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# On RG-spaces and the regularity degree

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Abstract. We continue the study of a lattice-ordered ring $G(X)$, associated with the ring $C(X)$. Following [10], $X$ is called RG when $G(X) = C(X_\delta)$. An RG-space must have a dense set of very weak P-points. It must have a dense set of almost-P-points if $X_\delta$ is Lindelöf, or if the continuum hypothesis holds and $C(X)$ has small cardinality. Spaces which are RG must have finite Krull dimension when taken with respect to the prime $z$-ideals of $C(X)$. There is a notion of regularity degree defined via the functions in $G(X)$. Pseudocompact spaces and metric spaces of finite regularity degree are characterized.

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## 1. Introduction and elementary results

Throughout $X$ will denote a Tychonoff space and $C(X)$ will be the ring of continuous real-valued functions on $X$. The need for examples of epimorphisms of commutative rings led early on to the consideration of rings of the form $C(X)$. Later, in connection with the study of the epimorphic hull (cf. [21]), a general lattice-ordered ring called $G(X)$ was defined. This algebra is an epimorphic extension of $C(X)$ and has many interesting properties. There is an associated notion of degree denoted $\mathrm{rg}(X)$. It is rarely a ring of continuous functions, but it is when $X$ is an RG-space. There are persistent open questions concerning these objects, the most stubborn being whether RG-spaces must have P-points. These have been previously studied in [21] and [10].

In this article we investigate these topics further. Here are the main results. An RG-spaces must have a dense set of what we call "very weak P-points". It

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must have a dense set of almost-P-points if $X_\delta$ is Lindelöf, or if the continuum hypothesis holds and $C(X)$ has cardinality $c$.

By studying the spectrum of $G(X)$ we show that RG spaces must have finite Krull dimension when said dimension is taken with respect to the prime $z$-ideals. This allow us to characterize RG spaces when $X$ is scattered and perfectly normal. This generalizes previous characterizations (see [10]) in the compact and metric cases.

The regularity degree is studied. The main results characterize (pseudo) compact spaces and metric spaces which are of finite regularity degree.

For background and notation one should consult [10], [8] and [19]. In particular $\beta X$ will denote the Stone-Cech compactification of $X$, the unique compactification of $X$ in which $X$ is $C^*$-embedded.

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Associated with a function $f : X \to \mathbb{R}$ is the function $f^* : X \to \mathbb{R}$ defined as follows: $f^*$ vanishes on $Z(f)$ and $f^*(x) = 1/f(x)$ for $x \in \mathrm{coz}(f)$. In the case when $f \in C(X)$ the function $f^*$ is rarely $X$- continuous, but it is continuous in the $G_\delta$-topology on $X$. (In ring theory $f^*$ is called the quasi-inverse of $f$ in $C(X_\delta)$)). The subalgebra of $C(X_\delta)$ generated by $C(X)$ and the quasi-inverses of the functions in $C(X)$ is denoted $G(X)$. Significantly, $G(X)$ is itself Von Neumann regular, which is to say that it is closed under taking quasi-inverses. As well $G(X)$ (like $C(X)$) is a lattice under point-wise operations.

By definition each function $h \in G(X)$ can be written in the form $\sum_{i=1}^{i=n} a_i b_i^*$, $a_i, b_i \in C(X)$. This representation is not unique but the least $n$ for which such a representation for $h$ exists is unique, and is called its regularity degree, $\mathrm{rg}(h)$. The regularity degree of $X$ is the supremum of $\{\mathrm{rg}(h) | h \in G(X)\}$. It can be infinite.

The following result summarizes some useful technical facts about the behaviour of the quasi-inverse and the regularity degree.

**Proposition 1.1.**

(i) *(bounded functions) Each function in $G(X)$ has a representation of the required form using functions from $C^*(X)$. Furthermore, the use of bounded functions does not change the value of the regularity degree.*

(ii) *(restrictions) $G(X)$ is well-behaved with respect to restrictions. In particular if $S \subset X$ and $h \in G(X)$ then $h|S \in G(S)$ and $\mathrm{rg}(h|S) \leq \mathrm{rg}(h)$.*

(iii) *(compression) If $\mathrm{rg}(X) = \infty$, then this can be witnessed by functions in $G(X)$ whose values lie between $0$ and $1$.*

(iv) *If $f, g \in C(X)$, $(fg^*)^* = f^* g$. In particular if $\mathrm{rg}(h) = 1$, then $\mathrm{rg}(h^*) = 1$.*

*Proof.* Claims (ii) and (iv) are easily verified.

(i) Let $h \in G(X)$, say $h = \sum_{1=1}^{1=n} a_i b_i^*, a_i, b_i \in C^*(X)$. Then replace $a_i b_i^*$ with $(a_i/c)(b_i/c)^*$ where $c = (\mathbf{1} + a_i^2) \vee (\mathbf{1} + b_i^2)$.

(iii) We claim that for each $n \in N$, there is a function $f_n \in G(X)$ such that $\mathbf{0} \leq f_n \leq \mathbf{1}$ and $\mathrm{rg}(f_n) > n$.

Suppose if possible that $\mathrm{rg}(X) = \infty$ but that there is a positive integer $N$ that globally bounds the number of terms needed to represent bounded functions in $G(X)$. Let $f \in G(X)$. Since $1 + f^2$ is not a zero divisor in the regular ring $G(X)$ its inverse $\mathbf{1}/(\mathbf{1} + f^2)$ also lies in $G(X)$. Then $f/(\mathbf{1} + f^2)$ and $\mathbf{1}/(\mathbf{1} + f^2)$ can be expressed in terms of $N$ terms. Now the formula for the quasi-inverse in [20, remark 3.4] shows that the number of terms (of the form $gk^*$) needed for the quasi-inverse of a function needing at most $N$ such terms is a function of $N$ alone. Thus $\mathbf{1} + f^2$ can be represented by $M$ terms, where $M$ is an integer that depends only on $N$. But $f$ is the product of $\mathbf{1} + f^2$ and $f/(\mathbf{1} + f^2)$ so it requires at most $MN$ terms.

Thus $G(X)$ has bounded functions of arbitrarily high regularity degree. Since $G(X)$ is a lattice, it also contains non-negative bounded functions of arbitrarily high regularity degree. Since multiplication of a function by a non-zero constant does not change its regularity degree, we can obtain our functions between $\mathbf{0}$ and $\mathbf{1}$ as claimed. □

## 2. The Space of Prime $z$-ideals of $C(X)$ versus the Stone Space of $Z(G(X))$

Let $\mathrm{PF}(X)$ denote the set of prime $z$-filters on $C(X)$. Clearly these are in 1-to-1 correspondence with the set $\mathrm{PZ}(X)$ of prime $z$-ideals on $X$ via the map $j : \mathrm{PZ}(X) \to \mathrm{PF}(X)$ given by: $j(P) = \{Z(f) : f \in P\}$ (see [8, chapter 2]). In [16, section 1], there is a discussion of the "patch topology" (although this term is not used) on $\mathrm{PZ}(X)$. Montgomery shows that $\mathrm{PZ}(X)$ with this patch topology is homeomorphic to a compactification of $X_\delta$.

We will present a proof of this result that is clearer than the proof in [16], and that moreover shows that the patch topology on $\mathrm{PZ}(X)$ is "naturally" homeomorphic to the space of maximal ideals of $G(X)$, or (equivalently) to the Stone space of the Boolean algebra of zerosets of members of $G(X)$. It will be technically useful to identify $\mathrm{PZ}(X)$, equipped with the patch topology, with $\mathrm{PF}(X)$, equipped with the topology induced by the bijection $j$ mentioned above.

### The Patch Topology.

Let $\mathbf{S}$ be a set of prime ideals of $C(X)$ and let $f \in C(X)$. In a slight variation of the notation used in [16, section1], we define: $h(f) = \{P \in S : f \in P\}$ and $h^c(f) = \mathbf{S} \backslash h(f)$. The patch topology on $\mathbf{S}$ is defined to be the topology for which $\{h(f) : f \in C(X)\} \bigcup \{h^c(g) : g \in C(X)\}$ is a subbase. Let $\mathbf{S}$ be $\mathrm{PZ}(X)$. Clearly $j[h(f)] = \{\alpha \in \mathrm{PF}(X) : Z(f) \in \alpha\}$ and $j[h^c(f)] = \{\alpha \in \mathrm{PF}(X) : Z(f) \notin \alpha\}$. Now $\{\alpha \in \mathrm{PF}(X) : Z(f) \in \alpha\}$ is closed under finite intersection (as $\alpha$ is a $z$ -filter and hence closed under finite intersection) and $\{\alpha \in \mathrm{PF}(X) : Z(f) \notin \alpha\}$ is closed under finite intersection (as $\alpha$ is a prime $z$-filter). Thus $\{j[h(f)] \cap j[h^c(g)] : f, g \in C(X)\}$ is an open base for the patch topology on $\mathrm{PF}(X)$. Clearly this family is just $\{\{\mathbf{F} \in \mathrm{PF}(X) : Z \in \mathbf{F}$ and $S \notin \mathbf{F}\} : Z, S \in Z(X)\}$.

**Stone duality and zero-dimensional compactifications.**

We briefly summarize how to construct zero-dimensional compactifications of zero-dimensional spaces. See [19, 3.2 and 4] for more details. If $\mathbf{A}$ is a Boolean algebra, denote by $\mathrm{S}(\mathbf{A})$ the set of all ultrafilters on $\mathbf{A}$. If $a \in \mathbf{A}$, denote by $\lambda(a)$ the set $\{\alpha \in S(\mathbf{A}) : a \in \alpha\}$. Then $\{\lambda(a) : a \in \mathbf{A}\}$ is a clopen base for a compact Hausdorff zero-dimensional topology on $\mathrm{S}(\mathbf{A})$, and thus topologized $\mathrm{S}(\mathrm{A})$ is called the *Stone space* of $\mathbf{A}$.

Suppose that $\mathbf{X}$ is a zero-dimensional Hausdorff space (i.e. the set $B(X)$ of all clopen sets of $X$ forms an open base of $X$). Suppose that $\mathbf{A}$ is a subalgebra of $\mathrm{B(X)}$ and that $\mathbf{A}$ is a base for the open sets of $\mathbf{X}$. In that case the map $i : X \to S(\mathbf{A})$ given by: $i(x) = \{A \in \mathbf{A} : x \in \mathbf{A}\}$ is a homeomorphism from $\mathbf{X}$ onto the subspace $i[X]$ of the Stone space $S(\mathbf{A})$. If we identify $X$ and $i[X]$, then $X$ is a dense subspace of $S(\mathbf{A})$ so $S(\mathbf{A})$ is a zero-dimensional compactification of $X$. Furthermore the map $A \to \lambda(A) = \mathrm{cl}_{S(\mathbf{A})}A$ is a Boolean algebra isomorphism from $\mathbf{A}$ onto $B(S(\mathbf{A}))$. Finally, every zero-dimensional compactification of $\mathbf{X}$ arises in this way: if $\mathbf{K}$ is such a compactification, then define $\mathbf{A}$ to be $\{A \cap X : A \in B(K)\}$; then $\mathbf{K}$ and $S(\mathbf{A})$ turn out to be naturally equivalent compactifications of $X$.

**Theorem 2.1.** *Let $X$ be a Tychonoff space. Then:*

(i) *The set $Z(G(X))$ of zero-sets of functions in $G(X)$ is a Boolean algebra $\boldsymbol{A}$ of clopen subsets of $X_\delta$ that forms an open base for $X_\delta$. Hence $S(\boldsymbol{A})$ is a zero-dimensional compactification of $X_\delta$.*

(ii) *If $\alpha \in S(\boldsymbol{A})$ define $\alpha^{\#}$ to be $\alpha \cap \boldsymbol{Z}(X)$. Then the map $k : S(\boldsymbol{A}) \to$ $\mathrm{PF}(X)$ given by: $k(\alpha) = \alpha^{\#}$ is a homeomorphism from $S(\boldsymbol{A})$ onto* $\mathrm{PF}(X)$ *(with the patch topology) whose restriction to $X_\delta$ is a bijection onto the fixed $z$-ultrafilters of $C(X)$.*

(iii) *$S(\boldsymbol{A})$ is the space of maximal ideals of $G(X)$ (with the hull-kernel topology).*

*Proof.* (i) We know that $Z(G(X))$ is the collection of finite unions of sets of the form $Z \cap C$, where $Z \in \mathbf{Z}(X)$ and $C \in \mathrm{coz}\, X$ (see 5.2 of [21]). This is in fact the Boolean subalgebra of the power set of $X$ (where sups are unions, infs are intersections, and Boolean-algebraic complements are set-theoretic complements) generated by $\mathbf{Z}(X)$. Let us denote it by $\mathbf{A}$. Clearly $\mathbf{Z}(X) \subseteq \mathbf{A}$ so $\mathbf{A}$ is a base for the open sets of $X_\delta$ (as $\mathbf{Z}(X)$ is). Hence as noted above $S(\mathbf{A})$ is a compactification of $X_\delta$.

(ii) First we must show that the map $k$ is well-defined, i.e. that if $\alpha \in S(\mathbf{A})$ then $\alpha^{\#} \in \mathrm{PF}(X)$. It is routine to show that if $\alpha \in S(\mathbf{A})$, then $\alpha^{\#}$ is a $z$-filter on $X$.

If $Z, S \in \mathbf{Z}(X)$ and $Z \cup S \in \alpha^{\#}$ then $Z \cup S \in \alpha$ and as $\alpha$ is prime (being an ultrafilter on $\mathbf{A}$), either $Z \in \alpha$ or $S \in \alpha$. Thus either $Z \in \alpha^{\#}$ or $S \in \alpha^{\#}$ and so $\alpha^{\#}$ is prime and hence in $\mathrm{PF}(X)$. Thus the map $k$ is well-defined.

