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On i-topological spaces: generalization of the concept of a topological space via ideals

Irina Zvina

Abstract. The aim of this paper is to generalize the structure of a topological space, preserving its certain topological properties. The main idea is to consider the union and intersection of sets modulo "small" sets which are defined via ideals. Developing the concept of an i-topological space and studying structures with compatible ideals, we are concerned to clarify the necessary and sufficient conditions for a new space to be homeomorphic, in some certain sense, to a topological space.

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1. Introduction

The use of the ideals in general topology historically developed along two main lines. The first line is concerned with the study of the local properties of topological spaces that may be extended to the global properties [7, 12]. The central concept in these investigations is the compatibility of an ideal with a topology. For example, the well-known Banach Category Theorem is an immediate corollary from the σ-Extension Theorem [7]. The works of the second line use ideals to generalize the certain properties of topological spaces, such as a compactness [9, 10, 11] and the separation axioms [1].

The aim of this paper is to generalize, via ideals, the concept of a topological space itself, thus it differs from the works of two mentioned lines. However, we make use of the ideas, developed in those works.

In the first section, we define an i-topological space, provide basic examples, introduce the notions of a subspace, an i-continuous mapping, a scattered set and a nowhere dense set for such spaces and study properties of these notions. We propose a sufficient condition for the existence of a supratopological space [8] that is i-homeomorphic to a given i-topological space. In the second section, we introduce the concept of a unified i-topological space and discuss two important examples.

In the last section, we study some properties of i-topological spaces with compatible ideals. We propose a construction which shows that for every such space there exists an i-homeomorphic supratopological space such that for every two its i-open subsets U and V there is an element A in the ideal for which $(U \cap V) \setminus A$ is i-open. On the other hand, we show that the compatibility is necessary for the existence of a topological space that is i-homeomorphic to a given i-topological space. We introduce a sufficient condition for the existence of such a space and leave as an open question whether there exists an i-topological space with a compatible ideal which is not i-homeomorphic to any topological space.

We refer a reader to the papers [3]-[6] by Dragan Janković and T.R. Hamlett for the survey on the use of ideals in topological spaces. These works also provide the historical background and contain many references to the related papers.

2. General notions of i-topological spaces

2.1. i-Topological spaces. Subspaces. A nonempty family $\mathcal{I}$ of subsets of a set X is called an *ideal on X* iff $X \notin \mathcal{I}$, $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, and $A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$. We define the relations $\leq$ and $\approx$ on $P(X)$ as follows:

(1) $A \leq B$ iff $A \setminus B \in \mathcal{I}$,
(2) $A \approx B$ iff $(A \setminus B) \cup (B \setminus A) \in \mathcal{I}$,

where $A, B \subseteq X$. One can easily check that $\leq$ and $\approx$ are a preorder and an equivalence on $P(X)$, respectively.

Definition 2.1. *Let X be a set and $\mathcal{I}$ be an ideal on X. Then an* i-topology on X *is a family $\mathcal{T}$ of subsets of X that satisfies the following conditions:*

(T1) $\varnothing, X \in \mathcal{T}$;
(T2) *for any $\mathcal{U} \subseteq \mathcal{T}$ there exists $U \in \mathcal{T}$ such that $\bigcup \mathcal{U} \approx U$;*
(T3) *for any $V, W \in \mathcal{T}$ there exists $U \in \mathcal{T}$ such that $V \cap W \approx U$;*
(T4) $\mathcal{T} \cap \mathcal{I} = \{\varnothing\}$.

The triple $(X, \mathcal{T}, \mathcal{I})$ is called an i-topological space *and the elements of $\mathcal{T}$ are called* i-open sets. *We will use the notation $\mathcal{T}(x) = \{\, U \in \mathcal{T} \mid x \in U \,\}$ for any $x \in X$.*

Let us give some basic examples.

Example 2.2. Suppose we are given a topological space $(X, \mathcal{T})$. Then $(X, \mathcal{T}, \{\varnothing\})$ and $(X, \mathcal{T}_r, \mathcal{I}_n)$ are i-topological spaces, where $\mathcal{T}_r$ is the family of all regular open sets and $\mathcal{I}_n$ is the family of all nowhere dense sets.

For the next example we need the following lemma.

Lemma 2.3. *Let $(X, \mathcal{T})$ be a topological space and $C \subseteq X$. Then there exist $A, B \subseteq X$ and an open set U such that $C = A \cup B$, the set A is nowhere dense and $cl(B) = cl(U)$.*

Proof. It is sufficient to take $U = int(cl(C))$, $B = C \cap U$ and $A = C \setminus B$. Then A is nowhere dense, since $A = C \setminus B = C \setminus (C \cap int(cl(C))) = C \setminus int(cl(C))$.

To complete the proof, it remains to show that $U_x \cap B \neq \varnothing$ for each $x \in U$ and any its open neighborhood U_x. Observe that $U_x \cap U$ is also an open neighbourhood of x. Since x belongs to the closure of C, it follows that $U_x \cap U \cap C \neq \varnothing$ and hence $U_x \cap B \neq \varnothing$. □

Example 2.4. Consider the real line $\mathbb{R}$ with the usual topology $\mathcal{T}_u$ and the family $\mathcal{I}_n$ of all nowhere dense sets. Then the triple $(\mathbb{R}, \mathcal{D}, \mathcal{I}_n)$ is an i-topological space, where $\mathcal{D} = \{ A \subseteq \mathbb{R} \mid \text{there exists } U \in \mathcal{T}_u \text{ such that } cl(A) = cl(U) \}$. Obviously, this can be generalized to any topological space. However, in our paper we will only use to the example based on the real line with the usual topology.

As we know the intersection of a family of ideals on the same set is an ideal. We provide an example where a family $\mathcal{T}$ is an i-topology with respect to two different ideals, but is not an i-topology with respect to their intersection.

Example 2.5. Let X be the union of $X_1 = \mathbb{R} \times \{1\}$ and $X_2 = \mathbb{R} \times \{2\}$. We construct a family $\mathcal{T}$ of subsets of X as follows:

$$\mathcal{T} = \{\varnothing, X\} \cup \mathcal{T}_1 \cup \mathcal{T}_2,$$

where

$$\mathcal{T}_1 = \{ \{x\} \cup B \mid x \in X_1 \text{ and } B \subseteq X_2 \},$$

$$\mathcal{T}_2 = \{ A \cup \{y\} \mid A \subseteq X_1 \text{ and } y \in X_2 \}.$$

Then $\mathcal{T}$ is an i-topology on X with respect to the ideals $P(X_1)$ and $P(X_2)$. However, $\mathcal{T}$ is not an i-topology with respect to $\{\varnothing\}$.

Now, our aim is to define the notion of a subspace. Suppose we are given an i-topological space $(X, \mathcal{T}, \mathcal{I})$ and a subset $M \subseteq X$. We will use the following notations:

$$\mathcal{T}|_M = \{ M \cap U \mid U \in \mathcal{T} \text{ and } M \cap U \notin \mathcal{I} \} \cup \{\varnothing, M\},$$

if $M \notin \mathcal{I}$ then

$$\mathcal{I}|_M = \{ M \cap A \mid A \in \mathcal{I} \}$$

and if $M \in \mathcal{I}$ then

$$\mathcal{I}|_M = \{\varnothing\}.$$

The triple $(M, \mathcal{T}|_M, \mathcal{I}|_M)$ is called a *subspace* of the i-topological space $(X, \mathcal{T}, \mathcal{I})$.

Proposition 2.6. *Every subspace of any i-topological space is an i-topological space.*

Proof. Fix an i-topological space $(X, \mathcal{T}, \mathcal{I})$ and its subspace $(M, \mathcal{T}|_M, \mathcal{I}|_M)$.

Let us prove that $\mathcal{I}|_M$ is an ideal. If $A, B \in \mathcal{I}$ then $(M \cap A) \cup (M \cap B) = M \cap (A \cup B) \in \mathcal{I}|_M$. Hence, we have finite additivity. Now, if $A \in \mathcal{I}$ and $B \subseteq M \cap A$ then $B \subseteq A$ and by heredity $B \in \mathcal{I}$. Hence, $B = M \cap B \in \mathcal{I}|_M$ and we have heredity.

