(X)$ for any space X.*

Proof. Let $\lambda = \mathrm{vet}(C_\alpha(X))$, and let $\{\mathcal{U}_\gamma\}_{\gamma<\lambda}$ be any family of open α-covers of X. For each $\gamma < \lambda$, put $A_\gamma = \{f \in C_\alpha(X) :$ there is $U \in \mathcal{U}_\gamma$ such that $f(X \setminus U) \subset \{0\}\}$. Then A_γ is dense in $C_\alpha(X)$. In fact, let $\bigcap_{i\leq m}[K_i, V_i]$ be a non-empty basic open set of $C_\alpha(X)$, fix $f \in \bigcap_{i\leq m}[K_i, V_i]$. There is $U \in \mathcal{U}_\gamma$ such that $\bigcup_{i\leq m} K_i \subset U$ because $\mathcal{U}_\gamma$ is an α-cover on X. Since $\bigcup_{i\leq m} K_i$ is compact in Tychonoff space X, there is $g \in C_\alpha(X)$ such that $g_{|\cup_{i\leq m} K_i} = f_{|\cup_{i\leq m} K_i}$ and $g(X \setminus U) \subset \{0\}$. Then $g \in A_\gamma \cap (\bigcap_{i\leq m}[K_i, V_i])$, and $\overline{A}_\gamma = C_\alpha(X)$.

Take $f_1 \in C(X)$ with $f_1(X) = \{1\}$, then $f_1 \in \bigcap_{\gamma<\lambda} \overline{A}_\gamma$. For each $\gamma < \lambda$ there is a subset $B_\gamma \subset A_\gamma$ with $|B_\gamma| < \lambda$ such that $f_1 \in \overline{\bigcup_{\gamma<\lambda} B_\gamma}$ by $\lambda = \mathrm{vet}(C_\alpha(X))$. Denote $B_\gamma = \{f_\kappa\}_{\kappa\in\Phi_\gamma}$, here $|\Phi_\gamma| < \lambda$. There is $U_\kappa \in \mathcal{U}_\gamma$ such that $f_\kappa(X \setminus U_\kappa) \subset \{0\}$ for each $\kappa \in \Phi_\gamma$. Put $\mathcal{U}'_\gamma = \{U_\kappa\}_{\kappa\in\Phi_\gamma}$. Then $\bigcup_{\gamma<\lambda} \mathcal{U}'_\gamma$ is an α-cover of X. In fact, for each $A \in \alpha$, since $f_1 \in [A, (0, 2)]$, there are $\gamma < \lambda$ and $\kappa \in \Phi_\gamma$ such that $f_\kappa \in [A, (0, 2)]$, then $A \subset U_\kappa$, so $\bigcup_{\gamma<\lambda} \mathcal{U}'_\gamma$ is an α-cover of X. This shows that $\alpha\mathrm{H}(X) \leq \mathrm{vet}(C_\alpha(X))$.

To show the reverse inequality, let $\lambda = \alpha\mathrm{H}(X)$. Since $C_\alpha(X)$ is a topological vector space, it is homogeneous. It suffices to show that $\mathrm{vet}(C_\alpha(X), f_0) \leq \lambda$, here $f_0 \in C(X)$ with $f_0(X) = \{0\}$. Suppose that $f_0 \in \bigcap_{\gamma<\lambda} \overline{A}_\gamma$ with each $A_\gamma \subset C_\alpha(X)$. For each $\gamma < \lambda$ and $n \in \mathbb{N}$, put $\mathcal{U}_{\gamma,n} = \{f^{-1}(O_n) : f \in A_\gamma\}$, here $\{O_n\}_{n\in\mathbb{N}}$ is a decreasing local base of 0 in $\mathbb{R}$. Then $\mathcal{U}_{\gamma,n}$ is an open α-cover of X. In fact, for each $A \in \alpha, f_0 \in [A, O_n]$, there is $f \in [A, O_n] \cap A_\gamma$, thus $A \subset f^{-1}(O_n) \in \mathcal{U}_{\gamma,n}$.