We show that $k$ is 1-to-1. Suppose that $\alpha, \beta \in S(\mathbf{A})$ and that $\alpha \neq \beta$. We show that $\alpha^{\#} \neq \beta^{\#}$. Without loss of generality suppose that $A \in \alpha - \beta$. As A is a union of finitely many sets of the form $Z \cap C$, where $Z \in \mathbf{Z}(X)$ and $C \in \text{coz } X$, and as $\alpha$ is an ultrafilter and hence prime, there exist $Z \in \mathbf{Z}(X)$ and $C \in \text{coz } X$ such that $Z \cap C \in \alpha - \beta$. Thus $Z \in \alpha$ and $C \in \alpha$. But $Z \cap C \notin \beta$ so either $Z \notin \beta$ or else $C \notin \beta$. If $Z \notin \beta$ then $Z \in \alpha^{\#} - \beta^{\#}$ and so $\alpha^{\#} \neq \beta^{\#}$. If $C \notin \beta$ then as $\beta$ is an ultrafilter on $\mathbf{A}$, it follows that $X - C \in \beta \cap \mathbf{Z}(X) = \beta^{\#}$. But $C \in \alpha$ so $X - C \notin \alpha$. Thus $X - C \notin \alpha^{\#}$ and again $\alpha^{\#} \neq \beta^{\#}$. Thus $k$ is 1-to-1 as claimed.

Next we show that $k$ maps S($\mathbf{A}$) onto PF($X$). Let $\mathbf{F} \in \text{PF}(X)$ and define $\alpha(\mathbf{F}) = \{A \in \mathbf{A}\}$ such that there exist $Z \in \mathbf{F}, S \in \mathbf{Z}(X) - \mathbf{F}$ such that $Z \cap (X - S) \subseteq A$. We will show that $\alpha(\mathbf{F}) \in S(\mathbf{A})$ and that $k(\alpha(\mathbf{F})) = \mathbf{F}$.

Clearly if $A \in \alpha(\mathbf{F})$ and $A \subseteq B \in \mathbf{A}$ then $B \in \alpha(\mathbf{F})$. If $A, B \in \alpha(\mathbf{F})$ then there exist $Z, T \in \mathbf{F}$ and $S, Y \in \mathbf{Z}(X) - \mathbf{F}$ such that $Z \cap (X - S) \subseteq A$ and $T \cap (X - Y) \subseteq B$. Then $Z \cap T \in F$ and $S \cup Y \notin \mathbf{F}$ as $\mathbf{F}$ is prime. Clearly $(Z \cap T) \cap (X - (S \cup Y)) \subseteq A \cap B$, so $A \cap B \in \alpha(\mathbf{F})$. Thus $\alpha(\mathbf{F})$ is a filter on $\mathbf{A}$.

Next we show that $\alpha(F)$ is an ultrafilter. It suffices to show that if $A \in \mathbf{A} - \alpha(F)$ then there is an $M \in \alpha(F)$ such that $M \cap A = \varnothing$. Now A has the form $A = \bigcup\{Z(i) \cap C(i) : Z(i), X - C(i) \in \mathbf{Z}(X), i = 1 \ldots n\}$. As $A \in \mathbf{A} - \alpha(\mathbf{F})$ for each $i$, $Z(i) \cap C(i) \notin \alpha(\mathbf{F})$ so either $Z(i) \notin \mathbf{F}$ or $X - C(i) \in \mathbf{F}$. Let $I = \{i : Z(i) \notin \mathbf{F}\}$ and $J = \{i : X - C(i) \in \mathbf{F}\}$. Then $I \cup J = \{1, \ldots, n\}$. Let $G = \bigcup((Z(i) : i \in I)$ and $H = X - \bigcup\{C(j) : j \in J\}$. Then $G \in \mathbf{Z}(X) - \mathbf{F}$ because $\mathbf{F}$ is prime, and $H \in \mathbf{F}$ because $\mathbf{F}$ is closed under finite intersection. Let $M = H \cap (X - G)$. Clearly $M \in \alpha(F)$. We claim that $M \cap A = \varnothing$. It suffices to show that for all $i, M \cap Z(i) \cap C(i) = \varnothing$. For any such i, as $I \cup J = \{1, \ldots, n\}$ either $i \in I$ or else $i \in J$. In the former case $Z(i) \cap (X - G) = \varnothing$ and in the latter case, $C(i) \cap H = \varnothing$. Thus $M \cap A = \varnothing$ and so $\alpha(\mathbf{F})$ is an ultrafilter as claimed, and therefore an element of S($\mathbf{A}$).

Next we show that $k(\alpha(\mathbf{F})) = \mathbf{F}$. If $Z \in \mathbf{F}$ it is clear that $Z \in \alpha(\mathbf{F}) \cap \mathbf{Z}(X) = \alpha^{\#} = k(\alpha(\mathbf{F}))$, so $\mathbf{F} \subseteq k(\alpha(\mathbf{F}))$. Conversely, suppose that $Z \in \alpha(\mathbf{F}) \cap \mathbf{Z}(X) = \alpha^{\#} = k(\alpha(\mathbf{F}))$. Thus there exist $T \in \mathbf{F}$ and $S \in \mathbf{Z}(X) - F$ such that $T \cap (X - S) \subseteq Z$. Thus $T \subseteq Z \cup S$. As $T \in \mathbf{F}$ it follows that $Z \cup S \in \mathbf{F}$. But $\mathbf{F}$ is prime and $S \notin \mathbf{F}$ so $Z \in \mathbf{F}$. Thus $k(\alpha(\mathbf{F})) \subseteq \mathbf{F}$ and so $k(\alpha(\mathbf{F})) = \mathbf{F}$.

We have established that $k$ is a well-defined bijection from S($\mathbf{A}$) onto PF($X$). It remains to show that $k$ is a homeomorphism.

We claim that $\mathbf{B} = \{\lambda(Z \cap C) : Z, X - C \in \mathbf{Z}(X)\}$ is an open base for $S(\mathbf{A})$. To see this, note that each member of $\mathbf{A}$ is the union of finitely many members of $\mathbf{B}$, so as $\lambda$ is a Boolean algebra isomorphism (the Stone duality ), it preserves finite unions and each $\lambda(A)$ is the union of finitely many members of $\mathbf{B}$. As each member of $\mathbf{B}$ is clopen in S($\mathbf{A}$), our claim follows.

We must show that the map $k$ is both continuous and open. By the above claim, to verify "open" it suffices to show that $k[\lambda(Z \cap C)]$ is open in PF($X$) whenever $Z \in \mathbf{Z}(X)$ and $C \in \text{coz } X$. But note that $k[\lambda(Z \cap C)] = \{\alpha \cap \mathbf{Z}(X) : \alpha \in \lambda(Z \cap C)\} = \{\alpha \cap \mathbf{Z}(X) : Z \cap C \in \alpha\} = \{\alpha \cap \mathbf{Z}(X) : Z \in \alpha, X - C \notin \alpha\} = \{\mathbf{F} \in \text{PF}(X) : Z \in \mathbf{F}, X - C \notin \mathbf{F}\}$ (because $k$ is onto).

But this latter set is open in the patch topology (see the above discussion), so $k$ is an open map. But as $k$ is a bijection, it follows that $k^{-1}[\{\mathbf{F} \in \mathrm{PF}(X) : Z \in F \text{ and} X - C \notin \mathbf{F}\}] = \lambda(Z \cap C)$ and because sets of the form $\{\mathbf{F} \in \mathrm{PF}(X) : Z \in F \text{and} X - C \notin F\}$ form an open base for the patch topology on $\mathrm{PF}(X)$ and $\lambda(Z \cap C)$ is open in $\mathrm{S}(\mathbf{A})$, it follows that $k$ is continuous and a homeomorphism as claimed.

Finally, if $x \in X_\delta$ then $x$ corresponds to the ultrafilter $\alpha(x) = \{A \in \mathbf{A} : x \in A\}$ and so $k(x) = k(\alpha(x)) = \alpha(x) \cap \mathbf{Z}(X) = \{Z \in \mathbf{Z}(X) : x \in Z\}$, which is the fixed $z$-ultrafilter at $x$.

(iii) This follows from the fact that the correspondence $M \to \{Z(F) : F \in M\}$ is a bijection from the set $\mathbf{M}$ of maximal ideals of the ring G(X) onto $\mathrm{S}(\mathbf{A})$, and if $\mathbf{M}$ is given the hull-kernel topology then this bijection is a homeomorphism. □

The following remark likely appears in [16], but we have not been able to pinpoint it. We retain the same notation as above.

**Proposition 2.2.** *Let $i : X_\delta \to X$ be the identity map on the underlying set of $X$. Then $i$ can be continuously extended to $i^\wedge : S(\boldsymbol{A}) \to \beta X$.*

*Proof.* If $\alpha \in S(\mathbf{A})$ let $i^\wedge(\alpha)$ be the unique point in $\cap\{\mathrm{cl}_{\beta X} Z : Z \in \alpha \cap \mathbf{Z}(X) = \alpha^\#\}$. (To see that there is indeed such a unique point, note that as $\alpha^\#$ is a prime $z$-ideal there is a unique $y \in \beta X$ such that $\mathbf{Z}(O^y) \subseteq \alpha^\# \subseteq \mathbf{Z}(M^y)$ (see [8, 7.15]).) To show that $i^\wedge$ is continuous, it suffices (because $\{\mathrm{cl}_{\beta X} Z : Z \in \mathbf{Z}(X)\}$) is a base for the closed sets of $\beta X$) to show that $(i^\wedge)^{-1}[\mathrm{cl}_{\beta X} Z]$ is closed in $\mathrm{S}(\mathbf{A})$ for each $Z \in \mathbf{Z}(X)$. But $(i^\wedge)^{-1}[\mathrm{cl}_{\beta X} Z] = \{\alpha \in S(\mathbf{A}) : i^\wedge(\alpha) \in \mathrm{cl}_{\beta X} Z\} = \{\alpha \in S(\mathbf{A}) : Z \in \alpha\} = \lambda(A)$ (by Stone duality). But $\lambda(A)$ is a clopen, hence closed, subset of $\mathrm{S}(\mathbf{A})$. Hence $i^\wedge$ is continuous. It also maps $\mathrm{S}(\mathbf{A})$ onto $\beta X$ as $j$ maps $X_\delta$ onto the dense subspace $X \subset \beta X$. □

Note that a portion of Proposition 2.2 has also appeared (with a different proof) in [4, 3.4, 3.9].

It may be worth noting that if $y \in \beta X$, then $(i^\wedge)^{-1}(y)$ consists of all prime $z$-filters $F \in \mathrm{PF}(X)$ for which $\mathbf{Z}(O^y) \subseteq F \subseteq \mathbf{Z}(M^y)$ (we are still identifying $\mathrm{S}(\mathbf{A})$ and $\mathrm{PF}(X)$ via $\alpha \to \alpha^\#$).

**Remark 2.3.** If $X$ is an RG-space then $G(X) = C(X_\delta)$, $\mathbf{Z}(G(X)) = \mathbf{Z}(X_\delta)$, and $S(\mathbf{Z}(G(X))) = \beta(X_\delta)$.

## 3. On regularity degree and prime $z$-ideal length

The following result relies on work in [4] where the Pierce sheaf was used to study the regularity degree in general commutative rings. Both [4] and [20] are useful references for the algebraic background needed. The following result is the key to theorem 3.4 below.

**Lemma 3.1.** *Let $X$ be Tychonoff. Suppose that $C(X)$ has a strict ascending chain of $k+1$ prime $z$-ideals. Then $G(X)$ has a function of regularity degree at least $k+1$. It can be constructed directly from the prime $z$-ideals in the ascending chain.*

*Proof.* (Cf. part (3) of [4, theorem 3.1] and the studies of the universal regular ring by Kennison [11], Olivier [18], and Wiegand [25]). Suppose that $P_0 \subset \cdots \subset P_k$ is a strict chain of prime $z$-ideals in $C(X)$. For $i = 0, \ldots, k-1$, choose $b_i \in P_{i+1} - P_i$ and let $b_k = 1$. By the proof of [4, theorem 3.1] the element $t = (b_0, \ldots, b_k) \in S' = Q_{cl}((C(X)/P_0)) \times Q_{cl}((C(X)/P_1)) \times \cdots \times Q_{cl}((C(X)/P_k))$ cannot be written as the sum of fewer than $k+1$ terms of the form $c_i d_i^*, c_i, d_i \in C(X)$. Now the fields $Q_{cl}(C(X)/P_i)$ are stalks (homomorphic images) of the universal regular ring $T(C(X))$ for $C(X)$ whose spectrum is the set of prime ideals in $C(X)$ under the patch topology. The spectrum of $G(X)$ (cf. 2.1 part (iii)) is the compact space of prime $z$-ideals of $C(X)$ under the patch topology. The universality of $T(C(X))$ and the fact that $C(X) \to G(X)$ is epic show that $G(X)$ is a homomorphic image of $T(C(X))$—the spectrum of the former ring is a subspace of the spectrum of the latter. (See also [4, Proposition 2.5]). Thus because the $P_i$ are prime $z$-ideals, the fields $Q_{cl}(C(X)/P_i)$ are homomorphic images of $G(X)$. Furthermore since $G(X)$ is a regular ring, the finite product $S'$ is also a homomorphic image of $G(X)$ (use the regularity of $G(X)$ to obtain a function $k_i \in ((\bigcap P_j, j \neq i) - P_i)$ and consider the basic idempotent $k_i k_i^*$). So the element $t$ is the image of say $h \in G(X)$. Clearly $\mathrm{rg}(h) \geq k+1$ because of the constraint on the regularity degree for $t$ in $S'$. □

**Definition 3.2.** *By the Krull $z$-dimension of $C(X)$ we will mean the supremum of the lengths of chains of prime $z$-ideals in $C(X)$. The Krull $z$-dimension of a maximal ideal will mean the supremum of the lengths of chains of prime $z$-ideals lying in it.*

**Remark 3.3.** It is interesting to compare the Krull $z$-dimension of fixed and free maximal ideals in $C(X)$ in different cases. When every fixed maximal ideal is a minimal prime, $X$ is necessarily a $P$-space, and each free maximal ideal is also a minimal prime. But in the examples of [9, 4.4, 7.4] each fixed maximal ideal has Krull $z$-dimension 1, and there is at least one free maximal ideal of Krull $z$-dimension greater than 1. Indeed, we will presently see (in the remarks after theorem 4.1) that in some models of $\Psi$, $C(\Psi)$ will have infinite Krull $z$-dimension.

**Theorem 3.4.** *If the Krull $z$-dimension of $C(X)$ is infinite then $X$ is not an* RG *space and* $\mathrm{rg}(X) = \infty$.

*Proof.* We will use the fact that if $X$ is RG then Spec$G(X)$, the space of prime $z$-ideals of $C(X)$ under the patch topology, is natually homeomorphic to $\beta(X_\delta)$ (see remark 2.3 and the discussion that precedes it).

Let us first recall some properties of $\beta(X_\delta)$.