Let us prove that $(M, \mathcal{T}|_M, \mathcal{I}|_M)$ is an i-topological space. It follows from the definition, that $\varnothing, M \in \mathcal{T}|_M$. Hence, we have (T1).

Suppose that $C, D \subseteq X$ and $C \approx D$. Then $M \cap C \approx_{\mathcal{I}|_M} M \cap D$. Indeed, there exist $A, B \in \mathcal{I}$ such that $D = (C \cup A) \setminus B$, since $C \approx D$. Hence, $M \cap D = M \cap ((C \cup A) \setminus B) = ((M \cap C) \cup (M \cap A)) \setminus (M \cap B) \approx_{\mathcal{I}|_M} M \cap C$.

It follows from the definition of subspace, that for any family $\mathcal{V} = \{V_s\}_{s \in S}$ of the elements of $\mathcal{T}|_M$ there exists a family $\mathcal{U} = \{U_s\}_{s \in S}$ of the elements of $\mathcal{T}$ such that $V_s = M \cap U_s$ for each $V_s \in \mathcal{V}$. Take U from $\mathcal{T}$ such that $\bigcup \mathcal{U} \approx U$. Then $\bigcup \mathcal{V} = M \cap (\bigcup \mathcal{U})$ and by the previous paragraph $M \cap (\bigcup \mathcal{U}) \approx_{\mathcal{I}|_M} M \cap U$. Therefore, $\bigcup \mathcal{V} \approx_{\mathcal{I}|_M} M \cap U$.

If $\mathcal{V}$ and $\mathcal{U}$ are finite, we take $W \in \mathcal{T}$ such that $W \approx \bigcap \mathcal{U}$ and in the same way show that $\bigcap \mathcal{U} \approx_{\mathcal{I}|_M} M \cap W$. Thus, we have (T2) and (T3).

To complete the proof, it remains to show that $\mathcal{T}|_M$ and $\mathcal{I}|_M$ satisfy (T4). Suppose that $V \in \mathcal{T}|_M \cap \mathcal{I}|_M$. Then it follows from the definition of subspace, that $V = \varnothing$, since $\mathcal{I}|_M \subseteq \mathcal{I}$. □

2.2. i-Continuous mappings. Suppose we are given two i-topological spaces $(X, \mathcal{T}_X, \mathcal{I}_X)$ and $(Y, \mathcal{T}_Y, \mathcal{I}_Y)$. We say that a mapping $f\colon X \to Y$ is *i-continuous* if it satisfies the following conditions:

(N1) for every family $\{V_s\}_{s \in S}$ of i-open subsets of Y satisfying

$$V_s \cap f(X) \notin \mathcal{I}_Y$$

for each $s \in S$ there exists a family $\{U_s\}_{s \in S}$ of nonempty i-open subsets of X such that

$$U_s \approx f^{-1}(V_s)$$

holds for each $s \in S$ and

$$\bigcup_{s \in S} U_s \approx f^{-1}\left(\bigcup_{s \in S} V_s\right);$$

(N2) $f^{-1}\mathcal{I}_Y \subseteq \mathcal{I}_X$.

Notice that our definition generalizes the usual definition of continuous mapping. Indeed, if we consider topological spaces with the ideal $\{\varnothing\}$ and in (N1) examine families which consist of only one i-open set, we obtain the usual definition.

The mapping f is called *i-homeomorphism* if f is a bijection and both f and f^{-1} are i-continuous. Two spaces are called *i-homeomorphic* if there exists an i-homeomorphism of one space to the other. The following proposition is a natural consequence from the definition of i-continuous mapping.

Proposition 2.7. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be i-continuous mappings of i-topological spaces. Assume that for any i-open set in Y its intersection with $f(X)$ does not lie in the ideal of the space Y. Then the composition $g \circ f$ is an i-continuous mapping.*

Suppose we are given a family $\mathfrak{F} = \{(X, \mathcal{T}_s, \mathcal{I}_s)\}_{s\in S}$ of i-topological spaces. An i-topological space $(X, \mathcal{T}_X, \mathcal{I}_X)$ is called *minimal in the family* $\mathfrak{F}$ if it is an element of $\mathfrak{F}$ and the identity mapping $id\colon (X, \mathcal{T}_s, \mathcal{I}_s) \to (X, \mathcal{T}_X, \mathcal{I}_X)$ is i-continuous for each $s \in S$.

Let $f_s\colon X \to Y_s$ and $(Y_s, \mathcal{T}_s, \mathcal{I}_s)$ be a mapping and an i-topological space for each $s \in S$, respectively. Then $(X, \mathcal{T}_X, \mathcal{I}_X)$ is called an *initial i-topological space generated by* $\{f_s\}_{s\in S}$ if it is minimal in the following family of i-topological spaces $\{\, (X, \mathcal{T}, \mathcal{I}) \mid f_s\colon X \to Y_s \text{ is } i\text{-continuous for each } s \in S \,\}$.

The next proposition shows that our definition of continuity generalizes the usual definition in the sense of initial topology. We omit the proof.

Proposition 2.8. *Let $(Y, \mathcal{T}, \mathcal{I})$ be an i-topological space and $f\colon X \to Y$ be a mapping. Consider families $\mathcal{T}_X = f^{-1}\mathcal{T}|_{f(X)}$ and $\mathcal{I}_X = f^{-1}\mathcal{I}|_{f(X)}$. Then $(X, \mathcal{T}_X, \mathcal{I}_X)$ is an initial i-topological space generated by $\{f\}$.*

2.3. Scattered and nowhere dense sets**.** First, let us recall the definition of the set operator ψ [3]. The various properties of this operator are to be found in [3]. Suppose we are given a topological space $(X, \mathcal{T})$ and an ideal $\mathcal{I}$ on X. Then $\psi\colon P(X) \to P(X)$ is defined as follows. For any $A \subseteq X$,

$$\psi(A) = \{\, x \in X \mid \text{ there exists } U \in \mathcal{T}(x) \text{ such that } U \setminus A \in \mathcal{I} \,\}.$$

We use this definition for i-topological spaces. One can easily check that $\psi(A) = \bigcup\{\, U \in \mathcal{T} \mid U \leq A \,\}$ holds for any $A \subseteq X$. If there is a chance for confusion, we will use indexes to specify the ideal or i-topology in the way we do it in Proposition 2.10 and Lemma 2.12.

Now, let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space and A be a subset of X. We say that $x \in X$ is an *isolated point of* A if there exists $U \in \mathcal{T}(x)$ such that $A \cap U \setminus \{x\} \in \mathcal{I}$. We say that A is *scattered* if every its subset contains an isolated point. We say that $A \subseteq X$ is *nowhere dense* if $\psi(A \cup B) = \varnothing$ for each $B \subseteq X$ such that $\psi(B) = \varnothing$.

In what follows we will use the notations $s(\mathcal{I}, \mathcal{T}) = \{\, A \subseteq X \mid A \text{ is scattered} \,\}$ and $n(\mathcal{I}, \mathcal{T}) = \{\, A \subseteq X \mid A \text{ is nowhere dense} \,\}$. Clearly, a subset is scattered or nowhere dense with respect to a certain i-topology and ideal. However, in case there is no chance for confusion, we will simply say scattered or nowhere dense and write $s(\mathcal{I})$ and $n(\mathcal{I})$ instead of $s(\mathcal{I}, \mathcal{T})$ and $n(\mathcal{I}, \mathcal{T})$.

What is stated in the following proposition is an easy corollary from the definition of a scattered set. The result of Proposition 2.10 will be used later in the next section.