Case 1. $\lambda > \omega$. For each $n \in \mathbb{N}$, since $\{\mathcal{U}_{\gamma,n}\}_{\gamma<\lambda}$ is a family of open α-covers of X, there is a subset $\mathcal{U}'_{\gamma,n} \subset \mathcal{U}_{\gamma,n}$ with $|\mathcal{U}'_{\gamma,n}| < \lambda$ for each $\gamma < \lambda$ such that $\bigcup_{\gamma<\lambda} \mathcal{U}'_{\gamma,n}$ is an open α-cover of X. Denote $\mathcal{U}'_{\gamma,n} = \{U_\tau\}_{\tau \in \Phi_{\gamma,n}}$. There is $f_\tau \in A_\gamma$ such that $U_\tau = f_\tau^{-1}(O_n)$ for each $\tau \in \Phi_{\gamma,n}$. Let $B_\gamma = \{f_\tau : \tau \in \Phi_{\gamma,n}, n \in \mathbb{N}\}$. Then $B_\gamma \subset A_\gamma$ and $|B_\gamma| < \lambda$. We show that $f_0 \in \overline{\bigcup_{\gamma<\lambda} B_\gamma}$. For arbitrary basic neighborhood $[A, V]$ of f_0 in $\mathrm{C}_\alpha(X)$, there is $n \in \mathbb{N}$ such that $O_n \subset V$. Since $\bigcup_{\gamma<\lambda} \mathcal{U}'_{\gamma,n}$ is an open α-cover of X, there are $\gamma < \lambda$ and $\tau \in \Phi_{\gamma,n}$ such that $A \subset U_\tau = f_\tau^{-1}(O_n)$, hence $f_\tau(A) \subset V$, i.e., $f_\tau \in [A, V]$, so $f_0 \in \overline{\{f_\tau : \tau \in \Phi_{\gamma,n}, n \in \mathbb{N}, \gamma < \lambda\}} = \overline{\bigcup_{\gamma<\lambda} B_\gamma}$.

Case 2. $\lambda = \omega$. Put $M = \{n \in \mathbb{N} : X \in \mathcal{U}_{n,n}\}$. If M is infinite, there is $m \in M$ such that $O_m \subset V$ for arbitrary basic neighborhood $[A, V]$ of f_0 in $\mathrm{C}_\alpha(X)$. By the definition of $\mathcal{U}_{m,m}$, there is $g_m \in A_m$ such that $X = g_m^{-1}(O_m)$, then $g_m(X) \subset V$, so $g_m \in [A, V]$, thus the sequence $\{g_m\}_{m \in M}$ converges to f_0. If M is finite, there is $n_0 \in \mathbb{N}$ such that for each $m \geq n_0$ and $g \in A_m$, $g^{-1}(O_m) \neq X$. Since $\{\mathcal{U}_{m,m}\}_{m \geq n_0}$ is a sequence of open α-covers of X, there is a finite subset $\mathcal{U}'_m$ of $\mathcal{U}_{m,m}$ for each $m \geq n_0$ such that $\bigcup_{m \geq n_0} \mathcal{U}'_m$ is an open α-cover of X. Denote $\mathcal{U}'_m = \{U_{m,j}\}_{j \leq i(m)}$. There is $f_{m,j} \in A_m$ such that $U_{m,j} = f_{m,j}^{-1}(O_m)$ for each $m \geq n_0, j \leq i(m)$. Next, we shall show that $f_0 \in \overline{\{f_{m,j} : m \geq n_0, j \leq i(m)\}}$. For arbitrary basic neighborhood $[A, V]$ of f_0 in $\mathrm{C}_\alpha(X)$, let $F = \{(m, j) \in \mathbb{N}^2 : m \geq n_0, j \leq i(m) \text{ and } A \subset U_{m,j}\}$. Obviously, $F \neq \varnothing$. If F is finite, take $x_{m,j} \in X \setminus U_{m,j}$ for each $(m, j) \in F$ because $U_{m,j} \neq X$. There is $K \in \alpha$ with $A \cup \{x_{m,j} : (m, j) \in F\} \subset K$. Then K is not contained by any element of $\bigcup_{m \geq n_0} \mathcal{U}'_m$, so $\bigcup_{m \geq n_0} \mathcal{U}'_m$ is not an α-cover of X, a contradiction. Hence F is infinite, and there are $m \geq n_0$ and $j \leq i(m)$ such that $A \subset U_{m,j} = f_{m,j}^{-1}(O_m)$ and $O_m \subset V$, so $f_{m,j}(A) \subset V$, i.e., $f_{m,j} \in [A, V]$. Thus $f_0 \in \overline{\{f_{m,j} : m \geq n_0, j \leq i(m)\}}$.

This shows that $\mathrm{vet}(\mathrm{C}_\alpha(X)) \leq \alpha\mathrm{H}(X)$. □

By Theorem 1, $\mathrm{C}_p(X)$ has countable fan tightness if and only if for each sequence $\{\mathcal{U}_n\}$ of open ω-covers of X there is a finite subset $\mathcal{U}'_n \subset \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $\bigcup_{n \in \mathbb{N}} \mathcal{U}'_n$ is an ω-cover of X.