1. There is a Boolean isomorphism between the algebra of clopen sets of $X_\delta$ and the algebra of clopen sets of $\beta(X_\delta)$ defined by taking traces in one direction

and closures in the other (see the discussion before 2.1). Furthermore, if we have a countable family of clopen sets $\{B_n\} \subset \beta(X_\delta)$ with traces $\{A_n\}$ in $X_\delta$, then if $S = \bigcup B_n$, with closure $S' \subset \beta(X_\delta)$ then $S'$ is clopen in $\beta(X_\delta)$ and has trace equal to $\bigcup A_n$. (Thus $\beta(X_\delta)$ is basically disconnected; see [8, 4K(7)]).

2. (All closures are taken in $\beta(X_\delta)$). If $h \in G(X)$ , if $B = \mathrm{cl}[\mathrm{coz}(h)]$ and if $A$ is the trace of $B$ in $X$, then $B = \mathrm{cl}(A)$. In particular, if $f \in C(X)$ then $\mathrm{coz}(f)$ is clopen in $X_\delta$ and $\mathrm{cl}[\mathrm{coz}(f)]$ is the clopen set of $\beta(X_\delta)$ that consists of the prime $z$-ideals of $C(X)$ that do not contain $f$.

3. Let $X$ be an RG space. Suppose that $B$ is clopen in $\beta(X_\delta)$ with trace $A$ in $X_\delta$. Let $P$ be a prime $z$-ideal in $B$. Suppose that $h \in G(X), a \in C(X)$ and $h|A = a|A$. Then $h$ and $a$ agree at $P$ in the field $C(X_\delta)/M^P$ where $M^P$ is the maximal ideal in $C(X_\delta)$ corresponding to the point $P \in \beta(X_\delta)$. (Reason: $h - a$ has a clopen zero set in $X_\delta$ , it vanishes on $A$ so it vanishes on its closure $B$).

4. Now assume that $C(X)$ has infinite Krull $z$-dimension. We could conclude that $\mathrm{rg}(X)$ is infinite directly from Lemma 3.1 but instead our strategy will be to define a family of pairwise disjoint clopen sets of $X_\delta$ and then use them to define a function on their union that is not in $G(X)$ because it is of "infinite" degree. The proof will also show that $C(X_\delta)$ has functions of arbitrarily large regularity degree and thus $\mathrm{rg}(X) = \infty$.

Since the ring $C(X)$ has infinite Krull $z$-dimension, there are two possibilites.

Case 1. There exists in $C(X)$ a maximal ideal that contains an infinite ascending sequence $\{P_n\}$ of prime $z$-ideals.

Case 2. Case 1 fails, so either there is an infinite descending sequence of prime $z$-ideals, or all ascending chains of prime $z$-ideals are finite, but their lengths are unbounded globally. One checks easily using [8, 14.8 (a)], if necessary, that in case 2 it is always possible to choose a countable sequence $\{C_k\}$ of finite ascending chains $\{Q_{k,t}\}$ of prime $z$-ideals with the following properties:

(i) distinct chains are disjoint,

(ii) chain $C_k$ is of length $s_k$, say, and the $s_k$ are strictly increasing,

(iii) for each $k$, $Q_{i,1} \subsetneq Q_{k,t}$ for all $i < k$, for all $t$.

We now contradict the RG property in both cases.

Case 1. Use the ascending sequence $\{P_n\}$ to choose for each $n$, $b_n \in P_{n+1} - P_n$. Now let $B_1 = \mathrm{coz}(b_1)$ and for $n > 1$, let $B_n = \mathrm{coz}(b_n) - [\bigcup_{i=1}^{n-1} \mathrm{coz}(b_i)]$. The $\{B_n\}$ are non-empty clopen and disjoint in $X_\delta$. Each $B_n$ has $P_n$ but no other $P_i$ its $\beta(X_\delta)$-closure. Define $h$ on $X$ by $h|B_n = b_n|B_n$ for each $n$ and $h[X - \bigcup B_i] = 0$. Then $h \in C(X_\delta)$ and for each $n, h(P_n) = b_n(P_n)$ by part 3. Now, suppose, if possible that $h \in G(X)$ say $h = \sum_{i=1}^{i=m} c_i d_i^*$. Take the first $m+2$ clopen sets $B_1, \ldots, B_{m+2}$ and their closures $\mathrm{cl}(B_1), \ldots, \mathrm{cl}(B_{m+2})$ which are disjoint in $\beta(X_\delta)$ [8, 6.5 III]. Let $W = \{P_1, \ldots, P_{m+2}\}$. (Observe the parallel with the situation of Lemma 3.1). The set $W$ is closed and discrete in $\mathrm{Spec}G(X)$ and $h|W$ can be written as the sum of $m$ terms. But this contradicts Lemma 3.1 since we have a chain of $m+2$ primes, and $h$ has been chosen so that $h|W$ plays the role of $t$ in 3.1.

Case 2. We inductively define a set of elements $\{b_{k,t}\}$, $t \in \{1, \ldots, s_k - 1\}$ so that $b_{k,t} \in Q_{k,t+1} - Q_{k,t}$, and $b_{k,t} \in Q_{i,t}$ for all $i < k$. Start in chain $C_1$ by choosing $s_1 - 1$ elements $a_{1,t} \in Q_{1,i+1} - Q_{1,_i}$. Suppose now that the choices have been made for the first $k-1$ chains, and consider $C_k$. Since $Q_{k,t+1}$ is a prime ideal, condition (iii) shows that there is an element $b_{k,t} \in Q_{k,t+1} - Q_{k,t}$, and $b_{k,t} \in Q_{i,1}$ for all $i < k$.

As in case 1, we want to get disjoint clopen sets and define a function on their union. Consider $\mathrm{cl}[\mathrm{coz}(b_{k,t})]$. It contains the primes $\{Q_{k,j}\}, j \leq t$, it excludes the primes $Q_{i,t}, t < k$, and possibly it contains primes from subsequent chains $\{C_r\}, r > k$.

For each $k$ and $t \in \{1, \ldots, s_k - 1\}$ let $B_{k,t} = \mathrm{coz}(b_{k,t}) - [\bigcup_{j=1}^{j=t-1} \mathrm{coz}(b_{k,j}) \cup (\bigcup_{r>k} \mathrm{coz}(b_{r,t}))]$. By construction the $B_{k,t}$ are disjoint clopen subsets of $X_\delta$ and $clB_{k,t}$ contains $Q_{k,t}$ and no other prime from the array. Now define $h$ on $\bigcup B_{k,t}$ to be $b_{k,t}$ on $B_{k,t}$ and let $h$ vanish off $\bigcup B_{k,t}$. Then $h \in C(X_\delta)$ and by property 3, $h$ and $b_{k,t}$ agree on $Q_{k,t}$. Now condition (ii) gives us chains of arbitrary length so we argue that $h \notin G(X)$ as in Case 1. $\square$

**Lemma 3.5.** *Suppose that $S$ is a subspace of $T$ that induces an epi $C(T) \to C(S)$. Then the preimage of a chain of prime $z$-ideals in $C(S)$ is a chain of prime $z$-ideals of $C(T)$. The lengths of such chains under taking preimages is either maintained or grows—it cannot decline.*

*Proof.* It is well-known that the preimage of a prime is a prime. It is also well known (cf. [14]) that taking preimages of primes under an epimorphism is $1-1$. So it suffices to check that the preimage of a $z$-ideal is a $z$-ideal. But this is straightforward: let $P$ be a prime $z$-ideal in $C(S)$ and let $Q$ be its preimage in $C(T)$. Suppose that $g \in Q$, $f \in C(T)$ with $Z(f) = Z(g)$. Now $f|S$ and $g|S$ have the same zero set. But $g|Y_n$ is in the $z$-ideal $P$ of $C(S)$. So $f|S \in P$, and therefore $f \in Q$. $\square$

**Theorem 3.6.** *A space $X$ cannot be* RG *and must be of infinite regularity degree if it contains an infinite strictly decreasing sequence of epi-inducing (for example $C^*$-embedded) subspaces $Y_n$ such that for each $n$, $Y_n - Y_{n+1}$ contains a cozero set of $Y_n$ that is dense in $Y_n$.*

*Proof.* We will show that $C(X)$ contains chains of prime $z$-ideals of arbitrary length. Then the conclusion follows from theorem 3.4.

Subspaces that, upon restriction, induce epimorphisms of rings were studied in [1]. It is immediate that the restriction map $C(Y_n) \to C(Y_{n+1})$ is an epi if the map $C(X) \to C(Y_{n+1})$ is an epi. Suppose that $C(Y_{n+1})$ has a strictly ascending chain of $k$ prime $z$-ideals $P_1, \ldots, P_k$. By lemma 3.5 this gives a strict chain of $k$ prime $z$-ideals in $C(Y_n)$, say $Q_1, \ldots, Q_k$, where $Q_i = \{f \in C(Y_n) : f|Y_{n+1} \in P_i\}$. Now by assumption, $Y_n - Y_{n+1}$ contains densely $\mathrm{coz}(h)$ for some $h \in C(Y_n)$. Since $h$ vanishes on $Y_{n+1}$, it lies in $Q_1$ because $Q_1$ contains the preimage of any function that is zero on $Y_{n+1}$. Since $\mathrm{coz}(h)$ is dense in $Y_n$ , $h$ is a non-zero divisor in $C(Y_n)$. But $h \in Q_1$, and it is elementary commutative algebra that non zero divisors cannot lie in minimal primes (cf. [13]). So $Q_1$ contains properly an additional prime $z$-ideal, and therefore $C(Y_n)$ has a chain of at least $k+1$ prime $z$-ideals.

Now we will see that $C(X)$ has chains of prime $z$-ideals of arbitrary length. Given any natural number $N$, consider $C(Y_N)$ which has a chain of prime $z$-ideals of length at least 1. By working back through the chain induced by the inclusions $X \subset Y_1 \subset Y_2 \subset Y_N$ one sees that $C(X)$ has a chain of prime $z$-ideals of length at least $N-1$. □

**Remark 3.7.** Here are some particular cases of the result or of its method.

(1) $X$ cannot be RG if it has a decreasing sequence $Y_n$ of closed epi-inducing subspaces, with the property that for infinitely many $n$, $Y_n$ is weakly Lindelöf and $Y_{n+1}$ is nowhere dense in $Y_n$. Reason: the open set $Y_n - Y_{n+1}$ is a union of cozero sets so there is a union $V$ of countably many of them that is dense in $Y_n - Y_{n+1}$. But $V$ is itself a cozero set.

(2) Suppose $X$ is normal, RG, scattered and of infinite CB-index. Then in the derived sequence of isolated points there cannot be infinitely many occurances of countable sets. Reason: this is a special case of (i). All of the remainders are closed and hence $C$-embedded, and at each stage the set of isolated points is dense.

In the scattered perfectly normal case we can get a characterization, as follows:

**Theorem 3.8.** *A scattered perfectly normal space is* RG *if and only if it is of finite* CB-*index.*

*Proof.* By [10, 2.12] finite CB-index implies RG. The converse holds by theorem 3.6 because at each stage the set of isolated points is a dense cozero set. □

## 4. Relationships that exist between $\mathrm{CB}(X), \mathrm{rg}(X)$, and Krull $z$-dimension

**Theorem 4.1.**

(i) *if $X$ is Tychonoff, then* $\mathrm{rg}(X) = \mathrm{rg}(\upsilon X)$,

(ii) *The map $\lambda : \sum a_i b_i^* \to \sum a_i^\upsilon (b_i^\upsilon)^*$ is a ring isomorphism from $G(X)$ onto $G(\upsilon X)$. In particular, if $X$ is pseudocompact then $G(X)$ and $G(\beta X)$ are naturally isomorphic. Furthermore* $\mathrm{rg}(X) = \mathrm{rg}(\upsilon X)$.

(iii) $\mathrm{rg}(X) \le \mathrm{rg}(\beta X)$.

*Proof.* (i) and (ii). If $f = \sum a_i b_i^* \in G(X)$ let $\lambda(f) = \sum a_i^\upsilon (b_i^\upsilon)^*$. Clearly $\lambda(f) \in G(\upsilon X)$ and $\lambda(f)|X = f$. Since $X$ is $G_\delta$-dense in $\upsilon X$, $X_\delta$ is dense in $(\upsilon X)_\delta$. Using this denseness one easily verifies that $\lambda$ is well-defined and onto. That $\lambda$ is $1-1$ and respects ring operations is immediate. Clearly $\mathrm{rg}(\lambda(f)) = \mathrm{rg}(f)$, so $\mathrm{rg}(X) = \mathrm{rg}(\upsilon X)$.

(iii). Let $f \in G(X)$, with $f = \sum_{i=1}^{i=n} a_i b_i^*, a_i, b_i \in C^*(X)$. Let $F = \sum_{i=1}^{i=n} a_i^\beta (b_i^\beta)^*$. Then $F \in G(\beta X)$. If $\mathrm{rg}(\beta X) = k$, then $\mathrm{rg}(F) \le k$. Thus $\mathrm{rg}(X) \le \mathrm{rg}(\beta X)$. □

**Example 4.2.** Spaces $X$ with $\mathrm{rg}(X) = 1$ but $\mathrm{CB}(X) > 1$ and even infinite.

The P-space S of [8, 4N] is scattered and of CB-index 2. There is also a P-space of infinite CB-index as follows : let $\omega_\omega$ denote the smallest ordinal of cardinality $\aleph_\omega$. Remove the ordinals of countable cofinality (i.e. its non P-points) and call the resulting subspace $X$. Then $X$ is a scattered P-space of CB-index $\omega_0$.

**Example 4.3.** Spaces with $\mathrm{CB}(X) = 2$ but with $\mathrm{rg}(X)$ varying.

These will all be achieved using the pseudocompact space $\Psi$ of [8] which is never RG because it is separable and has a zero-set which is an uncountable discrete set. The first class furnishes examples of the fact that finite regularity degree does not imply RG (see also example 8.8). The second case shows that one can have pathology even when all zero sets have discrete boundaries.

**First case (finite regularity degree).** Take any choice of a maximal almost disjoint family on $N$ yielding a $\Psi$ which is almost compact (see [8, 6J]). Such MAD families exist by [17] and [22]. Now $\beta\Psi$ is scattered of CB-index 3 so $\beta\Psi$ is RG and $\mathrm{rg}(\beta\Psi) \leq 7$. Let $f \in G(\Psi)$. By proposition 1.1 $f$ has a representation as a finite sum using functions from $C^*(\Psi)$. Each extends to $\beta\Psi$ so there exists $F \in G(\beta\Psi)$ such that $F|\Psi = f$. By [10, 2.12] $\mathrm{rg}(F) \leq 7$ so we also have $\mathrm{rg}(f) \leq 7$.

