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# Every infinite group can be generated by P-small subset

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Abstract. For every infinite group $G$ and every set of generators $S$ of $G$, we construct a system of generators in $S$ which is small in the sense of Prodanov.

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A subset $B$ of a group $G$ is called *large* if $G = F \cdot B = B \cdot F$ for some finite subset $F$ of $G$. A subset $S$ of a group $G$ is called *small* if the subset $G \backslash F \cdot S \cdot F$ is large for every finite subset $F$ of $G$.

V. Malykhin and R. Moresco [4] posed the following question: can ever infinite group by generated by small subset? This question was answered positively in [6] (see also [7, Theorem 13.1], some partial results were obtain also in [2]).

Following [2, §2.1] we call a subset $S$ of a group $G$ *left small in the sense of Prodanov* (briefly *left P-small*) if there exist an injective sequence $(a_n)_{n<\omega}$ such that the family $\{a_n \cdot S : n < \omega\}$ consists of pairwise disjoint subsets. Analogously, *right small in the sense of Prodanov* (briefly *right P-small*) is introduced. The set $S$ is called *P-small* when it is both left P-small and right P-small. Clearly, all these notions coincide in the abelian case. That was the case considered by Prodanov [5], who introduced the notion by noticing that if for a subset $A$ of an abelian group $G$ the difference set $A - A$ is not not large, then $A$ is P-small.

By [3, Theorem 4.2], every P-small subset of Abelian group is small, but there are small subsets of Abelian groups which are not P-small. On the other hand, the free group of rank 2 contains P-small subsets which are not small. It was proved in [2, Theorem 3.6] that every abelian group has a P-small set of generators. Furthermore, every free group (more generally, every group admitting an infinite abelian quotient) and every infinite symmetric group admit a P-small set of generators [2, Proposition 3.7, Theorem 3.11]. In this paper we offer a common generalization of all preceding results in our theorem below by proving that every set of generators of an infinite group contains a P-small subset of generators.

For a subset $A$ of a group $G$ we denote by $\langle A\rangle$ the subgroup generated by $A$.

**Theorem 1.** *Let $G$ be an infinite group, $A \subseteq G$, $G = \langle A\rangle$. Then there exists a small and P-small subset $X$ of $G$ such that $\langle X\rangle = G$ and $X \subseteq A$.*

*Proof.* If $G$ is finitely generated, the statement is trivial since every set of generators of $G$ contains a finite set of generators. We can take an arbitrary finite system $X$, $X \subseteq A$ of generators of $G$ and choose inductively the sequences $(y_n)_{n<\omega}$, $(z_n)_{n<\omega}$ such that

$$y_n \cdot X \cap y_m \cdot X = \varnothing, \quad X \cdot z_n \cap X \cdot z_m = \varnothing$$

for all $n, m$ such that $n < m < \omega$.

Assume that $G$ is not finitely generated and fix some minimal well-ordering $\{g_\alpha : \alpha < \kappa\}$ of $A \cup \{e\}$, $g_0 = e$, $e$ is the identity of $G$. Put $G_0 = \{e\}$ and $x_0 = g_1$. Suppose that, for some ordinal $\lambda < \kappa$, the elements $\{x_\alpha : \alpha < \lambda\}$ and the subgroup $\{G_\alpha : \alpha < \lambda\}$ have been chosen. If $\lambda$ is a limit ordinal, we put $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$, take the first element $g_\beta$ such that $g_\beta \notin G_\lambda$ and put $x_\lambda = g_\beta$. If $\lambda$ is a non-limit ordinal, we denote by $G_\lambda$ the subgroup generated by $G_{\lambda-1} \cup \{x_{\lambda-1}\}$, take the first element $g_\beta$ such that $g_\beta \notin G_\lambda$ and put $x_\lambda = g_\beta$. After $\kappa$ steps we get the subset $X = \{x_\alpha : \alpha < \kappa\}$ and the properly increasing chain $\{G_\alpha : \alpha < \kappa\}$ of subgroups of $G$ such that $X \subseteq A$, $G = \langle X\rangle$ and $x_\alpha \in D_\alpha := G_{\alpha+1} \setminus G_\alpha$ for every $\alpha < \kappa$. By [5, Theorem 13.1], $X$ is small.

