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Maximal balleans

Olga I. Protasova

Abstract. A ballean is a set X endowed with some family of subsets of X which are called the balls. We postulate the properties of the family of balls in such a way that the balleans with the appropriate morphisms can be considered as the asymptotic counterparts of the uniform topological spaces. The purpose of the paper is to find and study the asymptotic counterparts for maximal topological spaces and maximal topological groups.

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1. Ball structures and balleans

A *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radiuses.* Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, \quad B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure $\mathcal{B} = (X, P, B)$ is called

- *lower symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B^*(x, \alpha') \subseteq B(x, \alpha), \quad B(x, \beta') \subseteq B^*(x, \beta);$$

- *upper symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \quad B^*(x, \beta) \subseteq B(x, \beta');$$

- *lower multiplicative* if, for any α, $\beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x,\gamma),\gamma) \subseteq B(x,\alpha) \cap B(x,\beta);$$

- *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x,\alpha),\beta) \subseteq B(x,\gamma).$$

Let $\mathcal{B} = (X, P, B)$ be a lower symmetric, lower multiplicative ball structure. Then the family

$$\{\bigcup_{x \in X} B(x,\alpha) \times B(x,\alpha) : \alpha \in P\}$$

is a base of entourages for some (uniquely determined) uniformity on X. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x,U) = \{y \in X : (x,y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

A ball structure is said to be a *ballean* if it is upper symmetric and upper multiplicative. The balleans arouse independently in asymptotic topology [6], [19] under name of coarse structure and in combinatorics [7]. For good motivation to study the balleans related to metric spaces see the survey [3].

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a $\prec$-*mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f(B_1(x,\alpha)) \subseteq B_2(f(x),\beta).$$

If $f : X_1 \to X_2$ is a bijaction such that f and f^{-1} are the $\prec$-mappings, we say that the balleans $\mathcal{B}_1$ and $\mathcal{B}_2$ are *asymorphic*. If $X_1 = X_2$ and the identity mapping $id : X_1 \to X_2$ is a $\prec$-mapping, we write $\mathcal{B}_1 \subseteq \mathcal{B}_2$. If $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$, we write $\mathcal{B}_1 = \mathcal{B}_2$. If $\mathcal{B}_1 \subseteq \mathcal{B}_2$ but $\mathcal{B}_1 \neq \mathcal{B}_2$, we write $\mathcal{B}_1 \subset \mathcal{B}_2$ and say that $\mathcal{B}_2$ is stronger than $\mathcal{B}_1$.

By the definition, $\prec$-mappings can be considered as the asymptotic counterparts of the uniformly continuous mappings between the uniform topological spaces. The approach to study balleans with $\prec$-morphisms is reflected in the papers [11, 12, 13, 14].

To determine the subject of the paper we need some more definitions.

Let $\mathcal{B} = (X, P, B)$ be a ballean. A subset A of X is called *bounded* if there exist $x \in X$ and $\alpha \in P$ such that $A \subseteq B(x,\alpha)$. A ballean $\mathcal{B}$ is called bounded if its support X is bounded. A ballean B is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x,\alpha)$. A ballean $\mathcal{B}$ is called *proper* if $\mathcal{B}$ is connected and unbounded.

A proper ballean $\mathcal{B} = (X, P, B)$ is called *maximal* if every stronger ballean on X is bounded. By Zorn Lemma, every proper ballean can be strengthened to some maximal ballean. By [12, Theorem 5.1], a ballean $(X, \mathbb{R}^+, B_d)$ of unbounded metric space (X, d), where $B_d(x, r) = \{y \in X : d(x, y) \leq r\}$, is not maximal. For criterion of metrizability of balleans see [11].