**Second case (infinite regularity degree).** We shall use Teresawa's theorem [22, 2.1] choosing $[0,1]$ as our compact metric space without isolated points, and work with a $\Psi$ for which $\beta\Psi - \Psi$ is homeomorphic to $[0,1]$. Let $n \in N$. By theorem 10.1 below there exists $f_n \in G[0,1]$such that $\mathrm{rg}(f_n) \geq n$. Since $[0,1]$ is compact it is $G$-embedded in $\beta\Psi$ (cf. [10, 2.1 (c)] ) and there exists $F_n \in G(\beta\Psi)$ such that $F_n|[0,1] = f_n$. Thus $\mathrm{rg}(F_n) \geq n$ and $\mathrm{rg}(\beta\Psi) = \infty$. Now by theorem 4.1 that means that $\mathrm{rg}(\Psi) = \infty$.

**Remark 4.4.**

(1) The previous example reveals an interesting phenomenon in the behaviour of $\mathrm{rg}(X)$. A space has $\mathrm{rg}(X) = 1$ exactly when it is a P-space (see [10, 1.4]). So if $\mathrm{rg}(X) > 1$, $X$ has a non P-point $x$ and one easily verifies that for every neighbourhood $U$ of $x$, $\mathrm{rg}(U) > 1$. One might conjecture that in general if $\mathrm{rg}(X) \geq n$ then there must be a point $x \in X$ such that $\mathrm{rg}(U) \geq n$ for every neighbourhood $U$ of $x$. But this is false. In the example above $\mathrm{rg}(\Psi) = \infty$, but every point $x$ has a neighbourhood that is a one point compactification of a countable discrete set and hence of regularity degree 2.

(2) Note as well that the previous example gives a version of $\Psi$ for which $C(\Psi)$ has infinite Krull $z$-dimension because this holds for $C[0,1]$.

## 5. Almost-$P$-points and spaces X for which $|C(X)| = c$

Recall that a point of $X$ is an **almost-P-point** if it does not belong to a zero-set with empty interior. We denote the set of almost-P-points of $X$ by $gX$. Clearly $gX$ is the intersection of the dense cozero-sets of $X$. A space $X$ is called an **almost-P-space** if every non-empty zero-set has a non-empty interior. These are precisely the spaces $X$ for which $gX = X$.

The following facts about $gX$ are useful. The third replies to a question by M. Tressl (private communication) about powers of $\mathbb{R}$ .

**Proposition 5.1.**

(i) *$gX$ is the intersection of the dense $F_\sigma$-sets of $X$.*
(ii) *Let $U = W \cap S$, where $W$ is open in $X$ and $S$ is dense in $X$ . Then $gU = U \cap gX$.*
(iii) *If $X$ is realcompact, then $gX = g(\beta X)$. In particular, if $gX = \varnothing$ and $X$ is realcompact, then $g(\beta X) = \varnothing$.*

*Proof.* (i) Let $hX$ denote the intersection of the dense $F_\sigma$-sets of $X$. As each cozero-set of $X$ is an $F_\sigma$-set of $X$ , $hX \subset gX$. Conversely, suppose that $p \notin hX$. Then there is a dense $F_\sigma$-set $F$ of $X$ that excludes $p$. By [8, 3.11(b)] there exists a zero-set $Z$ of $X$ such that $p \in Z \subset X - F$. Thus $X - Z$ is a dense cozero-set of $X$ that excludes $p$, and $p \notin gX$.

(ii) First suppose that $U$ is open in $X$. Let $p \in U$. If $p \notin gX$, there exists a dense cozero-set $C$ of $X$ that excludes $p$. Then $U \cap C$ is a dense cozero-set of $U$ that excludes $p$. Thus $p \notin gU$, and so $gU \subset (gX) \cap U$.

Conversely, suppose that $p \in U - gU$. By (i) there is a dense $F_\sigma$-set $F$ of $U$ that excludes $p$. Clearly there exists an $F_\sigma$-set $A \subset X$ such that $F = U \cap A$. As $F$ is dense in $U$ and $U$ is open in $X$ , it follows that $(X - U) \cup A$ is a dense $F_\sigma$-set of $X$ to which $p$ does not belong. Thus $p \notin hX$, and by (i) $p \notin gX$. Thus $(gX) \cap U \subset gU$.

Next suppose that $U$ is dense in $gX$. It is routine to show that $gU \subset U \cap gX$. Conversely, suppose that $p \in U \cap gX$ and let $H$ be a dense $F_\sigma$-set of $U$. Then there exists an $F_\sigma$-set $A$ of $X$ such that $H = U \cap A$. As $U$ is dense in $X$, so is $H$ and hence $A$. As $p \in gX$, by (i) $p \in A$. Thus $p \in U \cap A = H$. Apply (i) again to conclude that $p \in gU$.

In the general case, observe that $W \cap S$ is dense in $W$, so by the results above we have $gU = g(W \cap S) = (W \cap S) \cap gW = (W \cap S) \cap (W \cap gX) = U \cap gX$.

(iii) That $gX \subset g(\beta X)$ is a special case of (ii). Conversely let $p \in g(\beta X)$. As $X$ is realcompact it is the intersection of the (necessarily dense) cozero-sets of $\beta X$ that contain $X$ (cf. [19, 5.11(c)]). Thus $p \in X$. If $C$ is a dense cozero-set of $X$ , then as $X$ is $C^*$-embedded in $\beta X$ , there is a cozero-set $V$ of $\beta X$ (necessarily dense in $X$) for which $C = V \cap X$. Thus $p \in V$ as $p \in g(\beta X)$. Hence $p \in V \cap X = C$ and $p \in gX$. $\square$

**Theorem 5.2.** *Let $j : X_\delta \to X$ be the identity map on the underlying set and let $j^\beta : \beta(X_\delta) \to \beta X$ be its Stone extension. The following are equivalent:*

(i) *$X$ is an almost-P-space,*
(ii) *$j^\beta$ is irreducible.*

*Proof.* (ii) $\Rightarrow$ (i). Let (i) fail. Then there exists a $Z \in Z(X)$ such that $Z \neq \varnothing$ and $\mathrm{int}_X Z = \varnothing$. Let $A = X_\delta - Z$. Thus $A$ is a proper clopen subset of $X_\delta$ and therefore $\mathrm{cl}_{\beta(X_\delta)} A$ is a proper compact subset of $\beta(X_\delta)$. But $j^\beta[\mathrm{cl}_{\beta(X_\delta)} A] = \mathrm{cl}_{\beta X}(j[A]) = \mathrm{cl}_{\beta X} A$. But $\mathrm{int}_X(X - A) = \varnothing$ so $A$ is dense in $X$ and therefore $\mathrm{cl}_{\beta X} A = \beta X$. Thus $j^\beta$ maps the proper closed subset $\mathrm{cl}_{\beta(X_\delta)} A$ onto $\beta X$ and therefore $j^\beta$ is not irreducible, and (ii) fails.

(i) $\Rightarrow$ (ii). Let (ii) fail. Thus there is a proper closed subset $K$ of $\beta(X_\delta)$ such that $j^\beta[K] = \beta X$. Since $\beta(X_\delta)$ is 0-dimensional, there exists a proper clopen subset of $\beta(X_\delta)$ that contains $K$, and this clopen subset will be of the form $\mathrm{cl}_{\beta(X_\delta)} A$ where $A$ is clopen in $X_\delta$ and $A \neq X_\delta$. Since $X_\delta - A \neq \varnothing$ and is open in $X_\delta$, there exists an $S \in Z(X)$ such that $\varnothing \neq S \subset X_\delta - A$.

Now $\beta X = j^\beta[\mathrm{cl}_{\beta(X_\delta)} A] = \mathrm{cl}_{\beta X} j[A] = \mathrm{cl}_{\beta X} A$. Therefore $A$ is dense in $X$, and $\mathrm{int}_X S = \varnothing$. But $S \neq \varnothing$, so $X$ is not an almost-P-space. $\square$

**Theorem 5.3.** *Let $X$ be a non-pseudocompact space with no almost-P-points. Then $X$ has a non-compact zero-set with empty interior, and hence $X$ has non-remote points.*

*Proof.* Since $X$ is not pseudocompact choose $f \in C(X) - C^*(X)$, without loss of generality $f \geq \mathbf{0}$. Hence there exists a countably infinite subset $D = \{d(n) : n \in \mathbf{N}\}$ of $X$ such that $f(d(n+1)) \geq f(d(n)) + 1$ for each $n$. Let $Z(n) = f^{-1}[f(d(n)) - 0.25, f(d(n)) + 0.25]$. Then $d(n) \in Z(n) \in \mathbf{Z}(X)$ and if $n \neq k$ then $Z(n) \cap Z(k) = \varnothing$. As $X$ has no almost-P-points for each $n \in \mathbf{N}$ there exists a nowhere dense zero-set $S(n)$ of $X$ such that $d(n) \in S(n)$. Let $T(n) = Z(n) \cap S(n)$. Then $T(n)$ is a nowhere dense zero-set of $X$ that contains $d(n)$, and clearly $f[T(n)] \subseteq [f(d(n)) - 0.25, f(d(n)) + 0.25]$ (since $T(n) \subseteq Z(n)$). Let $A(n) = X - f^{-1}[(f(d(n)) - 0.3, f(d(n)) + 0.3)]$. Then $A(n) \in \mathbf{Z}(X)$, $A(n) \cap T(n) = \varnothing$, and by the choice of the $d(n)$, if $n \neq k$ then $A(n) \cup A(k) = X$. Hence for each $n$ there is a $g_n \in C(X)$ such that $A(n) = Z(g_n)$, $T(n) = g_n^{-1}(1)$, and $\mathbf{0} \leq g_n \leq \mathbf{1}$. Now define $g : X \to \mathbf{R}$ by:

$$g(x) = \sum_{n \in \mathbf{N}} g_n(x) \tag{1}$$

We check whether this definition makes sense (i.e. whether $g(x) \in \mathbf{R}$). If $m \in \mathbf{N}$ and if $g_m(x) \neq 0$ then $x \notin A(m)$ so as noted above if $k \neq m$ then $x \in A(k)$; thus $g_k(x) = 0$ if $k \neq m$. Thus at most one term in the sum in the right hand side of (1) is non-zero, and $g$ is well-defined.

Now consider the family

$$\mathbf{B} = \{f^{-1}[(\infty, f(d(2)))]\} \bigcup \{f^{-1}[[f(d(n-1)), f(d(n+1))]] : n = 2, 3, 4, \ldots\}$$

Clearly $\mathbf{B}$ is a locally finite family of closed subsets of X, and the restriction of g to any one of them equals the restriction of no more than four different $g_n$ to that set. Thus for each $B \in \mathbf{B}$, $g|B \in C(\mathbf{B})$ and by [8, 1A(3)], $g \in C(X)$.

From our construction it is clear that $g_n^{-1}(1) = \bigcup\{T(n) : n \in \mathbf{N}\}$. Thus $D \subseteq g^{-1}(1)$ and so $g^{-1}(1)$ is a non-compact zero-set of $X$. Now $T(n) = g^{-1}(1) \cap f^{-1}[(f(d(n))-0.3, f(d(n))+0.3)]$ so each $T(n)$ is open in $g^{-1}(1)$. Hence for each $n \in \mathbf{N}$ there exists an open subset $W(n)$ of $X$ such that $W(n) \cap g^{-1}(1) = T(n)$. If $\mathrm{int}_X g^{-1}(1) \neq \varnothing$ there exists some $k \in \mathbf{N}$ such that $\mathrm{int}_X g^{-1}(1) \cap T(k) \neq \varnothing$. Then $\varnothing \neq \mathrm{int}_X g^{-1}(1) \cap W(k) \subseteq T(k)$, contradicting the construction of $T(k)$. Thus $g^{-1}(1)$ is a non-compact zero-set of $X$ with empty interior, as required. As $g^{-1}(1)$ is non-compact there is a point $p \in \mathrm{cl}_{\beta X} g^{-1}(1) - X$, and as $g^{-1}(1)$ is nowhere dense, such a $p$ is not a remote point of $X$. □

**Remark 5.4.** Interestingly, a pseudocompact space with no almost-P-points and non-measurable cellularity, must also have non-remote points because Terada [23] has shown that the points of $\upsilon X - X$ cannot be remote. Thus any Tychonoff space of non-measurable cardinality and with no almost-P-points has non-remote points.

**Lemma 5.5.** *Let $V$ be $z$-embedded in $X$, for example a cozero set of $X$. Then $|C(V)| \leq |C(X)|$.*

*Proof.* In [7, 1.4, 1.5] it is shown that for any space $X$, $|\mathbf{Z}(X)| = |\mathbf{Z}(X)|^{\aleph_0}$ and $|C(X)| = |\mathbf{Z}(X)|$. Now if $V$ is $z$-embedded, the map $Z \to Z \cap V$ maps $\mathbf{Z}(X)$ onto $\mathbf{Z}(V)$. Thus by the above $|C(V)| = |\mathbf{Z}(V)| \leq |\mathbf{Z}(X)| = |C(X)|$. □

**Lemma 5.6.** *Let $X$ be an* RG*-space. If $X$ is the union of $\aleph_1$ nowhere dense zero-sets, then $X_\delta$ can be partitioned into $\aleph_1$ non-empty clopen subsets.*

*Proof.* Assume that $X = \bigcup\{Z(\alpha) : \alpha < \omega_1\}$, where each $Z(\alpha)$ is a nowhere dense zero-set of $X$. Inductively define a family $\{A(\alpha) : \alpha < \omega\}$ as follows: $A(0) = Z(0)$. Now let $\beta < \omega_1$ and assume inductively that we have defined $\{A(\alpha) : \alpha < \beta\}$ as follows:

(a) $\{A(\alpha) : \alpha < \beta\}$ is a pairwise disjoint countable collection of clopen subsets of $X_\delta$,
(b) $\bigcup\{Z(\alpha) : \alpha < \beta\} = \bigcup\{A(\alpha) : \alpha < \beta\}$,
(c) if $\alpha < \beta$ then $A(\alpha) \subseteq Z(\alpha)$.