Proposition 2.9. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space. Then $s(\{\varnothing\}) \subseteq s(\mathcal{I})$ and $\mathcal{I} \subseteq s(\mathcal{I})$.*

Proposition 2.10. *Let $(X, \mathcal{T}_1, \mathcal{I}_1)$ and $(X, \mathcal{T}_2, \mathcal{I}_2)$ be i-topological spaces. Assume that, for any subset $A \subseteq X$, it holds that $\psi_1(A) = \varnothing$ iff $\psi_2(A) = \varnothing$ holds. Then $n(\mathcal{I}_1, \mathcal{T}_1) = n(\mathcal{I}_2, \mathcal{T}_2)$.*

Proof. Take $B \in n(\mathcal{I}_1, \mathcal{T}_1)$ and $A \subseteq X$ such that $\psi_2(A) = \varnothing$. Then by our assumption $\psi_1(A) = \varnothing$ and hence $\psi_1(A \cup B) = \varnothing$, since $B \in n(\mathcal{I}_1, \mathcal{T}_1)$. Thus, under the assumption $\psi_2(A \cup B) = \varnothing$ and we have $B \in n(\mathcal{I}_2, \mathcal{T}_2)$. In the similar way we can show that $B \in n(\mathcal{I}_2, \mathcal{T}_2)$ implies $B \in n(\mathcal{I}_1, \mathcal{T}_1)$ for each $B \subseteq X$. □

The converse statement of the previous Lemma does not hold. Indeed, consider the i-topological spaces $(\mathbb{R}, \mathcal{T}_u, \mathcal{I}_n)$ and $(\mathbb{R}, \mathcal{D}, \mathcal{I}_n)$ from Example 2.2 and Example 2.4. It is not difficult to check that $\mathcal{I}_n = n(\mathcal{I}_n, \mathcal{T}_u) = n(\mathcal{I}_n, \mathcal{D})$. However, $\psi_{\mathcal{T}_u}(\mathbb{Q}) = \varnothing$ and $\psi_{\mathcal{D}}(\mathbb{Q}) \neq \varnothing$.

The next proposition shows that in an i-topological space the ideal can be extended over the family of all nowhere dense sets such that together with the given i-topology it will satisfy the conditions of an i-topological space. Moreover, this process is finite.

Proposition 2.11. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space. Then*

(i) $\mathcal{I} \subseteq n(\mathcal{I})$;
(ii) $n(\mathcal{I})$ *is an ideal;*
(iii) $n(\mathcal{I}) \cap \mathcal{T} = \{\varnothing\}$;
(iv) $n(n(\mathcal{I})) = n(\mathcal{I})$.

Proof. Statement (i) is obvious. Let us prove (ii). Suppose that A, B and C are subsets of X such that A and B are nowhere dense and $\psi(C) = \varnothing$. Then $\psi(B \cup C) = \varnothing$ and $\psi(A \cup (B \cup C)) = \psi((A \cup B) \cup C) = \varnothing$. Hence, $A \cup B$ is nowhere dense. On the other hand, a subset of a nowhere dense set is nowhere dense. Hence, we have finite additivity and heredity.

Suppose that $U \in n(\mathcal{I}) \cap \mathcal{T}$. Then $\psi(U \cup \varnothing) = \psi(U) = \varnothing$, since $\psi(\varnothing) = \varnothing$. On the other hand, $\psi(V) \neq \varnothing$ for each nonempty $V \in \mathcal{T}$. Hence, $U = \varnothing$ and we have (iii).

The last statement follows from Lemma 2.12. □

Lemma 2.12. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space and $A \subseteq X$. Then $\psi_{\mathcal{I}}(A) = \varnothing$ iff $\psi_{n(\mathcal{I})}(A) = \varnothing$ for each $A \subseteq X$.*

Proof. Suppose that A is a subset of X, $\psi_{\mathcal{I}}(A) = \varnothing$ and U is an i-open set such that $U \subseteq \psi_{n(\mathcal{I})}(A)$. Then there exists $B \in n(\mathcal{I})$ such that $U \subseteq A \cup B$. Therefore, $U \subseteq \psi_{\mathcal{I}}(A \cup B)$. On the other hand, $\psi_{\mathcal{I}}(A \cup B) = \varnothing$, since $\psi_{\mathcal{I}}(A) = \varnothing$ and B is nowhere dense. Thus, $U = \varnothing$ and $\psi_{n(\mathcal{I})}(A) = \varnothing$.

Now, notice that $\psi_{\mathcal{I}}(A) \subseteq \psi_{n(\mathcal{I})}(A)$ follows from $\mathcal{I} \subseteq n(\mathcal{I})$ for any subset $A \subseteq X$. Hence, $\psi_{n(\mathcal{I})}(A) = \varnothing$ implies $\psi_{\mathcal{I}}(A) = \varnothing$ for any $A \subseteq X$. □

2.4. Existence of a supratopological space that is i-homeomorphic to a given i-topological space. Let us recall that $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a family of subsets of X, is called *supratopological space* if $X \in \mathcal{T}$ and $\bigcup \mathcal{U} \in \mathcal{T}$ for each $\mathcal{U} \subseteq \mathcal{T}$ [8].

Suppose we are given an i-topological space $(X,\mathcal{T},\mathcal{I})$. Then an operation $\alpha\colon P(\mathcal{T})\to\mathcal{T}$ such that $\alpha(\mathcal{U})\approx\bigcup\mathcal{U}$ holds for each $\mathcal{U}\subseteq\mathcal{T}$ is called *associative* if it satisfies

$$\alpha(\{\,\alpha\,(\mathcal{U}_s\,)\mid s\in S\,\})\approx\alpha\left(\bigcup\nolimits_{s\in S}\mathcal{U}_s\right)$$

for each collection $\{\mathcal{U}_s\}_{s\in S}$ of families of i-open sets. To simplify our notations, we write $\alpha_{s\in S}(\alpha(\mathcal{U}_s))$ instead of $\alpha(\{\,\alpha\,(\mathcal{U}_s\,)\mid s\in S\,\})$.

In what follows, we will use the notation $\mathcal{T}^{\cup}=\{\,\bigcup\mathcal{U}\mid\mathcal{U}\subseteq\mathcal{T}\,\}$.

Proposition 2.13. *Let $(X,\mathcal{T},\mathcal{I})$ be an i-topological space. Then $(X,\mathcal{T}^{\cup},\mathcal{I})$ is an i-topological space. If there exists an associative operation $\alpha\colon P(\mathcal{T})\to\mathcal{T}$ such that $\alpha(\mathcal{U})\approx\bigcup\mathcal{U}$ for each $\mathcal{U}\subseteq\mathcal{T}$ then the spaces $(X,\mathcal{T},\mathcal{I})$ and $(X,\mathcal{T}^{\cup},\mathcal{I})$ are i-homeomorphic.*

Proof. Obviously, (T1) and (T4) are satisfied for the triple $(X,\mathcal{T}^{\cup},\mathcal{I})$ and (T2) is satisfied, since the family $\mathcal{T}^{\cup}$ is closed under the arbitrary unions. Let us prove (T3). Suppose that $U,V\in\mathcal{T}^{\cup}$. Then there exist $\mathcal{U},\mathcal{V}\subseteq\mathcal{T}$ and $U_1,V_1\in\mathcal{T}$ such that $U=\bigcup\mathcal{U}\approx U_1$ and $V=\bigcup\mathcal{V}\approx V_1$. Take $W\in\mathcal{T}$ satisfying $W\approx U_1\cap V_1$. Then $W\approx U\cap V$ and (T3) is proved.

To prove that the spaces $(X,\mathcal{T},\mathcal{I})$ and $(X,\mathcal{T}^{\cup},\mathcal{I})$ are i-homeomorphic under the assumption that there exists an associative operation $\alpha\colon P(\mathcal{T})\to\mathcal{T}$, such that $\alpha(\mathcal{U})\approx\bigcup\mathcal{U}$ for each $\mathcal{U}\subseteq\mathcal{T}$, we have to show that the identity mapping $id\colon X\to X$ satisfies (N1). Fix some $\mathcal{U}\subseteq\mathcal{T}^{\cup}$. By the definition, there exists a family $\mathcal{V}_U\subseteq\mathcal{T}$, for each $U\in\mathcal{U}$, such that $U=\bigcup\mathcal{V}_U$. Consider the union $\mathcal{V}=\bigcup_{U\in\mathcal{U}}\mathcal{V}_U$ of all these families. Clearly, $\mathcal{V}\subseteq\mathcal{T}$. Then $U\approx\alpha(\mathcal{V}_U)$, for each $U\in\mathcal{U}$, and $\bigcup\mathcal{U}\approx\bigcup\mathcal{V}\approx\alpha(\mathcal{V})\approx\alpha_{U\in\mathcal{U}}(\alpha(\mathcal{V}_U))\approx\bigcup_{U\in\mathcal{U}}\alpha(\mathcal{V}_U)$. The proof is complete. □

3. Unified i-topological spaces

In this section, we will make use of the notion of a compatible ideal. Let us recall the definition [6]. In a topological space $(X,\mathcal{T})$ an ideal $\mathcal{I}$ of subsets of X is said to be *compatible with* $\mathcal{T}$, denote $\mathcal{I}\sim\mathcal{T}$, if it satisfies the following condition for every subset A of X and every subfamily $\mathcal{U}$ of $\mathcal{T}$

if $A\subseteq\bigcup\mathcal{U}$ and $U\cap A\in\mathcal{I}$ holds for every $U\in\mathcal{U}$ then $A\in\mathcal{I}$.