2. Criterion of maximality

Given an arbitrary ball structure $\mathcal{B} = (X, P, B)$, we say that the ballean *env* $\mathcal{B}$ is a *ballean envelope* of $\mathcal{B}$ if *env* $\mathcal{B}$ is the smallest ballean on X such that $\mathcal{B} \subseteq$ *env* $\mathcal{B}$. To describe *env* $\mathcal{B}$ more constructively, we consider the free semigroup $F(P)$ in the alphabet P and take $F(P)$ as the set of radiuses of *env* $\mathcal{B}$. Then, for every $x \in X$ and every $w \in F(P)$, we define the ball *env* $B(x, w)$ inductively by the length of the word w. If $w = \alpha$ and $\alpha \in P$, we put

$$envB(x, w) = B(x, \alpha) \bigcup B^*(x, \alpha).$$

If $w = v\alpha$ and $\alpha \in P$, we put

$$env\ B(x, w) = B(env\ B(x, v), \alpha) \bigcup B^*(env\ B(x, v), \alpha).$$

Note that $y \in env\ B(x, w)$ if and only if $x \in env\ B(y, \tilde{w})$, where $\tilde{w}$ is the word w written in the reverse order, so the ball structure $env\ \mathcal{B} = (X, F(P), env\ B)$ is upper symmetric. Since $env\ B(x, uv) = env\ B(env\ B(x, u), v)$, then *env* $\mathcal{B}$ is upper multiplicative. Hence, *env* $\mathcal{B}$ is a ballean. If $\mathcal{B}'$ is a ballean on X such that $\mathcal{B} \subseteq \mathcal{B}'$, it follows from the upper symmetry and upper multiplicativity of $\mathcal{B}'$, that *env* $\mathcal{B} \subseteq \mathcal{B}'$.

Let $\{\mathcal{B}_\lambda = (X, P_\lambda, B_\lambda) : \lambda \in \Lambda\}$ be a family of ball structures with common support X and pairwise disjoint family $\{P_\lambda : \lambda \in \Lambda\}$ of radiuses. We put $P = \bigcup\{P_\lambda : \lambda \in \Lambda\}$ and, for any $x \in X$ and $\alpha \in P$, $\alpha \in P_\lambda$, denote $B(x, \alpha) = B_\lambda(x, \alpha)$. Then $\mathcal{B} = (X, P, B)$ is the smallest ball structure on X such that $\mathcal{B}_\lambda \subseteq \mathcal{B}$ for every $\lambda \in \Lambda$, and *env* $\mathcal{B}$ is the smallest ballean on X such that $\mathcal{B}_\lambda \subseteq$ *env* $\mathcal{B}$ for every $\lambda \in \Lambda$.

We say that a ball structure $\mathcal{B} = (X, P, B)$ is a *monoball structure* if its set of radiuses P is a singleton.

Theorem 2.1. *A proper ballean $\mathcal{B} = (X, P, B)$ is maximal if and only if, for every monoball structure $\mathcal{B}'$ on X, either $\mathcal{B}' \subseteq \mathcal{B}$ or the ballean envelope of $\mathcal{B} \bigcup \mathcal{B}'$ is bounded.*

Proof. Suppose that $\mathcal{B}$ is maximal and $\mathcal{B}' \nsubseteq \mathcal{B}$. Then $\mathcal{B} \subset \mathcal{B} \bigcup \mathcal{B}'$ and the ballean *env* $(\mathcal{B} \bigcup \mathcal{B}')$ is stronger than $\mathcal{B}$. It follows that *env* $(\mathcal{B} \bigcup \mathcal{B}')$ is bounded.

Assume that $\mathcal{B}$ is not maximal and choose a proper ballean $\mathcal{B}'' = (X, P'', B'')$ such that $\mathcal{B} \subset \mathcal{B}'$. Then there exists $\beta \in P''$ such that, for every $\alpha \in P$, there exists $x(\alpha) \in X$ such that $B(x(\alpha), \alpha) \nsubseteq B''(x(\alpha), \beta)$. For every $x \in X$, we put $B'(x, \beta) = B''(x, \beta)$ and consider the monoball structure $\mathcal{B}' = (X, \{\beta\}, B')$. By the choice of β, we have $\mathcal{B}' \nsubseteq \mathcal{B}$. On the other hand, *env* $(\mathcal{B} \bigcup \mathcal{B}') \subseteq \mathcal{B}''$, therefore *env* $(\mathcal{B} \bigcup \mathcal{B}')$ is unbounded. □