Theorem 2. *$vet(C_p(X)) = sup\{H(X^n) : n \in \mathbb{N}\}$ for any space X.*

Proof. Let $\lambda = \mathrm{vet}(\mathrm{C}_p(X))$ and $n \in \mathbb{N}$. Suppose that $\{\mathcal{U}_\gamma\}_{\gamma<\lambda}$ is a family of open covers of the space X^n. For each $\gamma < \lambda$, a family $\mathcal{V}$ of subsets of X is called having a property $P_{n,\gamma}$ if for each $\{V_i\}_{i \leq n} \subset \mathcal{V}$ there is $U \in \mathcal{U}_\gamma$ such that $\prod_{i \leq n} V_i \subset U$. Denote by $\Gamma_{n,\gamma}$ the family of the all finite sets, which has the property $P_{n,\gamma}$, of open sets in X. For each $\mathcal{V} \in \Gamma_{n,\gamma}$, let $F_\mathcal{V} = \{f \in \mathrm{C}_p(X) : f(X \setminus \bigcup \mathcal{V}) \subset \{0\}\}$. We show that the set $A_\gamma = \bigcup\{F_\mathcal{V} : \mathcal{V} \in \Gamma_{n,\gamma}\}$ is dense in $\mathrm{C}_p(X)$.

Let $W(f, K, \varepsilon)$ be any basic neighborhood of f in $C_p(X)$. Since K is finite, there is a finite family $\mathcal{W}$ of open subsets in X such that for any $(x_1, x_2, ..., x_n) \in K^n$ there are $U \in \mathcal{U}_\gamma$ and a finite subset $\{W_i\}_{i \leq n} \subset \mathcal{W}$ such that $(x_1, x_2, ..., x_n) \in \prod_{i \leq n} W_i \subset U$. Then $K \subset \bigcup \mathcal{W}$. For each $x \in K$, put $V_x = \bigcap\{W \in \mathcal{W} : x \in W\}$, and $\mathcal{V} = \{V_x : x \in K\}$. Then $K \subset \bigcup \mathcal{V}$ and the family $\mathcal{V}$ has the property $P_{n,\gamma}$. In fact, take an arbitrary $(x_1, x_2, ..., x_n) \in K^n$, there are $\{W_i\}_{i\leq n} \subset \mathcal{W}$ and $U \in \mathcal{U}$ such that $(x_1, x_2, ..., x_n) \in \prod_{i\leq n} W_i \subset U$. Since each $V_{x_i} \subset W_i$, $\prod_{i\leq n} V_{x_i} \subset U$. Now, take $g \in \mathrm{C}_p(X)$ such that $f_{|K} = g_{|K}$ and $g(X \setminus \bigcup \mathcal{V}) = \{0\}$, then $g \in F_{\mathcal{V}} \subset A_\gamma$, so $W(f, K, \varepsilon) \cap A_\gamma \neq \varnothing$. Thus $\overline{A}_\gamma = \mathrm{C}_p(X)$.

Let $f_1 \in \mathrm{C}(X)$ with $f_1(X) = \{1\}$. Then $f_1 \in \bigcap_{\gamma<\lambda} \overline{A}_\gamma$. There is a subset $B_\gamma \subset A_\gamma$ with $|B_\gamma| < \lambda$ for each $\gamma < \lambda$ such that $f_1 \in \overline{\bigcup_{\gamma<\lambda} B_\gamma}$. Then there is a subset $\Delta_{n,\gamma} \subset \Gamma_{n,\gamma}$ with $|\Delta_{n,\gamma}| < \lambda$ such that $B_\gamma \subset \bigcup\{F_{\mathcal{V}} : \mathcal{V} \in \Delta_{n,\gamma}\}$. Let $\mathcal{V} \in \Delta_{n,\gamma}$. For each $\xi = (V_1, V_2, ..., V_n) \in \mathcal{V}^n$, take $G_\xi \in \mathcal{U}_\gamma$ such that $\prod_{i\leq n} V_i \subset G_\xi$. Put $\mathcal{G}_\gamma = \{G_\xi : \xi \in \mathcal{V}^n, \mathcal{V} \in \Delta_{n,\gamma}\}$. Clearly, $|\mathcal{G}_\gamma| < \lambda$ and $\mathcal{G}_\gamma \subset \mathcal{U}_\gamma$. We show that $\bigcup_{\gamma<\lambda} \mathcal{G}_\gamma$ covers X.