We preserve this definition for i-topologies: in i-topological space, *the ideal is compatible with the i-topology* if the above property is satisfied.

An i-topological space $(X,\mathcal{T},\mathcal{I})$ is called *unified* if $U\approx V$ implies $U=V$ for each i-open U and V. A triple $(X,\mathcal{T}_1,\mathcal{I})$ is called a *unification of* $(X,\mathcal{T},\mathcal{I})$ if $X\in\mathcal{T}_1$ and for any $U\in\mathcal{T}$ the set $\{\,V\in\mathcal{T}_1\mid U\approx V\,\}$ contains exactly one element.

We omit the proofs of the results from the following lemma. The statement about the nowhere dense sets is an easy consequence from Proposition 2.10.

Proposition 3.1. *Let* $(X, \mathcal{T}, \mathcal{I})$ *be an i-topological space and* $(X, \mathcal{T}_1, \mathcal{I})$ *be its unification. Then the following statements hold:*

(i) $(X, \mathcal{T}_1, \mathcal{I})$ *is a unified i-topological space;*
(ii) $n(\mathcal{I}, \mathcal{T}) = n(\mathcal{I}, \mathcal{T}_1)$.

If in additional $\mathcal{I} \sim \mathcal{T}$ *then:*

(iii) $\mathcal{I} \sim \mathcal{T}_1$;
(iv) $(X, \mathcal{T}, \mathcal{I})$ *and* $(X, \mathcal{T}_1, \mathcal{I})$ *are i-homeomorphic.*

The easiest way to obtain an i-homeomorphic unified i-topological space for a given i-topological space $(X, \mathcal{T}, \mathcal{I})$ with a compatible ideal is to choose one element from each class of equivalence which are considered with the respect to the equivalence relation $\approx$. Notice that we can construct such i-topology without using the Axiom of Choice. Indeed, we take $\bigcup \{ V \in \mathcal{T} \mid U \approx V \}$ for any $U \in \mathcal{T}$. The compatibility of the ideal implies that the union of any class of equivalence is equivalent, in the sense of $\approx$, to each its member. In the last section of this paper we will propose another way how to obtain such i-topology without using the Axiom of Choice (see Proposition 4.12).

Example 3.2. Consider the real line with the usual topology. Clearly, the partial orders $\subseteq$ and $\leq$, where $\leq$ is generated by the ideal of all nowhere dense sets, are not equivalent. Indeed, $A \leq B$ does not imply $A \subseteq B$ for each $A, B \subseteq \mathbb{R}$. Moreover, we will show that there does not exist family $\mathfrak{F}$ of subsets of $\mathbb{R}$ such that $\mathfrak{F}$ contains at least one element from each equivalence class generated by $\leq$, i.e. $\mathfrak{F} \cap [A] = \{ B \subseteq \mathbb{R} \mid A \approx B \} \neq \varnothing$, and the condition $A \leq B$ implies $A \subseteq B$ for each $A, B \in \mathfrak{F}$.

In the process of proving of this statement we will show that in any unification of the i-topological space $(\mathbb{R}, \mathcal{D}, \mathcal{I}_n)$ from Example 2.4 the ideal is not compatible with the i-topology. First, let us prove the following lemmas.

Lemma 3.3. *Consider the real line with the usual topology and the family* $\mathcal{D}$ *from Example 2.4. For any nonempty* $A \subseteq \mathbb{R}$*, whose closure is equal to the closure of some nonempty open set, there exist non-empty* $B, C \in \mathcal{D}$ *such that* $cl(A) = cl(B) = cl(C)$*,* $B \cup C \subseteq A$ *and* $B \cap C = \varnothing$.

Proof. Take $U \in \mathcal{T}_u$ such that $cl(A) = cl(U)$. To simplify the proof, we suppose that $U = (0, 1)$. In all other cases, the proof will be similar. Let us define the sets $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$ as follows. Put $B_0 = \varnothing$ and $C_0 = \varnothing$. Then

$$B_n = B_{n-1} \cup \left(\bigcup_{k=1}^{2^n-1} \{b_k^n\} \right) \quad \text{and} \quad C_n = C_{n-1} \cup \left(\bigcup_{k=1}^{2^n-1} \{c_k^n\} \right)$$

where

$$b_k^n \in I_k^n \cap (A_1 \setminus (B_{n-1} \cup C_{n-1})), \quad c_k^n \in I_k^n \cap (A_1 \setminus (B_n \cup C_{n-1})),$$
$$I_k^n = \left(\frac{k-1}{2^n}, \frac{k+1}{2^n} \right), \quad A_1 = A \setminus \{0, 1\} \quad \text{and} \quad k \in \{1, \ldots, 2^{n-1}\}.$$

Clearly, A_1 is dense in U. Thus, a complement in A_1 of a union of two finite sets is dense in U. The sets B_n and C_n are finite for any $n \in \mathbb{N}$. Hence, the intersections in the middle line above are not empty for any valid n and k. Therefore, we can take points b_k^{n+1} and c_k^{n+1} for any $k \in \{1, \dots, 2^{n-1}\}$. We define the sets B and C as follows: $B = \bigcup_{n\in\mathbb{N}} B_n$ and $C = \bigcup_{n\in\mathbb{N}} C_n$.

Clearly, $B \cup C \subseteq A$ and $B \cap C = \varnothing$. To complete the proof, it remains to show that B and C are dense in U. Fix $x \in U$ and $\varepsilon > 0$. We have to show that there exist $n, k \in \mathbb{N}$ such that $I_k^n \subseteq (x - \varepsilon, x + \varepsilon)$. Take $n \in \mathbb{N}$ such that $\frac{1}{2^{n-1}} < \varepsilon$. Now, we can choose the necessary k, since intervals I_k^n cover U, their lengths are equal to $\frac{1}{2^{n-1}}$ and length of two intervals is less than 2ε. □

Lemma 3.4. *Consider the i-topological space $(\mathbb{R}, \mathcal{D}, \mathcal{I}_n)$ from Example 2.4. Then in any its unification the ideal is not compatible with an i-topology.*

Proof. Suppose that $(\mathbb{R}, \mathcal{D}_1, \mathcal{I}_n)$ is a unification of $(\mathbb{R}, \mathcal{D}, \mathcal{I}_n)$. Take $U = (0, 1)$. Define the sets $\{B^n\}_{n\in\mathbb{N}}$ and $\{C^n\}_{n\in\mathbb{N}}$ as follows. Put $C^0 = U$. Then take B^n and C^n such that they are dense in U, $B^n \cap C^n = \varnothing$ and $B^n \cup C^n \subseteq C^{n-1}$ for each $n \in \mathbb{N}$. We can choose such B^n and C^n, since the previous Lemma holds. Now, take $\hat{B}^n \in \mathcal{D}_1$ such that $\hat{B}^n \approx B^n$ for each $n \in \mathbb{N}$. Then $\hat{B}^n \cup C^n \leq C^{n-1}$. Define A as follows:

$$A = \bigcup_{n\in\mathbb{N}} A_n, \quad \text{where} \quad A_0 = \varnothing, \quad A_n = \bigcup_{k=1}^{2^n-1} \{a_k^n\},$$

$$a_k^n \in I_k^n \cap \left(\hat{B}^n \setminus \left(\bigcup_{k=1}^{n-1} \hat{B}^k \right) \right) \quad \text{and} \quad k \in \{1, \dots, 2^{n-1}\}.$$