Example 2.2. Let X be an infinite set of regular cardinality k. Denote by $\mathcal{F}$ the family of all subsets of X of cardinality $< k$. Let P be the set of all mappings $f : X \to \mathcal{F}$ such that, for every $x \in X$, we have $x \in f(x)$ and

$$|\{y \in X : x \in f(y)\}| < k.$$

Given any $x \in X$ and $f \in P$, we put $B(x, f) = f(x)$ and consider the ball structure $\mathcal{B} = (X, P, B)$. Since $B^*(x, f) = \{y \in X : x \in f(y)\}$, we have that $\mathcal{B}$ is upper symmetric. Since k is regular, $\mathcal{B}$ is upper multiplicative. Hence, $\mathcal{B}$ is a ballean. Clearly, $\mathcal{B}$ is connected and unbounded, so $\mathcal{B}$ is proper. Using Theorem 2.1, we show that $\mathcal{B}$ is maximal. Let $\mathcal{B}' = (X, \{1\}, B')$ be a monoball structure on X such that $\mathcal{B}' \nsubseteq \mathcal{B}$. Then we have two possibilities.

Case 1. There exists $x \in X$ such that $B'(x, 1) = k$. Put $Y = X \setminus B'(x, 1)$ and choose a subset $Z \subseteq B'(x, 1)$ such that $|Z| = |Y|$. Fix an arbitrary bijection $h : Z \to Y$ and, for every $x \in X$, put

$$f(x) = \begin{cases} \{x\}, \; if \;\; x \notin Z; \\ \{x, h(x)\}, \; if \;\; x \in Z. \end{cases}$$

Clearly, $f \in P$ and $B(B'(x, 1), f) = X$. It means that *env* $(\mathcal{B} \bigcup \mathcal{B}')$ is bounded.

Case 2. There exists $x \in X$ such that the set $Y = \{y \in X : x \notin B'(y, 1)\}$ is of cardinality k. Then Y is a bounded subset in *env* $(\mathcal{B} \bigcup \mathcal{B}')$ and, repeating the argument of the Case 1, we conclude that *env* $(\mathcal{B} \bigcup \mathcal{B}')$ is bounded.

Let X be an infinite set, φ be a filter on X. For any $x \in X$ and $F \in \varphi$, we put

$$B_\varphi(x, F) = \begin{cases} \{x\}, \;\; if \;\; x \in F; \\ X \setminus F, \;\; if \;\; x \notin F; \end{cases}$$

and consider the ballean $\mathcal{B}(X, \varphi) = (X, \varphi, B_\varphi)$. Clearly, $\mathcal{B}(X, \varphi)$ is unbounded and $B(X, \varphi)$ is connected if and only if $\bigcap \varphi = \varnothing$. A ballean $\mathcal{B} = (X, P, B)$ is called *pseudodiscrete* if , for every $\alpha \in P$, there exists a bounded subset V of X such that $B(x, \alpha) = \{x\}$ for every $x \in X \setminus V$. By [13], $\mathcal{B}$ is pseudodiscrete if and only if there exists a filter φ on X such that $\mathcal{B} = \mathcal{B}(X, \varphi)$.

If φ, ψ be filters on X such that $\bigcap \varphi = \bigcap \psi = \varnothing$ and $\varphi \subset \psi$, then $\mathcal{B}(X, \psi)$ is stronger than $\mathcal{B}(X, \varphi)$. Hence, if φ is not ultrafilter, then $\mathcal{B}(X, \varphi)$ is not maximal.

Example 2.3. Let X be an infinite set, φ be a free ultrafilter on X. Using Theorem 2.1, we show that the ballean $\mathcal{B}(X, \varphi)$ is maximal. Let $\mathcal{B}' = (X, \{1\}, B')$ be a monoball structure. We put $Y = \{x \in X : B'(x, 1) \neq \{x\}\}$ and consider two cases.