Now define $A(\beta)$ to be $Z(\beta) - \bigcup[A(\alpha) : \alpha < \beta\}$. We see that our inductive hypotheses are satisfied for $\alpha < \beta + 1$. Thus $\{A(\alpha) : \alpha < \omega_1\}$ is a partition of $X$ into clopen subsets of $X_\delta$, each of which is a nowhere dense subset of $X$. Let $S = \{\alpha < \omega_1 : A(\alpha) \neq \varnothing\}$. First note that $S$ must be infinite as no topological space is the union of finitely many nowhere dense subsets. If $S$ were countably infinite then $X = \bigcup\{Z(\alpha) : \alpha \in S\}$ which would contradict [10, 2.2]. Thus $|S| = \aleph_1$, and $\{A(\alpha) : \alpha \in S\}$ is the desired partition of $X_\delta$. □

**Theorem 5.7.** *Suppose that $X$ is an* RG*-space that is the union of $\aleph_1$ nowhere dense zero-sets. Then $|C(X)| \geq 2^{\aleph_1}$.*

*Proof.* By the preceding lemma there is a partition $\{A(\alpha) : \alpha \in S\}$ of $X$ into $\aleph_1$ pairwise disjoint non-empty clopen subsets of $X_\delta$, where $|S| = \aleph_1$. If $T \subset S$ define $g_T : X_\delta \to \mathbb{R}$ to be the characteristic function of $\bigcup\{A(\alpha) : \alpha \in T\}$, which is clopen in $X_\delta$. Clearly $T \to g_T$ is a one-to-one map from the power set of $S$ into $C(X_\delta)$, so $|C(X_\delta)| \geq 2^{|S|} = 2^{\aleph_1}$. □

**Corollary 5.8.** *If the continuum hypothesis holds then an* RG*-space $X$ for which $|C(X)| = c$ must have a dense set of almost-*P*- points.*

*Proof.* If $gX$ is not dense in $X$ then there is a cozero-set $V$ of $X$ for which $V \cap gX = \varnothing$. Hence $gV = \varnothing$ by proposition 5.1 and by the continuum hypothesis $V$ is the union of $\aleph_1$ nowhere dense zero-sets. As $|C(V)| \leq |C(X)| = c < 2^c = 2^{\aleph_1}$, by theorem 5.7 $V$ cannot be an RG-space. Hence by [10, 2.3(f)], $X$ cannot be an RG-space either. □

**Remark 5.9.**

(1) "Dense" is the best that we can do in 5.8, as the one-point compactification of **N** satisfies the hypotheses of 5.8 but the unique non-isolated point is not an almost-P-point.
(2) Another way to prove 5.8 is to note that $|C(V)| = c$ (by the argument above) and that $V$ is an RG-space (by [10, 2.3 (f)]) so by 5.7, $V$ must have an almost-P-point, which will be an almost P-point of $X$ as $V$ is open in $X$.

## 6. Results and examples involving Lindelöf spaces

We begin with a result that does not require any assumptions on $X_\delta$.

**Theorem 6.1.** *Let $X = S \cup T$ where $S$ and $T$ are complementary dense realcompact $z$-embedded (for example, Lindelöf) suspaces of $X$. Then $X$ is not* RG.

*Proof.* By [3, 2.6 (b)] $S$ and $T$ are closed and hence clopen in $X_\delta$ so $\chi_S \in C(X_\delta)$. But since $S$ and $T$ are both dense in $X$, $\chi_S$ is not $X$-continuous at any point of $X$. Therefore $\chi_S \notin G(X)$, as functions in $G(X)$ are continuous on dense open subsets of $X$ ([10, 2.1 (a)] ). Thus $X$ is not RG. □

The following result in the positive direction is an improvement to [10, 2.3 (d)].

**Theorem 6.2.** *Suppose that $X$ is* RG*, and that $X_\delta$ is normal. Let $Y \subset X$ and let $Y$ be $z$-embedded and realcompact. Then $Y$ is* RG*. In particular $Y$ is* RG *if it is Lindelöf.*

*Proof.* By [3, 2.6] $Y_\delta$ is closed in $X_\delta$. Now apply [10, 2.3 (g)]. □

The following result is negative and considers disjoint dense subspaces of a space $X$. But we again need a constraint on $X_\delta$.

**Theorem 6.3.** *Let $S$ and $T$ be disjoint dense realcompact $z$-embedded (say Lindelöf) subspaces of $X$. If $X_\delta$ is normal then $X$ is not* RG.

*Proof.* Again $S$ and $T$ are disjoint and closed in $X_\delta$ and by the normality of $X_\delta$ there is a function $f \in C(X_\delta), \mathbf{0} \leq f \leq \mathbf{1}$ such that $f[S] = 0$ and $f[T] = 1$. Now $A = f^{-1}[0, 1/2] \in Z(X_\delta)$ and hence it is clopen in $X_\delta$. Again $\chi_A[S] = 1, \chi_A[T] = 0$. But $S$ and $T$ are dense in $X$ so $\chi_A$ is not continuous at any point of $X$. Hence as in 6.1 $\chi_A \notin G(X)$, and $X$ is not RG. □

**Corollary 6.4.** *[CH] Let $|X| = c$. Then if $X$ contains disjoint dense realcompact $z$-embedded subspaces then $X$ is not* RG.

*Proof.* By [15, 4] $X_\delta$ is paracompact and hence normal. Now apply the theorem. □

We can relax the demand on $X_\delta$ if we know that $S_\delta$ is Lindelöf.

**Theorem 6.5.** *Let $S$ and $T$ be disjoint dense subspaces of $X$ with $S_\delta$ Lindelöf and $T$ realcompact and $z$-embedded. Then $X$ is not* RG.

*Proof.* As above, $S$ and $T$ are disjoint closed sets of $X_\delta$. Now for all $x \in S$, there exists $V_x \in \operatorname{coz}(X_\delta)$ such that $x \in V_x$ and $V_x \cap T = \varnothing$ (because the point $x$ does not belong to the closed subset $T$ of $X_\delta$). Therefore $S_\delta \subset \{\bigcup V_x : x \in S\}$. As $S$ is a Lindelöf subset of $X_\delta$, we can take a countable subcover and then take its union to get $W \in \operatorname{coz}(X_\delta)$ such that $S \subset W$ and $W \cap T = \varnothing$. Thus $W$ is clopen in $X_\delta$ and again $\chi_W \in C(X_\delta) - G(X)$ so $X$ is not RG. □

One way to ensure that $S_\delta$ is Lindelöf is to know that $S$ is a union of countably many scattered Lindelöf spaces (cf. [15, 5.2]). So we have:

**Corollary 6.6.** *Let $S$ and $T$ be disjoint dense Lindelöf subspaces of $X$ with $S$ the union of countably many Lindelöf scattered subspaces. Then $X$ is not* RG.

**Remark 6.7.** We do not know if this corollary holds if $S$ is the union of uncountably many dense Lindelöf subspaces. Note that if $X$ satisfies the conditions of the preceding corollary then $X$ is weakly Lindelöf and that there are weakly Lindelöf spaces whose Lindelöf subspaces are nowhere dense—for example see [2, Ex. 2.2].

**Theorem 6.8.** *If $X$ is an* RG*-space and if $X$ has a dense subspace $A$ such that $A_\delta$ is Lindelöf then $gX$ is dense in $X$.*

*Proof.* Case 1. Suppose, if possible, that $gX = \varnothing$. Then $X = \bigcup\{Z_\alpha : \alpha \in I\}$, where each $Z_\alpha \in Z(X)$ and $\operatorname{int}_X[Z_\alpha] = \varnothing$. Therefore $A_\delta = \bigcup\{A_\delta \cap Z_\alpha : \alpha \in I\}$, so as $A_\delta$ is Lindelöf there exist $\{\alpha_i . i \in N\}$, such that $A_\delta = \bigcup\{A_\delta \cap Z_{\alpha_i} : i \in N\}$. Clearly $A = \cup\{A \cap Z_{\alpha_i} : i \in N\}$.

We claim that for each $i \in N, \operatorname{int}_A(A \cap Z_{\alpha_i}) = \varnothing$. For if not, there exists a $V$ open in $X$ such that $\varnothing \neq V \cap A \subset A \cap Z_{\alpha_i}$. As $Z_{\alpha_i}$ is closed nowhere dense in $X$, $V - Z_{\alpha_i}$ is a non-empty open set of $X$ so $(V - Z_{\alpha_i}) \cap A \neq \varnothing$ as $A$ is dense in $X$, contradicting the choice of $V$. Thus the claim holds.

Now by the claim, and the fact that $A \cap Z_{\alpha_i} \in Z(A)$ it follows by [10, 2.2] that $A$ is not an RG-space, and hence by [10, 2.3 (b)] that $X$ is not an RG-space.

Case 2. Suppose that $gX$ is not dense in $X$. Then there exists $V \in \mathrm{coz}(X)$ such that $(gX) \cap V = gV = \varnothing$. Now $A \cap V$ is dense in $V$ because $A$ is dense in $X$, and $(A \cap V)_\delta = A_\delta \cap V_\delta$. Since $V_\delta$ is clopen in $X_\delta$ and $A_\delta$ is Lindelöf, $(A \cap V)_\delta$ is a closed subspace of a Lindelöf space and hence Lindelöf itself. Thus by case 1 (with $V$ replacing $X$ and $A \cap V$ replacing $A$), we have that $V$ is not RG and hence that $X$ is not RG. □

**Corollary 6.9.** *If $X$ is* RG *and has a dense subspace that is a countable union of scattered Lindelöf spaces, then $gX$ is dense in $X$.*

**Theorem 6.10.** *If a space $X$ is the union of countably many scattered Lindelöf subspaces, then $X_\delta$ is Lindelöf and $|C(X_\delta)| = |C(X)|$.*

*Proof.* Let $X = \bigcup\{L(n) : n \in \mathbf{N}\}$, where each $L(n)$ is a scattered Lindelöf space. By [15, 5.2] each $(L(n))_\delta$ is Lindelöf, so as the subspace topology that $L(n)$ inherits from $X_\delta$ is the same as that of $(L(n))_\delta$ , it follows that $X_\delta$ is the union of countably many Lindelöf subspaces and hence is Lindelöf.

Let $w(Y)$ and $wL(Y)$ denote respectively the weight and the weak Lindelöf number of the space $Y$. In [5] it is proved that for any space $Y, |C(Y)| \leq \mathrm{w}(Y)^{\mathrm{w}L(Y)}$. We apply this to $X_\delta$. Since $X_\delta$ is Lindelöf we have $\mathrm{w}L(X_\delta) = \aleph_0$. Furthermore we have $\mathrm{w}(X_\delta) \leq |\mathbf{Z}(X)| = |C(X)|$ (the latter equality is part of [7, 1.4]). Hence we have $|C(X_\delta)| \leq |C(X)^{\aleph_0}|$. But it is well-known that $|C(X)|^{\aleph_0}| = |C(X)|$ (see [7]). Thus $|C(X_\delta)| \leq |C(X)|$. But the opposite inequality obviously holds as $X_\delta$ is just $X$ with a stronger topology. □

**Remark 6.11.** Since the Comfort/ Hager inequality in the proof of 6.10 uses the weak Lindelöf number, one might be tempted to try to strengthen 6.10 by looking for conditions on $X$ that would imply that $X_\delta$ is weakly Lindelöf (rather than Lindelöf). However by [21, 5.14] a weakly Lindelöf P-space is Lindelöf, so no additional generality can be gained this way.

## 7. Some properties incompatible with being RG

Let $i : X_\delta \to X$ be the identity map on the underlying set of $X$.

**Theorem 7.1.** *$X$ is not an* RG*-space if it satisfies the condition: there exist disjoint subsets $C$ and $D$ of $\beta(X_\delta)$ so that*

(i) *$i^\beta(C)$ and $i^\beta(D)$ are both dense in $\beta X$,*
(ii) *for some $h \in C(\beta(X_\delta)), h[C] = \{0\}$ and $D \subset \mathrm{coz}(h)$.*

*Proof.* Suppose that we have such a function $h \in C(\beta(X_\delta))$. We will show that $h|X \notin G(X)$, a contradiction. Throughout the superscript $\alpha$ will denote the Stone extension of a bounded function on $X$ and the superscript $\gamma$ will denote the Stone extension of a bounded function on $X_\delta$.

Suppose, if possible, that $h|X = \sum_{i=1}^n a_i b_i^*, a_i, b_i \in C^*(X)$ (using part (i) of 1.1). Let $b = \Pi b_i$. We will show that $b = 0$. Clearly $b^\gamma = b^\alpha \circ i^\beta$ . By computation, $(h|X)b^2 \in C^*(X)$ and it determines $[(h|X)b^2]^\alpha \in C(\beta X)$. Now the functions $[(h|X)b^2]^\alpha \circ i^\beta$ , $(h|X)b^2]^\gamma$ and $h(b^\gamma)^2$ are all in $C(\beta(X_\delta))$ and they agree on $X_\delta$ so they are equal. But $h$ vanishes on $C$ and therefore $[(h|X)b^2]^\alpha$

vanishes on $i^\beta[C]$ which is dense in $\beta X$. Thus $[(h|X)b^2]^\alpha$ is the zero function and therefore $h(b^\gamma)^2$ is the zero function on $\beta(X_\delta)$. But $h$ is non-zero on all elements of $D$, so $b^\gamma$ vanishes on $D$ which implies that $b^\alpha$ vanishes on $i^\beta[D]$, so again by denseness, $b^\alpha = 0$ and $b = 0$.

The above argument shows the product of all the $b_i$ equals 0. In fact, the techniques just used also show that the $b_i$ are pairwise orthogonal. For example, if $d_i$ is the product of the $(n-1)$ $b_j$ obtained by ignoring $b_i$, then in $(h|X)d_i^2$ the term $a_i b_i (d_i)^2$ vanishes and since $(h|X)d_i^2 \in C^*(X)$ we get $d_i = 0$ by again using the fact that $i^\beta[D]$ is dense in $\beta X$ and $D \subset \mathrm{coz}(h)$. Thus all $(n-1)$-fold products vanish. Now continue by finite descent to get that the $\{b_i\}$ are orthogonal.