Then A is dense in U and hence $A \notin \mathcal{I}$. Clearly, $A \subseteq \bigcup_{n\in\mathbb{N}} \hat{B}^n$ and $A \cap \hat{B}^n \in \mathcal{I}$ for each $n \in \mathbb{N}$. Thus, $\mathcal{I}_n \nsim \mathcal{D}_1$. □

Lemma 3.5. *Consider the i-topological space $(\mathbb{R}, \mathcal{D}, \mathcal{I}_n)$ from Example 2.4. Assume that $(\mathbb{R}, \mathcal{D}_1, \mathcal{I}_n)$ is its unification and $U \leq V$ implies $U \subseteq V$ for each $U, V \in \mathcal{D}_1$. Then $\mathcal{I}_n \sim \mathcal{D}_1$.*

Proof. Fix $U \in \mathcal{D}_1$ and $\mathcal{V} \subseteq \mathcal{D}_1$ such that $U \cap V \in \mathcal{I}_n$ for each $V \in \mathcal{V}$. It follows from Lemma 2.3, that there exists $W \in \mathcal{D}_1$ such that $U \cap W \in \mathcal{I}$ and $V \leq W$ for each $V \in \mathcal{V}$. Hence, $\bigcup \mathcal{V} \subseteq W$ and $(\bigcup \mathcal{V}) \cap U \in \mathcal{I}_n$. Observe that $\mathcal{I}_n = n(\mathcal{I}_n)$. Then $\mathcal{I}_n \sim \mathcal{D}_1$ is an immediate corollary from Lemma 4.4. □

Comparing the results of Lemma 3.4 and Lemma 3.5, we conclude that in any unification $(\mathbb{R}, \mathcal{D}_1, \mathcal{I}_n)$ of the i-topological space $(\mathbb{R}, \mathcal{D}, \mathcal{I}_n)$ the condition $U \leq V$ does not imply $U \subseteq V$ for each $U, V \in \mathcal{D}_1$.

To complete the proof that there does not exist family $\mathfrak{F}$ of subsets of $\mathbb{R}$ such that $\mathfrak{F}$ contains at least one element from each equivalence class generated by $\leq$ and the condition $A \leq B$ implies $A \subseteq B$ for each $A, B \in \mathfrak{F}$ it remains to observe that for any $\mathfrak{F}$ there exists a unification of $(\mathbb{R}, \mathcal{D}, \mathcal{I}_n)$ such that its i-topology is a subfamily of $\mathfrak{F}$. Thus, by the result from the previous paragraph such family $\mathfrak{F}$ does not exist.

Example 3.6. Consider the real line $\mathbb{R}$ with the usual topology $\mathcal{T}_u$. Clearly, an i-topological space $(\mathbb{R}, \mathcal{T}_u, \mathcal{I})$ is unified iff $\mathcal{I} = \{\varnothing\}$. Now, let us assume that $\mathcal{I}$ is an ideal of subsets of $\mathbb{R}$ such that $\bigcup \mathcal{I} = \mathbb{R}$ and there exists another topology $\mathcal{T}$ on the real line, which is a unified i-topology with respect to $\mathcal{I}$, and the spaces $(\mathbb{R}, \mathcal{T}_u, \mathcal{I})$ and $(\mathbb{R}, \mathcal{T}, \mathcal{I})$ are i-homeomorphic. Then $(\mathbb{R}, \mathcal{T}, \mathcal{I})$ possesses the following properties.

(i) $U \leq V$ implies $U \subseteq V$ for each $U, V \in \mathcal{T}$.

(ii) For any $x \in \mathbb{R}$, there exist $U, V \in \mathcal{T}$ such that $U \approx (-\infty, x)$, $V \approx (x, +\infty)$, $x \in U \cup V$ and $x \notin U \cap V$.

(iii) Define two mappings $f \colon \mathbb{R} \to \mathcal{T}$ and $g \colon \mathbb{R} \to \{0, 1\}$ as follows: for any $x \in \mathbb{R}$, $f(x) \approx (-\infty, x)$, and $g(x) = 0$, if $x \notin f(x)$, or $g(x) = 1$, if $x \in f(x)$. The both mappings are defined correctly.

(iv) For any $x \in \mathbb{R}$ satisfying $g(x) = 1$, there exists $\delta(x) > 0$ such that $x \in f(y)$ for each $y \in (x - \delta(x), x)$ and $x \notin f(y)$ for each $y \in (-\infty, x - \delta(x))$. We allow $\delta(x)$ to be equal to $+\infty$.

(v) At least one of the two sets $A = \{ x \in \mathbb{R} \mid g(x) = 1 \}$ and $B = \{ x \in \mathbb{R} \mid g(x) = 0 \}$ is dense.

(vi) If A is dense then for any $x \in \mathbb{R}$ satisfying $g(x) = 1$ there exist $y \in (x - \delta(x), x)$ and a decreasing sequence $(y_n)_{n \in \mathbb{N}} \subseteq (y, x)$ such that $(y_n)_{n \in \mathbb{N}}$ converges to y and $(y_n)_{n \in \mathbb{N}} \subseteq f(y)$.

However, we leave as an open question whether there exists such topological space $(\mathbb{R}, \mathcal{T}, \mathcal{I})$.

4. i-Topological spaces with compatible ideals

In this section, we concentrate our attention on i-topological spaces with compatible ideals. The corresponding definition is to be found at the beginning of the second section.

4.1. Some general properties. The propositions of this subsection provide some properties of i-topological spaces with compatible ideals that are well-known for topological spaces [2].

Suppose we are given an i-topological space $(X, \mathcal{T}, \mathcal{I})$. Then an operation $\beta \colon P(\mathcal{T}) \to \mathcal{T}$ such that $\beta(\mathcal{U}) \approx \bigcap \mathcal{U}$ for each finite $\mathcal{U} \subseteq \mathcal{T}$ is called *distributive over the union operation* if it satisfies

$$\bigcup\nolimits_{V \in \mathcal{V}} \beta(\{U, V\}) \approx U \cap \left(\bigcup \mathcal{V} \right)$$

for each i-open set U and each collection of i-open sets $\mathcal{V}$.

Proposition 4.1. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space satisfying $\mathcal{I} \sim \mathcal{T}$. Then:*

(i) $\bigcup \mathcal{U} \approx \bigcup \mathcal{V}$ *holds for any two families of i-open sets $\mathcal{U} = \{U_s\}_{s \in S}$ and $\mathcal{V} = \{V_s\}_{s \in S}$ which satisfy $U_s \approx V_s$ for each $s \in S$;*

(ii) *any operation $\alpha \colon P(\mathcal{T}) \to \mathcal{T}$ which satisfies $\alpha(\mathcal{U}) \approx \bigcup \mathcal{U}$ for each $\mathcal{U} \subseteq \mathcal{T}$ is associative;*

(iii) *any operation* $\beta\colon P(\mathcal{T}) \to \mathcal{T}$ *which satisfies* $\beta(\mathcal{U}) \approx \bigcap \mathcal{U}$ *for each finite* $\mathcal{U} \subseteq \mathcal{T}$ *is distributive over the union operation.*

Proof. The proofs of all statements are not difficult, so we demonstrate just one of them. Let us prove (iii).

Suppose we are given an i-open set U and a family of i-open sets $\mathcal{V}$. Clearly, $\beta(\{U,V\}) \le U$ and $\beta(\{U,V\}) \le \bigcup \mathcal{V}$ for each $V \in \mathcal{V}$. Since $\mathcal{I}$ is compatible with $\mathcal{T}$, it follows that $\bigcup_{V\in\mathcal{V}} \beta(\{U,V\}) \le U$ and $\bigcup_{V\in\mathcal{V}} \beta(\{U,V\}) \le \bigcup \mathcal{V}$. Therefore, $\bigcup_{V\in\mathcal{V}} \beta(\{U,V\}) \le U \cap (\bigcup \mathcal{V})$.