Case $Y \in \varphi$. For very $x \in Y$, we take an element $f(x) \in B'(x, 1)$ such that $x \neq f(x)$. For every $x \in X \setminus Y$, we put $f(x) = x$. Thus, we get the mapping $f : X \to X$. Endow X with the discrete topology and consider the Stone-Čech extension $f^\beta : \beta X \to \beta X$ of f. We take the elements of βX to be the ultrafilters on X. Since $\{x \in X : f(x) = x\} \notin \varphi$, we have $f^\beta(\varphi) \neq \varphi$. Choose $Z \in \varphi$ such that $Z \subseteq Y$ and $f(Z) \notin \varphi$. Then $f(Z)$ is bounded in $\mathcal{B}(X, \varphi)$ and Z is bounded in *env* $(\mathcal{B}' \bigcup \mathcal{B}(X, \varphi))$. Hence, *env* $(\mathcal{B}' \bigcup \mathcal{B}(X, \varphi))$ is bounded.

Case $Y \notin \varphi$. Then Y is bounded in $\mathcal{B}(X, \varphi)$. If $B'(Y, 1) \in \varphi$, then *env* $(\mathcal{B}' \bigcup \mathcal{B}(X, \varphi))$ is bounded. If $B'(Y, 1) \notin \varphi$, then $\mathcal{B}' \subseteq \mathcal{B}(X, \varphi)$.

3. Subsets of maximal balleans

Let $\mathcal{B} = (X, P, B)$ be a ballean. Given any $A \subseteq X$ and $\alpha \in P$, we put

$$Int(A, \alpha) = \{a \in A : B(a, \alpha) \subseteq A\}.$$

We use the following classification of subsets of a ballean by their size from [7].

A subset $A \subseteq X$ is called

- *large* if there exists $\alpha \in P$ such that $X = B(A, \alpha)$;
- *small* if $X \setminus B(A, \alpha)$ is large for every $\alpha \in P$;
- *piecewise large* if there exists $\beta \in P$ such that $Int(B(A, \beta), \alpha) \neq \varnothing$ for every $\alpha \in P$;
- *extralarge* if $Int(A, \alpha)$ is large for every $\alpha \in P$.

By [7, Theorem 11.1], a subset A of X is small if and only if A is not piecewise large. Some results from [7, Section 11] witness that the large, extralarge and small subsets of a ballean can be considered as the asymptotic duplicates of dense, open dense and nowhere dense subsets of a topological space respectively.

Theorem 3.1. *If a ballean $\mathcal{B} = (X, P, B)$ is maximal, then every unbounded subset Y of X is large.*

Proof. Assume, otherwise, that Y is unbounded but Y is not large. For every $x \in X$, we put

$$B'(x, 1) = \begin{cases} \{x\}, & if \;\; x \notin Y; \\ Y, & if \;\; x \in Y; \end{cases}$$

and consider the monoball structure $B' = (X, \{1\}, B')$. Since Y is unbounded in $\mathcal{B}$, we have $\mathcal{B}' \subseteq \mathcal{B}$. Since Y is not large, it follows from the above description of the ballean envelope, that $env(\mathcal{B} \bigcup \mathcal{B}')$ is unbounded, whence a contradiction to Theorem 2.1. □

Now we give an example of a proper non-maximal ballean in which every unbounded subset is large, so the converse statement to Theorem 3.1 is not true.