Now let $t = b_1^2 + \cdots + b_n^2$, and let $s = (b_1^*)^2 + \cdots + (b_n^*)^2$. Since the $\{b_i\}$ are orthogonal, so are the $b_i^*$ and by computation $(h|X)t \in C^*(X)$. Again the properties of $h$ show that $ht = 0$. But by computation, $(h|X) = (h|X)ts$, so $h|X = 0$, which implies that $h = 0$ and this is false since $h$ is non-zero at the elements of $D$. □

**Remark 7.2.**

(1) If $gX = \varnothing$ and $X$ is RG then there is a compact set in $\beta(X_\delta)$ disjoint from $X$ that contains the maximal ideals of $C(\beta(X_\delta))$ that correspond to the minimal prime ideals in $C(X)$. (For each $x \in X$, there is a non-zero divisor $f_x \in C(X)$ that lies in $M_x$). Let $L_x$ be the (patch-open) subset of $\beta(X_\delta)$ consisting of all prime $z$-ideals of $C(X)$ that contain $f_x$. No minimal prime lies in any $L_x$ and $X \cup L_x$ is open in $\beta(X_\delta)$ so one takes its complement.

(2) It is easy to derive the non-vanishing of $gX$ in 6.8 from theorem 7.1 using [8, Exercise 3B.1] and remark (i). Moreover the methods show that if in addition to the hypotheses of 6.8, $X$ is also cozero-complemented and an RG-space then $X$ must have a P-point (if not the minimal primes of $C(X)$ form a compact subset of $\beta(X_\delta)$ disjoint from $A_\delta$ and 7.1 gives a contradiction).

(3) It is interesting to compare the condition in the Theorem 7.1 with Smirnov's theorem that says that a space is Lindelöf if and only if it is normally placed in its Stone-Cech compactification. If $gX = \varnothing$ we can apply 7.1 if the compact subset of $\beta(X_\delta)$ from part (i) is a subset of a zero set disjoint from a dense subset of $X$. One suspects that there are spaces that satisfy the condition in 7.1 but do not satisfy the hypotheses of 6.8.

Our next result uses P-spaces that are functionally countable. These were characterized by A.W. Hager (cf. [15, 3.1,3.2]) and include (properly) all Lindelöf P-spaces.

**Theorem 7.3.** *Let $X$ be an* RG *space for which $X_\delta$ is functionally countable. Then no family of at most $\aleph_1$ nowhere dense zero-sets of $X$ can have $X$ as its union.*

*Proof.* Suppose that $X$ is an RG-space that is a union of $\aleph_1$ nowhere dense zero-sets. By lemma 5.6 there is a partition $\{A(\alpha) : \alpha < \omega_1\}$ of $X$ into $\aleph_1$ non-empty clopen subsets of $X_\delta$. Let $H = \{r_\alpha : \alpha < \omega_1\}$ be a subset of $\mathbb{R}$ of cardinality $\aleph_1$ and define $f : X \to \mathbb{R}$ by $f[A(\alpha)] = r_\alpha$. Then $f \in C(X_\delta)$ and $|f[X_\delta]| = \aleph_1$, so $X_\delta$ is not functionally countable. □

## 8. The structure of $G(X)$. $G$-embedded subspaces

**Proposition 8.1.** *($G(X)$ as a functor). Let $k : X \to Y$ be continuous. Then under composition $k$ induces a (natural) ring homomorphism $G(k) : G(Y) \to G(X)$. Also* $\mathrm{rg}(G(k))(f) \leq \mathrm{rg}(f)$

*Proof.* By a straightforward computation $(b \circ k)^* = b^* \circ k$ for any $b \in C(Y)$. Furthermore if $f \in G(Y)$, then $f \circ k \in G(X)$ and $\mathrm{rg}(f \circ k) \leq \mathrm{rg}(f)$ as follows. Let $\mathrm{rg}(f) = n, f = \sum_{i=1}^{i=n} a_i b_i^*, a_i, b_i \in C(X)$. So $f \circ k = (\sum_{i=1}^{i=n} a_i b_i^*) \circ k = \sum_{i=1}^{i=n} (a_i \circ k)(b_i^* \circ k) = \sum_{i=1}^{i=n} (a_i \circ k)(b_i \circ k)^*$. Now $a_i \circ k, b_i \circ k \in C(X)$ so $f \circ k \in G(X)$ and $\mathrm{rg}(f \circ k) \leq n = \mathrm{rg}(f)$. □

**Example 8.2.** Note that it is possible that $k$ be a surjection and that $\mathrm{rg}(f \circ k) < \mathrm{rg}(f)$. The perfect irreducible surjection $\beta N \to N^*$ is a simple example. More generally, suppose that $X$ is an almost-P-space with no P-points, and let $i : X_\delta \to X$ be the identity map on the underlying set. We know by 5.2 that $k = \beta i : \beta(X_\delta) \to \beta X$ is a perfect irreducbile continuous surjection, a very well-behaved map. Since $X$ has no P-points, $\mathrm{rg}(X) \geq 2$ so let $f \in G^*(X)$ with $\mathrm{rg}(f) \geq 2$. Now $f^\beta \in G(\beta X)$ and $\mathrm{rg}(f^\beta) \geq 2$ since $\mathrm{rg}(f) = \mathrm{rg}(f^\beta|X) \leq \mathrm{rg}(f^\beta)$. But $f \circ i \in C^*(X_\delta)$ as $G^*(X) \subset C^*(X_\delta)$. So $(f \circ i)^\beta \in C^*((\beta X)_\delta)$ and so $\mathrm{rg}((f \circ i)^\beta = 1$. But $f^\beta \circ \beta i|X_\delta = (f \circ i)^\beta|X_\delta = f \circ i$, so $f^\beta \circ \beta i = (f \circ i)^\beta$ and therefore $\mathrm{rg}(f^\beta \circ \beta i) = 1$ while $\mathrm{rg}(f^\beta) = 2$. Note that since $\beta X_\delta$ is not a P-space, (its regularity degree exceeds 1), yet for all $f \in G(X), f^\beta \circ \beta i \in G(\beta(X_\delta))$ and $\mathrm{rg}(f^\beta \circ \beta i) = 1$.

**Remark 8.3.** If $k : X \to Y$ is a continuous surjection and $S \subset Y$, then one easily has $\chi_S \circ k = \chi_{k^{-1}[S]}$. Thus if $S \in Z(Y)$ is not open, $\mathrm{rg}(\chi_S) = 2$, while $\mathrm{rg}(\chi_S \circ k) = \mathrm{rg}(\chi_{k^{-1}[S]})$. Now if $S \in Z(Y)$ then $k^{-1}[S] \in Z(X)$ so $\mathrm{rg}(\chi_{k^{-1}[S]}) = 1$ if and only if $k^{-1}[S]$ is clopen in $X$. Thus if $S \in Z(Y), \mathrm{rg}(\chi_S) = \mathrm{rg}(\chi_S \circ k)$ if and only if either $S$ and $k^{-1}[S]$ are both non-clopen in $Y$ and $X$ respectively, or else $S$ and $k^{-1}[S]$ are both clopen respectively in $Y$ and $X$.

**Lemma 8.4.** *Let $V$ be a cozero-set of $X$ and let $f : V \to \mathbb{R}$ be a function. Define $f^\wedge$ to coincide with $f$ on $V$ and to vanish on $X - V$. Then:*

(i) $(f^\wedge)^* = (f^*)^\wedge$,
(ii) *if $f \in G(V)$, then $f^\wedge \in G(X)$ and* $\mathrm{rg}(f) = \mathrm{rg}(f^\wedge)$. *(In particular $V$ is $G$-embedded in $X$).*

*Proof.* Result (i) is readily checked. (ii). First assume that $f \in C(V)$, and let $V = \mathrm{coz}(g), g \in C^*(X)$. Let $k = f[\mathbf{1} + |f|]^{-1}$. Then $k \in C^*(V)$ so by [3, 1.1] there exists $h \in C(X)$ such that $h[X - V] = 0$ and $h|V = kg$. Now $[\mathbf{1}+|f|]^{-1} \in C^*(V)$ so for the same reason there is a $t \in C(X)$ with $t[X-V] = 0$

and $t|V = g[\mathbf{1} + |f|]^{-1}$. Thus $ht^* \in G(X)$ and $\mathrm{rg}(ht^*) = 1$. One checks that $ht^* = f^\wedge$ so $\mathrm{rg}(f^\wedge) = 1 = \mathrm{rg}(f)$ establishing (b) when $f \in C(V)$.

Now assume that $f \in G(V)$ and that $\mathrm{rg}(f) = n$. There exist $a_i, b_i \in C(V)$ so that $f = \sum_{i=1}^{i=n} a_i b_i^*$. Using (i) one easily verifies that

$$f^\wedge = \sum a_i^\wedge (b_i^\wedge)^* \qquad (**)$$

By part (iv) of 1.1 and the previous paragraph each $(b_i^\wedge)^*$ is in $G(X)$ and has regularity degree 1. Similarly each $a_i^\wedge$ is in $G(X)$ and has regularity degree 1. Thus it follows from $(**)$ that $f^\wedge \in G(X)$ and $\mathrm{rg}(f^\wedge) \leq n = \mathrm{rg}(f)$. So by part (ii) of proposition 1.1 $\mathrm{rg}(f^\wedge) = \mathrm{rg}(f)$. □

**Lemma 8.5.** *Let $V \in \mathrm{coz}(X)$ and $V \subset T \subset X$. Then if $T - V$ is $G$-embedded in $X$, then $T$ is $G$-embedded in $X$.*

*Proof.* Let $S = T - V$. Let $f \in G(T)$. Now $f|S \in G(S)$ so there exists $k \in G(X)$ such that $k|S = f|S$. Let $V = \mathrm{coz}(g)$. By lemma 8.4 there exists $f^\wedge \in G(X)$ such that $f^\wedge|V = f$ and $\mathrm{rg}(f^\wedge) = 1$. Let $h = f^\wedge gg^* + k(1 - gg^*)$. Then $h \in G(X)$ and one easily verifies that $h|T = f$ and $T$ is $G$-embedded in $X$. □

**Theorem 8.6.** *Let $X$ be a cozero-complemented space. Then:*

(i) *If $f \in G(X)$ there is a dense cozero-set $V$ of $X$ such that $f|V \in C(V)$.*
(ii) *If $X$ is also normal and the nowhere dense zero-sets of $X$ are* RG-*spaces then $G(X)$ is the set of $f \in C(X_\delta)$ for which there exists a dense cozero-set $V$ of $X$ such that $f|V \in C(V)$.*
(iii) *If $X$ is normal, and there is an integer $k$ such that* $\mathrm{rg}(Z) \leq k$ *for each nowhere dense zero-set $Z \in \mathbf{Z}(X)$ then* $\mathrm{rg}(X) \leq 2k + 1$.

*Proof.* (i) Let $f \in G(X)$. By the proof [10, 2.1(a)] there is a finite family $\{Z(i) : i = 1, \ldots, n\}$ of zero-sets of $X$ such that $f$ is continuous on $\bigcap\{(X - Z(i)) \cup \mathrm{int}_X Z(i)\}$. As $X$ is cozero-complemented, for each $i$ there exists a cozero-set $V(i)$ of $X$ that is dense in $\mathrm{int}_X Z(i)$. (If $\mathrm{int}_X Z(i)$ is empty then so is $V(i)$, but that will cause no problems.) Let $W(i) = (X - Z(i)) \cup V(i)$ and let $V = \bigcap\{W(i)\}$. Then $V$ is a dense cozero-set of $X$ contained in $\bigcap\{(X - Z(i)) \cup \mathrm{int}_X Z(i)\}$ and $f|V \in C(W)$ as required.

(ii) Suppose that $f \in C(X_\delta)$ and there exists $g \in C^*(X)$ such that coz $g$ is dense in X and $f|\mathrm{coz}\ g \in C(\mathrm{coz}\ g)$. By the proof of lemma 8.4 there exists $f^\wedge \in G(X)$ such that $f^\wedge|\mathrm{coz}\ g = f|\mathrm{coz}\ g$ and $\mathrm{rg}(f^\wedge) = 1$. By hypothesis $Z(g)$ is an RG-space, and $f|Z(g) \in C((Z(g))_\delta)$, so $f|Z(g) \in G(Z(g))$. Hence there exist $n \in N, a_i, b_i \in C(Z(g))$ such that $f|Z(g) = \sum_1^n a_i b_i^*$. As $X$ is normal, $Z(g)$ is $C$-embedded in $X$ and so for each $i$ there exist $j_i, k_i \in C(X)$ such that $j_i|Z(g) = a_i$ and $k_i|Z(g) = b_i$. Let $m = j_1 {k_1}^* \cdots + j_n {k_n}^*$. Then $m \in G(X)$ and $m|Z(g) = f|Z(g)$. We now claim that

$$f = f^\wedge gg^* + (1 - gg^*)m, \qquad (\sharp)$$

(where $f^\wedge$ is as defined in 8.4).

For if $x \in \operatorname{coz} g$ then $(\mathbf{1}-gg^*)(x) = 0$ and $f^\wedge gg^*(x) = f(x)g(x)/g(x) = f(x)$, so the right hand side of equation $\sharp$ evaluated at $x$ gives $f(x) + 0 = f(x)$. If $x \in Z(g)$ then $g^*(x) = 0$ and the right hand side of equation $\sharp$ evaluated at $x$ gives $= 0 + (1)m(x) = f(x)$. Our claim is verified and so $f \in G(X)$.

(iii) Let $f \in G(X)$. By (i) there exists $g \in C^*(X)$ such that $Z(g)$ is nowhere dense in $X$ and $f|X - Z(g) \in C(X - Z(g))$. As $f|Z \in G(Z)$, $\operatorname{rg}(f|Z) \leq k$ so there are $a_i, b_i \in C(Z)$ so that $f|Z = \sum_1^k a_i b_i^*$. As $X$ is normal $Z$ is $C$-embedded in $X$ so there exist $A_1, \dots, A_k, B_1, \dots, B_k \in C(X)$ such that $A_i|Z = a_i$ and $B_i|Z = b_i$. By part (ii) of 8.4 $(f|X - Z(g))^\wedge \in G(X)$ and $\operatorname{rg}(f|X - Z(g))^\wedge = 1$. Let $m = A_1 B_1^* + \cdots + A_k B_k^*$; then by part(ii) of 8.4 $m \in G(X)$ and $\operatorname{rg}(m) \leq k$. It is routine to show that $f = (f|X - Z(g))^\wedge gg^* + (\mathbf{1} - gg^*)m$, from which it follows that $\operatorname{rg}(f) \leq 2k + 1$. □

**Remark 8.7.**

(1) Example [10, 2.10] (call it $X$) satisfies all the hypotheses of 8.6(ii) except normality, but it does not satisfy its conclusion. (As usual, let $I(X)$ denote the set of isolated points of $X$. Observe that every nowhere dense set of $X$ is discrete and hence an RG-space, and that if $V \in \operatorname{coz} X$ then $I(X) - V$ is the complementary cozero-set.) As $f|I(X)$ is continuous for each $f \in C(X_\delta)$, the set occuring in part (b) of the statement of 8.6 is just $C(X_\delta\ )$, but this does not equal $G(X)$ because as noted in [10, 2.10] $X$ is not an RG-space. This shows that the hypothesis of normality cannot be dropped from 8.6(ii).