For an arbitrary $(x_1, x_2, ..., x_n) \in X^n$, let $F = \{f \in \mathrm{C}_p(X) : f(x_i) > 0$ for each $i \leq n\}$. Then F is an open neighborhood of f_1 in $\mathrm{C}_p(X)$. Since $f_1 \in \overline{\bigcup_{\gamma<\lambda} B_\gamma}$, there is $\gamma < \lambda$ such that $F \cap B_\gamma \neq \varnothing$. Then $F \cap F_{\mathcal{V}} \neq \varnothing$ for some $\mathcal{V} \in \Delta_{n,\gamma}$. Let $g \in F \cap F_{\mathcal{V}}$. Then $g(X \setminus \bigcup \mathcal{V}) = 0$ and $g(x_i) > 0$ for each $i \leq n$. Take $V_i \in \mathcal{V}$ such that $x_i \in V_i$ for each $i \leq n$, then there is $G_\xi \in \mathcal{G}_\gamma$ such that $(x_1, x_2, ..., x_n) \in \prod_{i\leq n} V_i \subset G_\xi$. So $(x_1, x_2, ..., x_n) \in \bigcup(\bigcup_{\gamma<\lambda} \mathcal{G}_\gamma)$. Hence $\mathrm{H}(X^n) \leq \mathrm{vet}(\mathrm{C}_p(X))$.

Conversely, suppose $\lambda = \sup\{\mathrm{H}(X^n) : n \in \mathbb{N}\}$. Fix $f \in \mathrm{C}_p(X)$ and a family $\{A_\gamma\}_{\gamma<\lambda}$ of subsets in $\mathrm{C}_p(X)$ such that $f \in \bigcap_{\gamma<\lambda} \overline{A}_\gamma$. For each $n \in \mathbb{N}$, $\gamma < \lambda$ and $x = (x_1, x_2, ..., x_n) \in X^n$, there is $g_{x,\gamma} \in W(f, \{x_1, x_2, ..., x_n\}, 1/n) \bigcap A_\gamma$. For each $i \leq n$, since $|g_{x,\gamma}(x_i) - f(x_i)| < 1/n$, by the continuity of f and $g_{x,\gamma}$, there is an open neighborhood O_i of x_i in X such that $|g_{x,\gamma}(y_i) - f(y_i)| < 1/n$ if $y_i \in O_i$. The set $U_{x,\gamma} = \prod_{i\leq n} O_i$ is a neighborhood of x in X^n. Thus $\mathcal{U}_{n,\gamma} = \{U_{x,\gamma} : x \in X^n\}$ covers X^n, and $|g_{x,\gamma}(y_i) - f(y_i)| < 1/n$ for each $(y_1, y_2, ..., y_n) \in U_{x,\gamma}$.

Case 1. $\lambda > \omega$. Since $\mathrm{H}(X^n) \leq \lambda$, there is a family $\{S_{n,\gamma}\}_{\gamma<\lambda}$ of subsets in X^n with $|S_{n,\gamma}| < \lambda$ for each $\gamma < \lambda$ such that $\bigcup_{\gamma<\lambda} \mathcal{S}_{n,\gamma}$ covers X^n, here each $\mathcal{S}_{n,\gamma} = \{U_{x,\gamma} : x \in S_{n,\gamma}\}$. For each $\gamma < \lambda$, let $B_{n,\gamma} = \{g_{x,\gamma} : x \in S_{n,\gamma}\}$, and $B_\gamma = \bigcup_{n\in\mathbb{N}} B_{n,\gamma}$. Then $B_\gamma \subset A_\gamma, |B_\gamma| < \lambda$, and $f \in \overline{\bigcup_{\gamma<\lambda} B_\gamma}$.

In fact, let $W(f, \{y_1, y_2, ..., y_n\}, \varepsilon)$ be a basic neighborhood of f in $\mathrm{C}_p(X)$ with $1/n < \varepsilon$. There is $\gamma < \lambda$ such that $(y_1, y_2, ..., y_n) \in \bigcup \mathcal{S}_{n,\gamma}$, thus there is $x \in S_{n,\gamma}$ such that $(y_1, y_2, .., y_n) \in U_{x,\gamma}$, so $g_{x,\gamma} \in B_{n,\gamma}$ and $|g_{x,\gamma}(y_i) - f(y_i)| < 1/n < \varepsilon$ for each $i \leq n$, hence $g_{x,\gamma} \in W(f, \{y_1, y_2, ..., y_n\}, \varepsilon) \cap B_\gamma$. This shows that $f \in \overline{\bigcup_{\gamma<\lambda} B_\gamma}$.

Case 2. $\lambda = \omega$. Replace $\gamma < \lambda$ by $k \geq n$, and let $B_k = \bigcup_{n\leq k} B_{n,k}$ in the proof of Case 1, then B_k is finite subset of A_k and $f \in \overline{\bigcup_{k\in\mathbb{N}} B_k}$.

In a word, $\mathrm{vet}(\mathrm{C}_p(X)) \leq \sup\{\mathrm{H}(X^n) : n \in \mathbb{N}\}$. □

The following result obtained by A. Arhangel'skiĭ[1] is generalized: $\mathrm{C}_p(X)$ has countable fan tightness if and only if X^n is a Hurewicz space for each $n \in \mathbb{N}$.

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