Example 3.2. Let X be an infinite set, S be a group of all substitution of X, e be the identity substitution. We denote by $\mathcal{F}(S)$ the family of all finite subsets of S containing e. Given any $x \in X$ and $F \in \mathcal{F}(S)$, we put

$$B(x, F) = \{f(x) : f \in F\}$$

and consider the ballean $\mathcal{B} = (X, \mathcal{F}(S), B)$. Clearly, a subset Y of X is bounded if and only if Y is finite. By [18, Theorem 1.2], a subset Y is large if and only if $|Y| = |X|$. Now assume that X is countable. Then every unbounded subset of X is large. We show that $\mathcal{B}$ is not maximal. Partition $X = \bigcup_{n \in \mathbb{N}} X_n$ so that $|X_n| = n$ for every $n \in \mathbb{N}$. Then, for every $x \in X$, we choose $n \in \mathbb{N}$ such that $x \in X_n$ and put $B'(x, 1) = X_n$. In this way we get the monoball structure $\mathcal{B}' = (X, \{1\}, B')$. Since $|B(x, F)| \leq |F|$ and $|X_n| = n$ for every $n \in \omega$, we have $\mathcal{B}' \nsubseteq \mathcal{B}$. On the other hand, every ball in *env* $(\mathcal{B} \bigcup \mathcal{B}')$ is finite, so *env* $(\mathcal{B} \bigcup \mathcal{B}')$ is proper. By Theorem 2.1, $\mathcal{B}$ is not maximal.

Let $\mathcal{B} = (X, P, B)$ be a proper ballean. A function $h : X \to [0, 1]$ is called *slowly oscillating* if, for every $\alpha \in P$ and every $\varepsilon > 0$, there exists a bounded subset V of X such that, for every $x \in X \setminus V$,

$$diam\ h(B(x, \alpha)) < \varepsilon,$$

where $diam A = sup\ \{|a - b| : a, b \in A\}$. We denote by $X^\sharp$ the set of all ultrafilters φ on X such that every member of φ is unbounded and consider $X^\sharp$ as a subspace of the Stone-Čech compactification βX of the discrete space X. Given any $\varphi, \psi \in X^\sharp$, we put $\varphi \sim \psi$ if and only if $h^\beta(\varphi) = h^\beta(\psi)$ for every slowly oscillating function $h : X \to [0, 1]$. Then $\sim$ is a closed (in $X^\sharp \times X^\sharp$) equivalence on $X^\sharp$. The factor-space $X^\sharp/ \sim$ is called the corona of $\mathcal{B}$ and is denote by $\nu(\mathcal{B})$. For coronas of balleans see [15].

Theorem 3.3. *Let $\mathcal{B} = (X, P, B)$ be a proper ballean. If every unbounded subset of X is large, then*

(i) *every small subset of X is bounded;*
(ii) *every piecewise large subset of X is large;*
(iii) *$\nu(\mathcal{B})$ is a singleton.*

Proof. (i) Assume, otherwise, that some small subset Y of X is unbounded. Then Y is large, but by the definition a small subset can not be large.

(ii) Let Y be a piecewise large subset of X. If Y is unbounded, then Y is large by the assumption. Otherwise, Y is bounded, but every bounded subset of a proper ballean is small, so Y is not piecewise large.

(iii) Suppose that $|\nu(\mathcal{B})| > 1$ and choose two distinct elements $a, b \in \nu(\mathcal{B})$. By the definition of $\nu(\mathcal{B})$, there exists a slowly oscillating function $h : X \to [0, 1]$ such that $h^\beta(a) = 0,\ h^\beta(b) = 1$. Put $A = \{x \in X : h(x) \in [0, \frac{1}{3}]\},\ B = \{x \in X : h(x) \in [\frac{2}{3}, 1]\}$. Clearly, A, B are unbounded, but $B(A, \alpha) \setminus B \neq \varnothing$ for every $\alpha \in P$, so A is not large. □

The next three example show that every condition from (i), (ii), (iii) separately does not imply that every unbounded subset of X is large.

Example 3.4. Let X be an infinite set and let φ be a filter of all cofinite subsets of X. Then a subset Y is small in the ballean $\mathcal{B}(X, \varphi)$ if and only if Y is finite, so every small subset is bounded. On the other hand, a subset Y is large if and only if $X \setminus Y$ is finite. Thus, if Y and $X \setminus Y$ are infinite, then Y is unbounded, but Y is not large.