On the other hand, if we take $A = (U \cap (\bigcup \mathcal{V})) \setminus (\bigcup_{V\in\mathcal{V}} \beta(\{U,V\}))$ then $A \subseteq \bigcup \mathcal{V}$ and $V \cap A \in \mathcal{I}$ for each $V \in \mathcal{V}$. Since $\mathcal{I}$ is compatible with $\mathcal{T}$, it follows that A belongs to the ideal. Thus, $\bigcup_{V\in\mathcal{V}} \beta(\{U,V\}) \le U \cap (\bigcup \mathcal{V})$ and we complete the proof. □

A family $\mathcal{B} \subseteq \mathcal{T}$ is called a *base* for the i-topology $\mathcal{T}$ if for any $U \in \mathcal{T}$ there exists $\mathcal{U} \subseteq \mathcal{B}$ such that $U \approx \bigcup \mathcal{U}$. A family $\mathcal{C} \subseteq \mathcal{T}$ is called a *subbase* for the i-topology $\mathcal{T}$ if there exists $\mathcal{B} \subseteq \mathcal{T}$ such that $\mathcal{B}$ is a base for $\mathcal{T}$ and for any $V \in \mathcal{B}$ there is a finite family $\mathcal{V} \subseteq \mathcal{C}$ such that $V \approx \bigcap \mathcal{V}$.

We omit the proof of the following proposition.

Proposition 4.2. *Let* $(X, \mathcal{T}_X, \mathcal{I}_X)$ *and* $(Y, \mathcal{T}_Y, \mathcal{I}_Y)$ *be* i*-topological spaces with compatible ideals and* $f\colon X \to Y$ *be a mapping such that* $f\mathcal{I}_X = \mathcal{I}_Y$*. Then the following conditions are equivalent:*

(i) f *is* i*-continuous;*
(ii) *inverse image of any member of a subbase* $\mathcal{C}$ *for* $\mathcal{T}_Y$ *such that its intersection with* $f(X)$ *does not lie in* $\mathcal{I}_Y$ *is equivalent to some nonempty* i*-open set from* $\mathcal{T}_X$*;*
(iii) *inverse image of any member of a base* $\mathcal{B}$ *for* $\mathcal{T}_Y$ *such that its intersection with* $f(X)$ *does not lie in* $\mathcal{I}_Y$ *is equivalent to some nonempty* i*-open set from* $\mathcal{T}_X$*;*
(iv) *inverse image of any* i*-open set from* $\mathcal{T}_Y$ *such that its intersection with* $f(X)$ *does not lie in* $\mathcal{I}_Y$ *is equivalent to some nonempty* i*-open set from* $\mathcal{T}_X$*.*

4.2. Scattered and nowhere dense sets. The aim of this subsection is to show some properties of families of scattered and nowhere dense sets in i-topological spaces with compatible ideals.

Proposition 4.3. *Let* $(X, \mathcal{T}, \mathcal{I})$ *be an* i*-topological space such that* $\bigcup \mathcal{I} = X$ *and* $\mathcal{I} \sim \mathcal{T}$*. Then* $\mathcal{I} = s(\mathcal{I})$*.*

Proof. The inclusion $\mathcal{I} \subseteq s(\mathcal{I})$ is obvious. Let us prove $s(\mathcal{I}) \subseteq \mathcal{I}$. Consider $A \in s(\mathcal{I})$. Let $B \subseteq A$ be a set of all points of A which are isolated from A. Then $B \in \mathcal{I}$. By the definition of scattered set there exists a point $x \in A \setminus B$ which is isolated from $A \setminus B$. Since $A \approx A \setminus B$, it follows that any point isolated from $A \setminus B$ is isolated from A, too. Then $A \setminus B = \varnothing$ and hence $A \in \mathcal{I}$. □

Lemma 4.4. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space. Then the following statements are equivalent:*

(i) *if $A \subseteq X$ and there exists $\mathcal{U} \subseteq \mathcal{T}$ such that $A \subseteq \bigcup \mathcal{U}$ and $U \cap A \in \mathcal{I}$ for each $U \in \mathcal{U}$ then A is nowhere dense;*
(ii) *if $V \in \mathcal{T}$ and there exists $\mathcal{U} \subseteq \mathcal{T}$ such that $U \cap V \in \mathcal{I}$ for each $U \in \mathcal{U}$ then $(\bigcup \mathcal{U}) \cap V$ is nowhere dense;*
(iii) *if $A \subseteq X$ and there exists $\mathcal{U} \subseteq \mathcal{T}$ such that $A \subseteq \bigcup \mathcal{U}$ and $U \cap A$ is nowhere dense for each $U \in \mathcal{U}$ then A is nowhere dense.*

Proof. To prove that (i) implies (ii), it is sufficient to observe that the condition from (i) is satisfied for $\bigcup_{U \in \mathcal{U}}(V \cap U)$.

Let us prove that (ii) implies (iii). Suppose that $B \subseteq X$ and $\psi(B) = \varnothing$. Then $\psi((A \cap U) \cup B) = \varnothing$ for each $U \in \mathcal{U}$. Take $V \in \mathcal{T}$ such that $V \leq A \cup B$. Then $U \cap V \leq (A \cap U) \cup B$ and hence $U \cap V \in \mathcal{I}$. It follows form (ii), that $(\bigcup \mathcal{U}) \cap V$ is nowhere dense. Under the assumption that $\psi(B) = \varnothing$ we conclude that $V = \varnothing$, since $V \leq ((\bigcup \mathcal{U}) \cap V) \cup B$.

The fact that (iii) implies (i) is obvious, since every element of an ideal is nowhere dense. □

The next corollary is an immediate consequence from the previous Lemma.

Corollary 4.5. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space. Then $n(\mathcal{I}) \sim \mathcal{T}$ if one of the following conditions is satisfied:*

(1) *$\mathcal{I} \sim \mathcal{T}$;*
(2) *$U \cap V \in \mathcal{I}$ implies $U \cap V = \varnothing$ for each $U, V \in \mathcal{T}$;*
(3) *there exists an operation $\beta\colon P(\mathcal{T}) \to \mathcal{T}$ such that $\beta(\mathcal{U}) \approx \bigcap \mathcal{U}$ for each finite $\mathcal{U} \subseteq \mathcal{T}$ which is distributive over the union operation.*

4.3. Existence of a topological space that is *i*-homeomorphic to a given *i*-topological space. In this subsection, we provide necessary and sufficient conditions for the existence of a topological space that is *i*-homeomorphic to a given *i*-topological space.

In what follows, we will need the next lemma.

Lemma 4.6. *Let $(X, \mathcal{T}_X, \mathcal{I}_X)$ and $(Y, \mathcal{T}, \mathcal{I})$ be i-topological spaces, $f\colon X \to Y$ be a surjective i-continuous mapping such that $f\mathcal{I}_X = \mathcal{I}$. Assume that $\mathcal{I}_X \sim \mathcal{T}_X$. Then $\mathcal{I} \sim \mathcal{T}$.*

Proof. Consider a subset $A \subseteq Y$ and a family $\mathcal{V} = \{V_s\}_{s \in S}$ of *i*-open subsets of Y such that $A \subseteq \bigcup \mathcal{V}$ and $A \cap V_s \in \mathcal{I}$ for each $V_s \in \mathcal{V}$. Then there exists a family $\mathcal{U} = \{U_s\}_{s \in S}$ of *i*-open subsets of X such that $f^{-1}(\bigcup \mathcal{V}) \approx \bigcup \mathcal{U}$ and $f^{-1}(V_s) \approx U_s$ for each $U_s \in \mathcal{U}$. Clearly, $f^{-1}(A) \cap U_s$ lies in the ideal $\mathcal{I}_X$ for every $s \in S$. Then it follows from $\mathcal{I}_X \sim \mathcal{T}_X$, that $f^{-1}(A) \cap (\bigcup \mathcal{U}) \in \mathcal{I}_X$. On the other hand, $f^{-1}(A) \subseteq f^{-1}(\bigcup \mathcal{V})$ and $f^{-1}(\bigcup \mathcal{V}) \approx \bigcup \mathcal{U}$. Consequently, $f^{-1}(A) \in \mathcal{I}_X$ and by the assumption $A \in \mathcal{I}$. □

Now, we refer the reader to the paper by Dragan Janković and T.R. Hamlett [6]. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{I}$ be an ideal on X satisfying $\mathcal{T} \cap \mathcal{I} = \{\varnothing\}$. Then there exists an extension $\tilde{\mathcal{I}}$ of the ideal $\mathcal{I}$ such that $\mathcal{T} \cap \tilde{\mathcal{I}} = \{\varnothing\}$ and the ideal $\tilde{\mathcal{I}}$ is compatible with $\mathcal{T}$. This is an immediate corollary from Theorems 3.1. and 3.5. in [6]. We use this result to prove the following proposition.