(2) As zero-sets of RG-spaces are RG-spaces, one consequence of 8.6 part (a) is that if $X$ is a normal cozero-complemented RG-space, and if $f \in C(X_\delta)$, then there exists a dense cozero-set $V$ of $X$such that $f|V \in C(V)$. Thus if $X$ is a normal basically disconnected RG-space and if $f \in C(X_\delta)$, then there exists a dense cozero-set $V$ of $X$ such that $f|V \in C(V)$.

(3) In [10, 5.6] it is shown that nowhere dense $z$-embedded zero-sets of quasi-P spaces are P-spaces. Thus a normal cozero-complemented quasi-P space $X$ will have $\operatorname{rg}(X)$ finite (see the argument in the remarks below). In particular:

**Example 8.8.** A countable nodec space of finite regularity degree.

The space $X$ presented in [9, 5.10] is a countable nowhere locally compact nodec (i.e. closed nowhere dense subspaces are discrete) extremally disconnected quasi-P space without P-points. Thus it satisfies the hypotheses of 8.6 and because its nowhere dense zero-sets are RG-spaces, the function $m$ constructed in the proof of 8.6 can be taken to be in $C(X)$ (as the function $f|Z(g)$ will be in $C(Z(g))$). From equation ($\sharp$) in 8.6 it follows that if $f \in G^*(X)$ then $\operatorname{rg}(f) \leq 3$. As remarked in 8.6 part (ii) above, that means that there is an integer $k$ such that $\operatorname{rg}(f) \leq k$ for all $f \in G(X)$. But $X$ is not an RG-space by [10, 2.2]. This is yet another example of a space $X$ with $\operatorname{rg}(X)$ finite without $X$ being an RG-space.

## 9. Very weak P-points and RG spaces

**Definition 9.1.** *A point $p \in X$ is called a* **very weak** P**-point** *if $p$ is not a limit point of any countable discrete subset of $X$. We denote the set of very weak* P*-points of $X$ by* $\mathrm{vwP}(X)$.

Recall that $p$ is a weak P-point of $X$ if $p$ is not a limit point of any countable subset of $X$. Weak P-points and very weak P-points were introduced (with different names) by Kunen [12] who showed that the implications P-point $\Rightarrow$ weak P-point $\Rightarrow$ very weak P-point are strict. (See also [24]). Note that each point of the space of example 8.8 is a very weak P-point, but no point is a weak P-point. The goal of this section is to show that if $\mathrm{vwP}(X)$ is not dense in $X$ then $X$ is not an RG space.

**Theorem 9.2.** *Let $X$ be any Tychonoff space and let $S$ be any countable scattered subspace of $X$ such that $\mathrm{CB}(S)$ is finite. Then $S$ is $G$-embedded in $X$.*

*Proof.* We induct on $\mathrm{CB}(S)$. If $\mathrm{CB}(S) = 1$ then $S$ is discrete. If $S$ is finite it is $C$-embedded and we are done. So suppose $S = \{s(i) : i \in \mathbf{N}\}$. As $X$ is Tychonoff there is a pairwise disjoint family $\{V(i) : i \in \mathbf{N}\}$ of cozero-sets of $X$ such that $s(i) \in V(i)$ for each $i$. Let $f \in C(S)$. Let $V = \bigcup\{V(i) : i \in \mathbf{N}\}$ and extend $f$ to $F \in C(V)$ by letting $F[V(i)] = \{f(s(i))\}$. As $V \in \mathrm{coz}\ X$ by 8.4 (b) $F$ can be extended to a member of $G(X)$. Hence $S$ is $G$-embedded in $X$ if $\mathrm{CB}(S) = 1$.

Now suppose that $T$ is $G$-embedded in $X$ if $T$ is any countable scattered subspace of $X$ and if $\mathrm{CB}(T) \leq n$. Suppose that $S$ is a countable scattered subspace of $X$ and that $\mathrm{CB}(S) = n + 1$. Let $L = S - I(S)$. Then $L$ is countable, scattered, and $\mathrm{CB}(L) = n$. As $I(S)$ is a countable open subset of $S$ there is a cozero-set $\mathrm{coz}\ g$ of $X$ for which $S \cap \mathrm{coz}\ g = I(S)$. Let $f \in C(S)$. By the preceding paragraph there is an $F \in G(X)$ such that $F|S = f|S$. By the induction hypothesis there is a $K \in G(X)$ such that $K|L = f|L$. A straightworward computation shows that $Fgg^* + K(1 - gg^*)|S = f$. As $Fgg^* + K(1 - gg^*) \in G(X)$ we are done. $\square$

The following result generalizes [10, 3.3] and also gives a shorter proof.

**Lemma 9.3.** *Let $X$ be a zero-dimensional Tychonoff space. Let $D = \{d(i) : i \in \mathbf{N}\}$ be a countable discrete subset of $X$. Let $q$ be a limit point of $D$. Suppose that $f \in G(X)$ and that $f$ satisfies these conditions:*

(i) *if $i \in \mathbf{N}$ and $d(i) \in A$ and $A$ is a clopen subset of $X$ then* $\mathrm{rg}(f|A) \geq n$,
(ii) *there exists $r > 0$ such that if $i \in \mathbf{N}$ then $f(d(i)) = r$,*
(iii) *there exists $s > 0$ such that $s \neq r$ and $f(q) = s$.*

*Then* $\mathrm{rg}(f) \geq n + 1$.

*Proof.* Suppose to the contrary that $f = h_1k_1^* + \cdots + h_nk_n^*$ where the $h_i, k_i \in C(X)$. Let $J = \{j \in \{1, \ldots, n\} : k_j(q) \neq 0\}$. Observe that $J \neq \varnothing$ or else $k_j^*(q) = 0$ for each $j \in \{1, \ldots, n\}$ and so $f(q) = 0$, contradicting (c). As $k_j^*$ is continuous on the open set $X - Z(k_j)$, if $W = \bigcap\{X - Z(k_j) : j \in J\}$ then

$q \in W$ and if $t$ is defined by $t = \sum\{h_j k_j^* : j \in J\}$ then $t$ is continuous on $W$ and hence at $q$.

Let $E$ be a clopen subset of $X$ contained in $W$ and containing $q$. Let $I = \{1, \ldots, n\} - J$. Then $|I| < n$ and

$$f|E = t|E + \sum\{h_j k_j^* : j \in I\} \tag{1}$$

Claim 1 : $f|(D \cap E) \cup \{q\}$ is not continuous at $q$.

To prove claim 1, note that since E is open and contains $q$ and since $q \in \mathrm{cl}_X D$ it follows that $q \in \mathrm{cl}_X(D \cap E)$. But $f(q) = s$ and $f[D \cap E] = r \neq s$ so $f$ cannot be continuous at $q$.

Claim 2: There exists $m \in \mathbf{N}$ such that $d(m) \in D \cap E$ and $f(d(m)) \neq t(d(m))$.

To prove claim 2, note that by the definition of $J$ and $I$ that $k_j(q) = 0$ ( and hence $k_j^*(q) = 0$) for every $j \in I$. Thus $f(q) = t(q)$. Hence if our claim failed then $f|(D \cap E) \cup \{q\} = t|(D \cap E) \cup \{q\}$, which contradicts claim 1 since $t|(D \cap E) \cup \{q\}$ is continuous at $q$ by our choice of $E$.

So, let $m$ be as in claim 2. By (i) there exists $i \in I$ such that $k(i)^*(d(m)) \neq 0$. As above there is a clopen subset $T$ of $E$ containing $d(m)$ and such that $k(i)^*$ is continuous on $T$. By (1) this means that

$$f|T = (t + h_i k_i^*)|T + \sum\{h_j k_j^* : j \in I - \{i\}\} \tag{2}$$

But $|I - \{i\}| \leq n-2$ and $(t + h_i k_i^*)|T \in C(T)$ so $\mathrm{rg}(f|T) \leq n-1$, contradicting assumption 1. Hence $\mathrm{rg}(f) \geq n+1$ as claimed. □

**Theorem 9.4.** *Let $X$ be a countable scattered space for which* $\mathrm{CB}(X) = n$. *Using the notation of* [10, 3.2], *define $f_n : \mathbf{X} \to \mathbf{R}$ by: $f_n[I_i(X)] = \{i+1\}, i = 0, \ldots, n-1)$. Then:*

(i) *$f_n \in G(X)$.*

(ii) *If $x \in D_{n-1}(X)$ and $A$ is a clopen set containing $x$ then* $\mathrm{rg}(f_n|A) \geq n$. *In particular* $\mathrm{rg}(X) \geq n$.

*Proof.* Induct on $n$. If $n = 1$ the result is trivial. Suppose that $f_k$ satisfies (i) and (ii) for each countable scattered space $X$ for which $\mathrm{CB}(X) \leq k$. Let $Y$ be a countable scattered space for which $\mathrm{CB}(Y) = k+1$. Then $D_k(Y) \neq \varnothing$ and $D_{k+1}(Y) = \varnothing$ (see [10, 3.2]). Evidently $\mathrm{CB}(Y - D_k(Y)) = k$ and $f_{k+1}|Y - D_k(Y) = f_k$. Hence by the induction hypotheses $f_{k+1}|Y - D_k(Y) \in G(Y - D_k(Y))$. Also $f_{k+1}|D_k(Y) \in G(D_k(Y))$ as $D_k(Y)$ is discrete. Thus by 9.2 there exist functions $F, G \in G(Y)$ such that $F|Y - D_k(Y) = f_{k+1}|X - D_k(Y)$ and $G|D_k(Y) = f_{k+1}|D_k(Y)$. As $D_k(Y)$ is closed in the countable space $Y$ there is a cozero-set coz $g$ of $Y$ for which $Y - \mathrm{coz}\ g = D_k(Y)$. Then $f_{k+1} = Fgg^* + G(\mathbf{1} - gg^*)$ and so $f_{k+1} \in G(Y)$. This, together with our induction hypotheses, shows that $f_{k+1}$ satisfies the hypotheses of 9.3. Hence by 9.3 $\mathrm{rg}(f_{k+1}) \geq k+1$. It remains to show that if $p \in D_k(Y)$ and if $A$ is any clopen subset of $X$ that contains $p$, then $\mathrm{rg}(f_{k+1}|A) \geq k+1$. But for such an

$A, \mathrm{CB}(A) = k+1$ and $f_{k+1}|A$ has exactly the same properties (re A) as $f_{k+1}$ has. So the proof that $\mathrm{rg}(f_{k+1}) \geq k+1$ also shows that $\mathrm{rg}(f_{k+1}|A) \geq k+1$. □

**Lemma 9.5.** *Let $S$ be a space for which* $\mathrm{vwP}(S) = \varnothing$*. Then for each $n, S$ has a countable scattered subspace $S(n)$ with* $\mathrm{CB}(S(n)) = n$*.*

*Proof.* The argument is by induction. Suppose that we have such a space $S(k)$ for the integer $k$. Let $I(S(k)) = \{d_n : n \in N\}$. Since the set $\{d_n\}$ is discrete, there exist pairwise disjoint cozero-sets $\{V_n\}$ in $X$ with $d_n = V_n \cap S(k)$. By hypothesis $d_n$ is the limit of a countable discrete set $T(n)$ and if we let $H(n) = V_n \cap T(n)$ then $d_n$ is a limit point of the countable discrete set $H(n)$. Let $S(k+1) = S(k) \cup (\bigcup H(n), n \in N)$. One verifies that $I(S(k+1)) = S(k+1) - S(k)$ and $S(k+1)$ is countable scattered of CB-index $k+1$. □

**Corollary 9.6.** *Suppose that $X$ is a space whose countable subspaces are scattered, and suppose that* $\mathrm{vwP}(X)$ *is empty. Then for each $n \in \mathbf{N}$ there is a countable scattered subspace $S(n)$ such that* $\mathrm{CB}(S(n)) = n$*.*

**Theorem 9.7.** *Let $X$ be a space with a subspace $S$ such that* $\mathrm{vwP}(S)$ *is empty. Then :*

(i) $\mathrm{rg}(X) = \infty$,
(ii) *$X$ is not an* RG*-space.*

*Proof.* (i) As $S$ has no isolated points it has a countable discrete subset $\{d(n) : n \in N\}$. As $X$ is Tychonoff there exists a pairwise disjoint family $\{V(n) : n \in N\}$ of cozero-sets of $X$ with $d(n) \in V(n)$. Let $n \in N$. Now $\mathrm{vwP}(V(n) \cap S) = V(n) \cap \mathrm{vwP}(S) = \varnothing$ so by 9.5 $V(n) \cap S$ has a countable scattered subspace $S(n)$ for which $\mathrm{CB}(S(n)) = n$. By 9.4 there exists $f(n) \in G(S(n))$ such that $\mathrm{rg}(f(n)) \geq n$. By 9.2 there exists $k(n) \in G(X)$ such that $k(n)|S(n) = f(n)$. Thus $\mathrm{rg}(k(n)) \geq \mathrm{rg}(f(n))$. It follows that $\mathrm{rg}(X) = \infty$.

(ii) Define $H : X \to \mathbb{R}$ by: $H|V(n) = k(n)|V(n)$ and $H[X - \bigcup\{V(n) : n \in N\}] = 0$, each set in question being clopen in $X_\delta$. Since $k_n|V(n) \in G(V(n))$ the restriction of $H$ to each set in the partition is continuous so $H \in C(X_\delta)$. But $\mathrm{rg}(H|V(n)) = \mathrm{rg}(k_n|V(n)) \geq n$ for each $n$ so $H \notin G(X)$ and $X$ is not an RG-space. □

**Corollary 9.8.** *If the set of very weak* P*-points of $X$ is not dense in $X$ then* $\mathrm{rg}(X) = \infty$ *and $X$ is not an* RG*-space.*

*Proof.* If $\mathrm{vwP}(X)$ is not dense there exists a cozero-set $V$ of $X$ disjoint from it. It follows that $\mathrm{vwP}(V) = \varnothing$. By 9.7 $\mathrm{rg}(V) = \infty$ and $V$ is not an RG space. Since $V$ is $G$-embedded in $X$, $\mathrm{rg}(X) = \infty$, and $X$ is not an RG-space since $V$ is not one by [10, 2.3(f)]. □

**Corollary 9.9.** *If $X$ has a first countable subspace without isolated points then* $\mathrm{rg}(X) = \infty$ *and $X$ is not an* RG*-space.*

*Proof.* If $S$ is first countable without isolated points then $\mathrm{vwP}(S) = \varnothing$. Now apply the theorem. □

## 10. Regularity degree for compact and related spaces

We now apply the above methods to some other spaces.