Example 3.5. Let X be an uncountable set, S be a group of all permutations of X, $\mathcal{B} = (X, \mathcal{F}(X), B)$ be a ballean defined in Example 3.2. By [18, Theorem 1.2], a subset Y of X is large if and only if $|Y| = |X|$, a subset Y is small if and only if $|Y| < |X|$. Hence, every piecewise large subset of X is large. On the other hand, a subset Y of X is bounded if and only if Y is finite. Thus, if Y is an infinite subset of X and $|Y| < |X|$, then Y is unbounded, but Y is not large.

Example 3.6. Let G be an infinite group with the identity e. Denote by $\mathcal{F}$ the family of all finite subsets of G containing e. Given any $g \in G$ and $F \in \mathcal{F}$, we put

$$B_l(g, F) = gF, \; B_r(g, F) = Fg,$$

and consider the balleans $\mathcal{B}_l(G) = (G, \mathcal{F}, B_l)$, $\mathcal{B}_r(G) = (G, \mathcal{F}, B_r)$. Clearly, $\mathcal{B}_l(G)$ and $\mathcal{B}_r(G)$ are proper. Now let G be an uncountable Abelian group. By [15, Theorem 4], $\nu(\mathcal{B}_l(G))$ is a singleton. A subset Y of G is large if and only if Y is finite. If a subset Y of G is large, then $|Y| = |G|$. Hence, if Y is infinite and $|Y| < |G|$, then Y is unbounded, but Y is not large.

Let $\mathcal{B} = (X, P, B)$ be a proper ballean satisfying the both conditions (i) and (ii) of Theorem 3.3. Then, clearly, every unbounded subset of X is large. It follows from Example 3.2 and Example 3.4, that the classes of proper balleans satisfying the conditions (i) and (ii) of Theorem 3.3 respectively are not incident.

4. Maximal balleans on groups

Let G be a group with the identity e, $\mathcal{B} = (G, P, B)$ be a ballean. Following [17], we say that $\mathcal{B}$ is

- *left* (resp. *right*) *invariant* if all the shifts $x \mapsto gx$ (resp. $x \mapsto xg$) are $\prec$-mappings;
- *uniformly left* (resp. *right*) invariant if, for every $\alpha \in P$, there exists $\beta \in P$ such that $gB(x, \alpha) \subseteq B(gx, \beta)$ (resp. $B(x, \alpha)g \subseteq B(xg, \beta)$) for all $x, g \in G$;
- *a group ballean* if it is uniformly left and right invariant.

A family $\mathcal{I}$ of subsets of a set X is called an *ideal* if, for any subsets $A, B \in \mathcal{I}$ and $A' \subseteq A$, we have $A \bigcup B \in \mathcal{I}$ and $A' \in \mathcal{I}$. A subset $\mathcal{I}' \subseteq \mathcal{I}$ is called a *base* for $\mathcal{I}$ if, for every $A \in \mathcal{I}$, there exists $A' \in \mathcal{I}'$ such that $A \subseteq A'$.

We say that an ideal $\mathcal{I}$ on a group G is a *group ideal* if, for any subsets $A, B \in \mathcal{I}$, we have $AB \in \mathcal{I}$ and $A^{-1} \in \mathcal{I}$.

Given a ballean $\mathcal{B} = (G, P, B)$, we say that a subset $A \subseteq G$ is *bounded from the identity* if there exists $\alpha \in P$ such that $A \subseteq B(e, \alpha)$.