Proposition 4.7. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space. Assume that there is a topological space, i-homeomorphic to $(X, \mathcal{T}, \mathcal{I})$. Then there exists a compatible extension $\tilde{\mathcal{I}}$ of the ideal $\mathcal{I}$ such that $\tilde{\mathcal{I}} \cap \mathcal{T} = \{\varnothing\}$.*

Proof. Let $(Y, \mathcal{T}_Y, \mathcal{I}_Y)$ and $h\colon X \to Y$ be the corresponding topological space and i-homeomorphism, respectively. Take $\tilde{\mathcal{I}} = h^{-1}\tilde{\mathcal{I}}_Y$, where $\tilde{\mathcal{I}}_Y$ is a compatible extension of the ideal $\mathcal{I}_Y$ defined in [6]. The statements that $\tilde{\mathcal{I}}$ is an ideal, $\mathcal{I} \subseteq \tilde{\mathcal{I}}$ and $\tilde{\mathcal{I}} \cap \mathcal{T} = \{\varnothing\}$ are the consequences from the facts that $\tilde{\mathcal{I}}_Y$ is an ideal satisfying $\tilde{\mathcal{I}}_Y \cap \mathcal{T}_Y = \{\varnothing\}$ and h is an i-homeomorphism. The fact that $\tilde{\mathcal{I}}$ is compatible with $\mathcal{T}$ is an immediate corollary from Lemma 4.6. $\square$

Given an i-topological space $(X, \mathcal{T}, \mathcal{I})$, we say that a family $\mathcal{A} \subseteq P(X)$ *covers a set* $A \subseteq X$ if $A \leq \bigcup \mathcal{A}$. The following corollary is a natural consequence from the previous Proposition.

Corollary 4.8. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space with a compatible ideal. Assume that there is a nonempty i-open set U such that it can be covered with i-open sets for which the intersection of U and each element of the cover lies in the ideal. Then there does not exist a topological space, i-homeomorphic to $(X, \mathcal{T}, \mathcal{I})$.*

Consider $(\mathbb{R}, \mathcal{D}, \mathcal{I}_n)$ from Example 2.4. The family every element of which is of the form $(\mathbb{R} \setminus \mathbb{Q}) \cup \{x\}$, $x \in \mathbb{Q}$, consists of open sets. Clearly, this family covers $\mathbb{Q}$ and every intersection of $\mathbb{Q}$ and an element of the family belongs to the ideal. Thus, by the previous corollary $(\mathbb{R}, \mathcal{D}, \mathcal{I}_n)$ is not i-homeomorphic to any topological space.

Now, we provide one particular construction for i-topological spaces with compatible ideals and prove a sufficient condition for the existence of a topological space that is i-homeomorphic to a given i-topological space. First, let us recall two related notions [5]. For a topological space $(X, \mathcal{T})$ and an ideal $\mathcal{I}$ on X, an operator ${}^*\colon P(X) \to P(X)$ is said to be a *local function of $\mathcal{I}$ with respect to $\mathcal{T}$* iff $A^* = \{\, x \in X \mid U \cap A \notin \mathcal{I}$ for each $U \in \mathcal{T}(x) \,\}$ for any $A \subseteq X$. The various properties of this operator are to be found in [5]. An extension $\mathcal{T}^*$ of the topology $\mathcal{T}$ is defined as follows: $\mathcal{T}^* = \{\, U \setminus A \mid U \in \mathcal{T}$ and $A \in \mathcal{I} \,\}$. We preserve these definitions for i-topological spaces.

We will exploit an operator ${}^{\sim}\colon P(X) \to P(X)$ satisfying $\tilde{A} = \psi(A) \cap A^*$ for each $A \subseteq X$ and use the notation $\tilde{\mathcal{T}} = \{\, \tilde{U} \mid U \in \mathcal{T} \,\}$. Notice that our use of tilde differs from the notation $\tilde{\mathcal{I}}$ for ideals in [6]. We hope that there will be no chance for confusion.

Lemma 4.9. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space satisfying $\mathcal{I} \sim \mathcal{T}$. Then the following statements hold for any i-open U, V, W and $\mathcal{U} \subseteq \tilde{\mathcal{T}}$:*

(i) $U \approx \tilde{U}$;
(ii) *if* $\tilde{U} \leq \tilde{V}$ *then* $\tilde{U} \subseteq \tilde{V}$;
(iii) *if* $\tilde{U} \approx \tilde{V}$ *then* $\tilde{U} = \tilde{V}$;
(iv) *if* $\tilde{U} \cap \tilde{V} \in \mathcal{I}$ *then* $\tilde{U} \cap \tilde{V} = \varnothing$;
(v) *if* $\tilde{U} \approx \tilde{V} \cap \tilde{W}$ *then* $\tilde{U} \subseteq \tilde{V} \cap \tilde{W}$;
(vi) *if* $\tilde{U} \approx \bigcup \mathfrak{U}$ *then* $\bigcup \mathfrak{U} \subseteq \tilde{U}$.

Proof. Statement (i) is obvious, since $U \in \psi(U)$, $\psi(U) \setminus U \in \mathcal{I}$ and $U \setminus U^* \in \mathcal{I}$ for each i-open U.

First, we show that $\tilde{U} \subseteq \tilde{V}$ follows from $U \leq V$. Clearly, $W_x \cap U \notin \mathcal{I}$ implies $W_x \cap V \notin \mathcal{I}$ for each $x \in U^*$ and any its i-open neighbourhood W_x. Hence, $U^* \subseteq V^*$. On the other hand, for any $x \in \psi(U)$ there exists its i-open neighbourhood W_x such that $W_x \leq U$. Then $W_x \leq V$ for each such W_x and hence $\psi(U) \subseteq \psi(V)$. Thus, we have $\tilde{U} \subseteq \tilde{V}$.

It follows from (i), that $\tilde{U} \leq \tilde{V}$ implies $U \leq V$. Thus, by the previous paragraph we have (ii) and hence (iii).

Let us prove (iv). Suppose that $\tilde{U} \cap \tilde{V} \in \mathcal{I}$ and $x \in \tilde{U} \cap \tilde{V}$. Then there is an i-open neighbourhood W_x of x such that $W_x \leq U$. It follows from (i), that $W_x \leq \tilde{U}$. Hence, $W_x \cap \tilde{V} \in \mathcal{I}$ and $W_x \cap V \in \mathcal{I}$. Then by the definition $x \notin V^*$ and hence $x \notin \tilde{V}$. The last statement contradicts our assumption, thus $\tilde{U} \cap \tilde{V} = \varnothing$.

Now, let us prove (v). Suppose that $\tilde{U} \approx \tilde{V} \cap \tilde{W}$. Then it follows from (i), that $U \approx V \cap W$ and hence $U \leq V$ and $U \leq W$. By the second paragraph of this proof we conclude that $\tilde{U} \subseteq \tilde{V}$ and $\tilde{U} \subseteq \tilde{W}$. Therefore, $\tilde{U} \subseteq \tilde{V} \cap \tilde{W}$.