To show that $X$ is P-small, we build a sequence sequences $(y_n)_{n<\omega}$ of elements of $G$ such that

$$y_n \cdot X \cap y_i \cdot X = \varnothing \tag{1}$$

for every $i < n$. To this end we use the following easy to see properties of the sets $D_\alpha$:

(a) $G = \bigcup_{\alpha<\kappa} D_\alpha$ is a partition with $D_\alpha \cap G_\lambda = \varnothing$ whenever $\lambda \le \alpha < \kappa$;
(b) $G_\alpha \cdot D_\alpha = D_\alpha \cdot G_\alpha = D_\alpha$ for every $\alpha < \kappa$;
(c) $|D_m| \ge |G_m| \ge 2^m$, for all $m < \omega$.

For every $m < \omega$ let $X_m = \{x_0, x_1, ..., x_m\}$.

Put $y_0 = e$. Suppose that, for some natural number $n$, the elements $y_0, y_1, ..., y_{n-1}$ have been chosen so that $\{y_0, y_1, ..., y_{n-1}\} \subset G_\omega$ and

$$y_i \cdot X \cap y_j \cdot X = \varnothing$$

for all $i, j$ such that $i < j \le n - 1$.

To determine $y_n$, we take a natural number $m$ such that $\{y_0, y_1, ..., y_{n-1}\} \subset G_m$ and

$$2^m > n(m + 1)^2.$$

By (c) and by the inequality $|\{y_0, y_1, ..., y_{n-1}\} \cdot X_m \cdot X_m^{-1}| \leq n(m+1)^2$ we can take the element $y_n \in D_m$ such that

$$\{y_0, y_1, ..., y_{n-1}\} \cdot X_m \cap y_n \cdot X_m = \varnothing.$$

By the choice of $y_n$, we have

$$y_n X_m \cap y_i \cdot X_m = \varnothing$$

for every $i < n$. If $k, l < \omega$, $k > m$, then $y_j x_k \in D_k$ for every $j \leq n$. Hence $y_n x_k = y_j x_l$ with $k, l > m$ yields $k = l$ and $n = j$. Now assume that $y_i x_k = y_j x_l$ holds with $k > m$, $i, j \leq n$ and $l \leq m$. Then according to (a) and (b) this is not possible as $y_n \cdot x_k \in D_k$, while $y_j \cdot x_l \in G_{m+1}$. Analogously, $y_n \cdot x_k = y_j \cdot x_l$ is not possible with $k \leq m$ and $l > m$. This proves that

$$y_n \cdot X \cap y_i \cdot X = \varnothing$$

for every $i < n$. After $\omega$ steps we get the sequence $(y_n)_{n<\omega}$ such that the family $\{y_n \cdot X : n < \omega\}$ consists of pairwise disjoint subsets. Applying these arguments to the set $X^{-1}$, we get the sequence $(z_n)_{n\in\omega}$ such that the family $\{X \cdot z_n : n \in \omega\}$ consists of pairwise disjoint subsets. Hence, $X$ is $P$-small. □

**Question 2.** *Let $G$ be an infinite group of cardinality $\kappa$. Does there exist a subset $X$ of $G$ and a $\kappa$-sequence $(y_\alpha)_{\alpha<\kappa}$ such that the family $\{y_\alpha \cdot X : \alpha \in \kappa\}$ consists of pairwise disjoint subsets and $G = \langle X \rangle$?*

If $G$ is Abelian the answer is positive (see the proof of Theorem 3.6 from [2]).

Finally, we offer also the following

**Question 3.**

(a) *Let $X$ be a subset of $G$ such that, for every natural number $n$ there exits a subset $Y_n$ of $G$ such that $|Y_n| = n$ and the family $\{y \cdot X : y \in Y_n\}$ is disjoint. Is $X$ left $P$-small ?*

(b) *By* [7, Theorem 12.10], *every infinite group can be partitioned into countably many small subsets. Can every infinite group be partitioned into countably many $P$-small subsets?*

(c) *Let $G$ be an infinite group. Does there exist a system $S$ of generators of $G$ such that $G \neq (S \cdot S^{-1})^n$ for every natural number $n$?*

Note added in November 2006. Recently T. Banakh and N. Lyaskovska answered negatively item (a) of Question 3.

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