For every uniformly left invariant ballean $\mathcal{B} = (G, P, B)$, the family $\mathcal{I}$ of all subsets of G bounded from the identity is a group ideal on G. Moreover, the identity mapping $id : G \to G$ is an asymorphism between $\mathcal{B}$ and the ballean $(G, \mathcal{I}, B_l)$, where $B_l(g, A) = gA \bigcup \{g\}$ for all $g \in G, A \in \mathcal{I}$. On the other hand, for every group ideal $\mathcal{I}$ on G, $(G, \mathcal{I}, B_l)$ is a uniformly left invariant ballean. Thus, we get the natural correspondence between the family of all uniformly left invariant balleans on G and the family of all group ideals on G. Following this correspondence, given an arbitrary group ideal $\mathcal{I}$ on G, we write $(G, \mathcal{I})$ instead of $(G, \mathcal{I}, B_l)$. We say that a group ideal is *proper* if $(G, \mathcal{I})$ is a proper ballean. It is easy to see that a group ideal $\mathcal{I}$ is proper if and only if $G \notin \mathcal{I}$ and $\mathcal{F}(G) \subseteq \mathcal{I}$, where $\mathcal{F}(G)$ is the ideal of all finite subsets of G.

Let G be an infinite group of regular cardinality. We put $X = G$ and consider the ballean $\mathcal{B}$ on G defined in Example 2.2. It is easy to see that $\mathcal{B}$ is left (and right) invariant.

Theorem 4.1. *Let G be an infinite group and let $\mathcal{I}$ be a proper group ideal on G. If the ballean $\mathcal{B} = (G, \mathcal{I})$ is maximal, then the subset $\{g^2 : g \in G\}$ is bounded in $(G, \mathcal{I})$.*

Proof. For every $x \in G$, we put $B'(x, 1) = \{x, x^{-1}\}$. Assume that $\mathcal{B}' \nsubseteq \mathcal{B}$, where $\mathcal{B}'$ is the monoball structure $(G, \{1\}, B')$. By Theorem 2.1, *env* $(\mathcal{B} \bigcup \mathcal{B}')$ is bounded. Note that every bounded subset of *env* $(\mathcal{B} \bigcup \mathcal{B}')$ is bounded in $\mathcal{B}$, so $\mathcal{B}$ is bounded, whence a contradiction. Hence, $\mathcal{B}' \subseteq \mathcal{B}$. Pick $A \in \mathcal{I}$ such that $\{x, x^{-1}\} \in xA$ for every $x \in G$, so $x^{-2} \in A$ and $\{g^2 : g \in G\}$ is bounded. □

It follows from Theorem 4.1, that if an Abelian group G admits a maximal group ballean, then the factor group $G/2G$ is infinite. In particular, there are no maximal group balleans on $\mathbb{Z}$. We show (Example 4.3) that, under CH, there are maximal group ballean on the countable Abelian group of exponent 2, but we begin with more simple example.

Example 4.2. Let G be a countable Abelian group of exponent 2. Under CH we construct a proper group ideal $\mathcal{J}$ on G such that every unbounded subset of $(G, \mathcal{J})$ is large. We use the following auxiliary statement.

If $\mathcal{I}$ is a proper group ideal with countable base on G and A is an unbounded subset of $(G, \mathcal{I})$, then there exists a proper group ideal $\mathcal{I}'$ with countable base on G such that $\mathcal{I} \subseteq \mathcal{I}'$ and A is large in $(G, \mathcal{I}')$. Let $\{B_n : n \in \omega\}$ be a countable base for $\mathcal{I}$ such that $0 \in B_0$, $B_n \subseteq B_{n+1}$ for every $n \in \omega$. Let $\{g_n : n \in \omega\}$ be a numeration of G. To construct $\mathcal{I}'$, we choose inductively an injective sequence $(c_n)_{n\in\omega}$ in G such that $c_n + g_n \in A$ for each $n \in \omega$, and the group ideal $\mathcal{I}' = \mathcal{I} \bigcup \mathcal{I}(C)$ is proper, where $C = \{c_n : n \in \omega\}$, $\mathcal{I}(C)$ is the smallest group ideal containing C, $\mathcal{I}'$ is the smallest group ideal such that $\mathcal{I} \subseteq \mathcal{I}', \mathcal{I}(C) \subseteq \mathcal{I}'$.