Finally, let us prove (vi). Suppose $\tilde{U} \approx \bigcup \mathfrak{U}$. Then $\tilde{V} \leq \tilde{U}$ for each $\tilde{V} \in \mathfrak{U}$. It follows from (ii), that $\tilde{V} \subseteq \tilde{U}$ for each $\tilde{V} \in \mathfrak{U}$. Thus, $\bigcup \mathfrak{U} \subseteq \tilde{U}$. □

Proposition 4.10. *Let* $(X, \mathcal{T}, \mathcal{I})$ *be an* i*-topological space satisfying* $\mathcal{I} \sim \mathcal{T}$. *Then* $\left(X, \tilde{\mathcal{T}}, \mathcal{I}\right)$ *is a unified* i*-topological space such that* $\mathcal{I} \sim \tilde{\mathcal{T}}$ *and the spaces* $(X, \mathcal{T}, \mathcal{I})$ *and* $\left(X, \tilde{\mathcal{T}}, \mathcal{I}\right)$ *are* i*-homeomorphic.*

Proof. The facts that (T1), (T3) and (T4) hold for $\left(X, \tilde{\mathcal{T}}, \mathcal{I}\right)$ are obvious.

Let us prove (T2). Suppose that $\mathfrak{U} \subseteq \mathcal{T}$ and $\tilde{\mathfrak{U}} = \{\, \tilde{U} \mid U \in \mathfrak{U} \,\}$. Then $(\bigcup \mathfrak{U}) \setminus \left(\bigcup \tilde{\mathfrak{U}}\right)$ belongs to the ideal $\mathcal{I}$, since $U \setminus \tilde{U} \in \mathcal{I}$ for each $U \in \mathfrak{U}$ and $\mathcal{I}$ is compatible with $\mathcal{T}$. On the other hand, for any $x \in \tilde{U} \setminus U$ there is an i-open neighbourhood U_x such that $U_x \leq U$. Take $\mathfrak{U}_x = \{\, U_x \mid x \in \tilde{U} \text{ and } U \in \mathfrak{U} \,\}$ and $A = \left(\bigcup \tilde{\mathfrak{U}}\right) \setminus (\bigcup \mathfrak{U})$. It follows that $U_x \cap A \in \mathcal{I}$ for each $U_x \in \mathfrak{U}_x$ and, since $\mathcal{I}$ is compatible with $\mathcal{T}$, $(\bigcup \mathfrak{U}_x) \cap A$ belongs to $\mathcal{I}$. Thus, we have (T2).

The fact that (N2) holds for the identity mapping $id\colon X \to X$ is obvious and (N1) follows from the previous paragraph of this proof. Then it follows from Lemmas 4.9 and 4.6, respectively, that $\left(X, \tilde{\mathcal{T}}, \mathcal{I}\right)$ is unified and the ideal $\mathcal{I}$ is compatible with $\tilde{\mathcal{T}}$. □

Notice that $\tilde{\tilde{\mathcal{T}}} = \tilde{\mathcal{T}}$ holds for each i-topological space $(X, \mathcal{T}, \mathcal{I})$ with a compatible ideal. This is an immediate consequence from the definition of $\tilde{\mathcal{T}}$, the previous Proposition and Lemma 4.9.

Proposition 4.11. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space satisfying $\mathcal{I} \sim \mathcal{T}$. Then $(X, \mathcal{T}^*, \mathcal{I})$ is an i-topological space such that $\mathcal{I} \sim \mathcal{T}^*$ and the spaces $(X, \mathcal{T}, \mathcal{I})$ and $(X, \mathcal{T}^*, \mathcal{I})$ are i-homeomorphic.*

Proof. Suppose that $\mathcal{U} = \{U_s\}_{s \in S} \subseteq \mathcal{T}^*$. Then there are an i-open set U and a collection $\mathcal{A} = \{A_s\}_{s \in S}$ of the elements of the ideal such that $U_s \cup A_s \in \mathcal{T}$ for each $s \in S$ and $U \approx \bigcup_{s \in S}(U_s \cup A_s)$. It follows from $\mathcal{I} \sim \mathcal{T}$, that $\bigcup \mathcal{A} \setminus \bigcup \mathcal{U}$ belongs to $\mathcal{I}$. Hence, $\bigcup \mathcal{U} \approx \bigcup_{s \in S}(U_s \cup A_s) \approx U$. Observe that $U = U \setminus \varnothing$ is an element from $\mathcal{T}^*$. Thus, we have (T2). All other facts which it remains to show to complete the proof are obvious. □

The next proposition is a natural corollary from Propositions 4.12 and 4.11. In what follows, all results hold if we replace $\left(\tilde{\mathcal{T}}\right)^*$ with $\left(\tilde{\mathcal{T}}\right)^\cup$. Notice that by Lemma 4.9 $\left(\tilde{\mathcal{T}}\right)^\cup \subseteq \left(\tilde{\mathcal{T}}\right)^*$.

Proposition 4.12. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological satisfying $\mathcal{I} \sim \mathcal{T}$. Assume that $\mathcal{P} = \left(\tilde{\mathcal{T}}\right)^*$. Then the following statements hold:*

(i) *$(X, \mathcal{P}, \mathcal{I})$ is an i-topological space with a compatible ideal;*
(ii) *$(X, \mathcal{T}, \mathcal{I})$ and $(X, \mathcal{P}, \mathcal{I})$ are i-homeomorphic;*
(iii) *$U \cap V \in \mathcal{I}$ implies $U \cap V = \varnothing$ for each $U, V \in \mathcal{P}$;*
(iv) *$\bigcup \mathcal{U} \in \mathcal{P}$ for each $\mathcal{U} \subseteq \mathcal{P}$;*
(v) *for any $U, V \in \mathcal{P}$ there is $A \in \mathcal{I}$ such that $(U \cap V) \setminus A \in \mathcal{P}$.*

The following proposition gives us a sufficient condition for the existence of a topological space i-homeomorphic to a given i-topological space. Notice that this condition implies compatibility of an ideal and it holds for any topological space.

Proposition 4.13. *Let $(X, \mathcal{T}, \mathcal{I})$ be an i-topological space. Assume that $\mathcal{I} = n(\mathcal{I})$ and for any $x \in X$ and any $U, V \in \mathcal{T}(x)$ there is $W \in \mathcal{T}(x)$ such that $W \leq U \cap V$. Then there exists a topological space, i-homeomorphic to $(X, \mathcal{T}, \mathcal{I})$.*

Proof. It follows from Corollary 4.5, that $\mathcal{I} \sim \mathcal{T}$.

First, let us show that $\tilde{U}_1 \cap \tilde{U}_2 \in \tilde{\mathcal{T}}$ for each $\tilde{U}_1, \tilde{U}_2 \in \tilde{\mathcal{T}}$. Take $x \in \tilde{U}_1 \cap \tilde{U}_2$. Then there are two i-open neighbourhoods W_1 and W_2 of x satisfying $W_1 \leq U_1$ and $W_2 \leq U_2$. By the assumption there is the third i-open neighbourhood W of x such that $W \leq W_1 \cap W_1$. Then $W \leq V$, where V is an i-open set such that $\tilde{V} \approx \tilde{U}_1 \cap \tilde{U}_2$. Thus, $x \in \psi(V)$.

On the other hand, $W_x \cap W \notin \mathcal{I}$ and hence $W_x \cap V \notin \mathcal{I}$ for each i-open neighbourhood W_x of x, since $x \in U_1^*$. Hence, $x \in V^*$ and we have $x \in \tilde{V}$. The point x is an arbitrary point from $\tilde{U}_1 \cap \tilde{U}_2$, thus $\tilde{U}_1 \cap \tilde{U}_2 \subseteq \tilde{V}$ and by Lemma 4.9 $\tilde{V} = \tilde{U}_1 \cap \tilde{U}_2$.

We observe that $U \cap V = W$ implies $(U \setminus A) \cap (V \setminus A) = W \setminus A$ for each subsets U, V, W and A. Suppose that $\mathcal{P} = \left(\tilde{\mathcal{T}}\right)^*$. Then for any $U, V \in \mathcal{P}$ there exists $W \in \mathcal{P}$ such that $U \cap V = W$. Finally, it follows from Proposition 4.12 that $(X, \mathcal{P}, \mathcal{I})$ is a topological space, i-homeomorphic to $(X, \mathcal{T}, \mathcal{I})$. □

We leave as an open question whether there exists an i-topological space with a compatible ideal that is not i-homeomorphic to any topological space.

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Irina Zvina (IrinaZvina@inbox.lv)
Institute of Mathematics of Latvian Academy of Sciences and University of Latvia, Akademijas laukums 1, Riga, LV-1050, Latvia.