We now use 9.4 to characterize compact (and related) spaces of infinite regularity degree. (Our original proof was different and used the fact that a compact space without P-points contains a compact zero set with an infinite boundary).

**Theorem 10.1.** *Let $K$ be a compact space that is not scattered. Then for each $n$, there exists a countable scattered subspace $S(n) \subset K$ such that $\mathrm{CB}(S(n)) = n$. Consequently $\mathrm{rg}(K) = \infty$ and the Krull $z$-dimension of $C(K)$ is infinite.*

*Proof.* Since $K$ is not scattered it has a compact subspace $A$ without isolated points. As in [15, 3.1], there is a compact subspace $L \subset A$ without isolated points that maps continuously onto the Cantor set $\mathbb{C}$. If $\mathrm{rg}(L) = \infty$ then $\mathrm{rg}(K) = \infty$ because $L$ is $C^*$-embedded in $K$. Thus it suffices to assume that there exists a continuous surjection $f : K \to \mathbb{C}$.

Although the following is likely folklore, we will now in detail construct a family $\mathbb{F}$ of countably many non-empty clopen subsets of $\mathbb{C}$ which, when partially ordered by inclusion, forms a tree of height $n$ with the property

(*) for all $F \in \mathbb{F}, |\{G \in \mathbb{F} : G \subset F\}| = \aleph_0$.

Let $\underline{s} = (s_1, \ldots, s_k) \in N^k$ be an ordered $k$-tuple of positive integers. Let $|\underline{s}|$ denote the number of components in $\underline{s}$. If $\underline{s} = (s_1, \ldots, s_k) \in N^k$ and if $j \in N$ we will denote the element $(s_1, \ldots, s_k, j) \in N^{k+1}$ by $\underline{s} + j$. If $\underline{s}$ is an initial sequence of $\underline{t}$ (i.e. $\underline{t} = (s_1, \ldots, s_n, a, b, \ldots)$) we will write $\underline{s} \leq \underline{t}$. (By convention $\underline{s} \leq \underline{t}$ if $\underline{s} = \underline{t}$). Now we construct $\mathbb{F}$.

Let $\{C_{\underline{s}} : |\underline{s}| \in N^1\}$ be a pairwise disjoint family of clopen non-empty subsets of $C$. If $\underline{s} \in N^1$ let $\{C_{\underline{s}+j} : j \in N\}$, be a pairwise disjoint family of clopen non-empty subsets of $C_{\underline{s}}$.

Now let $k < n$ and suppose that we have defined $\{C_{\underline{s}} : \underline{s} \in N^m, 1 \leq m \leq k\}$ with the following properties:

(1) for all $m \in \{1, \ldots, k-1\}$, for all $\underline{s} \in N^m, \{C_{\underline{s}+j} : j \in N\}$ is a pairwise disjoint family of clopen subsets of $\mathbb{C}$.
(2) If $|\underline{s}| < m \leq k$ and $j \in N$, then $C_{\underline{s}+j} \subseteq C_{\underline{s}}$.

Now define $\{C_{\underline{s}+j} : |\underline{s}| = k, j \in N\}$ as follows. The set $\{C_{\underline{s}+j} : j \in \omega\}$ is a pairwise disjoint family of non-empty clopen subsets of $C_{\underline{s}}$. Then (1) and (2) are satisfied when $k$ is replaced by $k+1$ and $\mathbb{F} = \{C_{\underline{s}} : 1 \leq \underline{s} \leq n\}$ is the required family.

We have a continuous surjection $f : K \to \mathbb{C}$. Then $\{f^{-1}[F] : F \in \mathbb{F}\} = \{C'_{\underline{s}} : \underline{s} \in \bigcup_{m=1}^{m=n} N^m\}$ (where $C'_{\underline{s}} = f^{-1}[C_{\underline{s}}]$) satisfies (1) and (2) when $C_{\underline{s}}$ is replaced by $C'_{\underline{s}}$ and $\mathbb{C}$ is replaced by $K$.

Now we define $S(n)$ inductively from the "bottom up". (1) If $|\underline{s}| = n$ choose $d(\underline{s}) \in C_{\underline{s}}$. If $|\underline{s}| = n-1$ , let $d(\underline{s})$ be a limit point of $\{d(\underline{s}+j), j \in N\}$. (There will be such a limit point as $K$ is compact and $\{d(\underline{s}) : |\underline{s}| = n\}$ is discrete by (1)).

If we have chosen $\{d(\underline{s}) : \underline{s} \in \bigcup_{i=m+1}^{n} N^i\}$ and if $\underline{s} \in N^m$, choose $d(\underline{s})$ to be a limit point of $\{d(\underline{s}+j) : j \in N\}$. Let $S(n) = \{d(\underline{s}) : \underline{s} \in \bigcup_{i=1}^{n} N^i\}$. Then $I(S(n)) = \{d(\underline{s}) : |\underline{s}| = n\}$, $I(S(n) - I(S(n))) = \{d(\underline{s}) : |\underline{s}| = n-1\}$ and so on. We see that $\mathrm{CB}(S(n)) = n$ and that $S(n)$ is countable and scattered.

The fact that $\mathrm{rg}(K)$ is infinite now follows from 9.2 and 9.4. The fact that $K$ is of infinite Krull $z$-dimension also follows from Theorem 3.6. □

**Corollary 10.2.** *Let $K$ be a compact space. Then the following are equivalent:*
(i) *$K$ is scattered and* $\mathrm{CB}(K) < \infty$*,*
(ii) $\mathrm{rg}(K) < \infty$*,*
(iii) *$K$ is an* RG *space.*

*Proof.* The equivalence (i) ⇔ (ii) is in [10, 3.4] as is the implication (iii) ⇒ (ii). Suppose (*ii*). Then $K$ is scattered by the above theorem, and $\mathrm{CB}(K)$ is finite by Theorem 9.4 so $K$ is RG by [10, 2.12]. Thus (ii) ⇒ (iii). □

**Corollary 10.3.** *Any space containing a compact space that is not* RG *is of infinite regularity degree and of infinite Krull $z$-dimension. In particular this applies to a locally compact space that is not scattered.*

*Proof.* The first assertion is immediate. For the second, suppose that $X$ is locally compact and not scattered. Then its Cantor-Bendixon kernel $A$ is non-empty, locally compact, and has no isolated points. Local compactness shows that $A$ has a compact subset $K$ that has no isolated points. Thus $K$ is not RG, and the conclusion follows. □

The following is immediate from our theorem and Theorem 4.1.

**Corollary 10.4.** *Any pseudocompact space without isolated points is of infinite regularity degree and of infinite Krull $z$-dimension.*

**Corollary 10.5.** *Let $K$ be a metric space. Then the following are equivalent:*
(i) *$K$ is scattered and* $\mathrm{CB}(K) < \infty$*,*
(ii) $\mathrm{rg}(K) < \infty$*,*
(iii) *$K$ is an* RG *space.*

*Proof.* Theorem [10, 3.4] shows that (i) ⇔ (iii) and that (iii) ⇒ (ii). To see that (ii) ⇒ (i) observe that, with (*ii*), $K$ is scattered by part (i) of theorem 9.7, and $\mathrm{CB}(K)$ is finite by Theorem [10, 3.3]. □

**Remark 10.6.** The example before Theorem 9.2 and [10, 2.4] show that we cannot generalize Theorem 10.1 to Lindelöf spaces. The space $\Psi$ is always scattered locally compact, of finite CB-index, and never RG so there is no possibility of a generalizing 10.2 from compact to locally compact spaces.

## 11. Does RG imply finite regularity degree?

By [10] and our work above we already have a positive answer to the above question in the pseudocompact, metric, scattered Lindelöf, and scattered perfectly normal cases. We now give two other instances where the answer is positive. Since we don't know whether perfectly normal RG-spaces are scattered, the first result is pertinent.

**Theorem 11.1.** *Let $X$ be perfectly normal and* RG. *Then* $\mathrm{rg}(X)$ *is finite.*

*Proof.* Let $f \in G(X)$. By [10, 2.1 (a)] there is a dense open subset of $X$ on which $f$ is continuous. Let $V_0$ be the union of the open subsets of $X = X_0$ on which $f$ is continuous. Then $V_0$ is a dense cozero set of $X_0$ whose complement is the nowhere dense zero-set $Z_1 = X_0 - V_0$. Since $Z_1$ is perfectly normal and also RG (by [10, 2.3(f)]) the process can be repeated— $f|Z_1$ is continuous on a dense cozero set $V_1$ of $Z_1$ with complement $Z_2$ nowhere dense in $Z_1$. The process can be continued to get subsets $Z_n$ with dense cozero sets $V_n$. Since the $Z_n$ are $C$-embedded in $X$, by theorem 3.6 the process must stop with an $N$ for which $Z_{N+1} = \varnothing$. Furthermore by theorem 3.4 the number $N$ is global—it works for all $f \in G(X)$.

Now we need a representation for $f$. We know that $X$ is the union of the $V_n$ and that $f$ is continuous on each of them. Suppose that $V_n = \mathrm{coz}(g_n)$. We can assume that each $g_n \in C(X)$ because each $Z_n$ is $C$-embedded in $X$. One readily verifies that $f = fg_0g_0^* + \sum_{j=1}^{j=N}[\Pi_{k=0}^{k=j-1}(1 - g_kg_k^*)]g_jg_j^*f$. Since it uses a number of terms that depends only on $N$, $\mathrm{rg}(X)$ is finite. □

**Proposition 11.2.** *Suppose that $X$ is an* RG*-space. Then $X$ must be of finite regularity degree, if it contains countably many pairwise disjoint subspaces $Y_n$ each homeomorphic to $X$ with the property that each $Y_n$ is clopen in $X_\delta$.*

*Proof.* Assume if possible that $rg(X) = \infty$. As each $Y_n$ is homeomorphic to $X$, there exists for each $n$, a function $f_n \in G(Y_n)$ such that $\mathrm{rg}(f_n) \geq n$. Now define $F : X_\delta \to \mathbb{R}$ by $F|Y_n = f_n$, $F|[X - \cup Y_n] = 0$. Since each $Y_n$ is clopen in $X_\delta$ (as are countable unions of clopen sets), $F \in C(X_\delta)$ by [8, 1A]. Since $X$ is RG, we have a representation $F = \sum_{i=1}^{i=k} a_ib_i^*, a_i, b_i \in C(X)$. But if one restricts this representation of $F$ to $Y_n$ one sees that the constant $k$ globally bounds the regularity degrees of the $\{f_n\}$, and this is not possible. □

**Corollary 11.3.** *If $X$ is* RG *and* $\mathrm{rg}(X) = \infty$*, then the free union of $\omega$ copies of $X$ is not* RG.

**Corollary 11.4.** *Suppose that $X$ is an* RG*-space of countable pseudocharacter. Then $X$ must be of finite regularity degree, if it contains countably many disjoint subspaces $Y_n$ each homeomorphic to $X$.*

**Remark 11.5.** It is natural to question the restrictiveness of the hypothesis of proposition 11.2. If such subspaces $Y_n$ exist then there is a constraint on the P-space $X_\delta$—it must have countably many pairwise disjoint clopen subsets each a copy of itself. This demand holds in some but certainly not all P-spaces. For example, under Martin's Axiom $\beta N - N$ will have a dense set of P-points, and if one takes a maximal family of pairwise disjoint clopen subsets $\{A_\alpha\}$ in $\beta N - N$, then the set of P-points in their union has this property since it is the free union of $c$ pairwise disjoint clopen copies of itself. There are many similar P-spaces (cf. [6]) where the constraint holds. It clearly fails if $X_\delta$ is the one-point Lindelöfization of an uncountable discrete set.

## 12. Miscellaneous

**Theorem 12.1.**

(i) *TFAE for a subspace $Y$ of a space $X$:*
  (a) *If $f \in G(X)$ then $f|Y \in C(Y)$*
  (b) *If $V \in \operatorname{coz} X$ then $V \cap Y$ is clopen in $Y$.*

(ii) *If $Y$ satisfies the (equivalent) conditions of (1) and $Y$ is $z$-embedded in $X$ then $Y$ is a P-space.*

*Proof.* To show that (a) implies (b) let $V \in \operatorname{coz} X$. Then the characteristic function $\chi_V$ of V belongs to $G(X)$. Now $\chi_V|Y = \chi_{V\cap Y}$ so by (a) $\chi_{V\cap Y} \in C(Y)$. This immediately implies (b).

To show that (b) implies (a), let $f \in G(X)$.Then $f = g_1h_1^* + \cdots + g_nh_n^*$, $g_i, h_i \in C(X)$. To show that $f|Y \in C(Y)$ it clearly suffices to show that $h_i^*|Y \in C(Y)$ for each $i$. But $h_i^*|\operatorname{coz}(h_i) \in C(\operatorname{coz}(h_i))$ so $h_i^*|Y\cap\operatorname{coz}(h_i) \in C(Y)\cap\operatorname{coz}(h_i)$. Similarly $h_i^*|Z(h_i) \in C(Z(h_i))$ (as $h_i^*[Z(h_i)] = 0$) so $h_i^*|Y \cap Z(h_i) \in C(Y \cap Z(h_i))$. Thus (by (b)) $h_i^*|Y$ is continuous on each of complementary clopen subsets of $Y$. Hence $h_i^*|Y \in C(Y)$ and we are done.

If $Y$ is $z$-embedded in $X$ then by (b) every cozero-set of $Y$ is clopen and so $Y$ is a P-space. □

## 13. Open questions

(1) If $X$ is RG, is $\operatorname{rg}(X)$ finite?
(2) (cf. 1.1 and the proof of proposition 11.2) If $X$ has infinite regularity degree, can this be witnessed by a family of functions in $G(X)$ that are pairwise orthogonal?
(3) Must an RG-space have a dense set of P-points?

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