We fix an arbitrary sequence $(x_n)_{n\in\omega}$ in A going to infinity with respect to $\mathcal{I}$ (i.e. for every $F \in \mathcal{I}$, there exists $m \in \omega$ such that $x_n \notin F$ for every $n \geq m$). Clearly, the sequence $(x_n + g_m)_{n\in\omega}$ is going to infinity with respect to $\mathcal{I}$ for every $m \in \omega$. Therefore we can choose inductively a subsequence $(d_n)_{n\in\omega}$ of $(x_n)_{n\in\omega}$ and a sequence $(y_n)_{n\in\omega}$ in G going to infinity with respect to $\mathcal{I}$ such that, for each $n \in \omega$, we have

$$y_n \notin B_n + \{\sum_{i\in\omega} m_i(d_i + g_i) : m_i \in \{0, 1\}, \sum_{i\in\omega} m_i \leq n\}.$$

After that we put $c_n = d_n + g_n$ and note that, for every $n \in \omega$,

$$y_n \notin B_n + \underbrace{C + \cdots + C}_{n},$$

where $C = \{c_i : i \in \omega\}$. Hence, $\mathcal{I}' = \mathcal{I} \bigcup \mathcal{I}(C)$ is a proper group ideal and A is large in $(G, \mathcal{I}')$.

To construct the ideal $\mathcal{J}$, we enumerate $\{A_\alpha : \alpha < \omega_1\}$ the family of all subset of G. Fix an arbitrary proper group ideal $\mathcal{I}_0$ with countable base on G. If A_0 is bounded in $(G, \mathcal{I}_0)$, we put $\mathcal{I}'_0 = \mathcal{I}_0$. Otherwise, we choose a proper group ideal $\mathcal{I}'_0$ with countable base such that $\mathcal{I}_0 \subseteq \mathcal{I}'_0$ and A_0 is large in $(G, \mathcal{I}'_0)$. Assume that, for some ordinal $\alpha < \omega_1$, we have chosen the proper group ideals $\mathcal{I}'_\beta,\ \beta < \alpha$ with countable bases. Put $\mathcal{I}_\alpha = \bigcup_{\beta<\alpha} \mathcal{I}'_\beta$ and note that $\mathcal{I}_\alpha$ is a proper group ideal with countable base. If A_α is bounded in $(G, \mathcal{I}_\alpha)$, we put $\mathcal{I}'_\alpha = \mathcal{I}_\alpha$. Otherwise, we choose a proper group ideal $\mathcal{I}'_\alpha$ with countable base such that $\mathcal{I}_\alpha \subseteq \mathcal{I}'_\alpha$ and A_α is large in $(G, \mathcal{I}'_\alpha)$. By the construction, the family $\{\mathcal{I}'_\alpha : \alpha < \omega_1\}$ is well-ordered by inclusion. Put $\mathcal{J} = \bigcup_{\alpha<\omega_1} \mathcal{I}'_\alpha$.

Example 4.3. Let G be a countable Abelian group of exponent 2. Under CH, we construct a proper group ideal $\mathcal{J}$ on G such that $(G, \mathcal{J})$ is maximal. We use the following auxiliary statement. Let $\mathcal{I}$ be a proper group ideal with countable base on G, $\mathcal{B}' = (G, \{1\}, B')$ be a monoball structure such that $\mathcal{B}' \nsubseteq \mathcal{B}$. Then there exists a proper group ideal $\mathcal{I}'$ with countable base such that $\mathcal{I} \subseteq \mathcal{I}'$ and *env* $(\mathcal{B}' \bigcup (G, \mathcal{I}'))$ is bounded.

Let $F \in \mathcal{I},\ X_F = \bigcup\{B'(x,1) : B'(x,1) \bigcap F \neq \varnothing\}$. If X_F is unbounded in $(G, \mathcal{I})$, we take a proper group ideal $\mathcal{I}'$ with countable base such that $\mathcal{I} \subseteq \mathcal{I}'$ and X_F is large in $(G, \mathcal{I}')$ (see Example 4.2). Since X_F is bounded in *env*$(\mathcal{B}' \bigcup (G, \mathcal{I}))$, we have that *env*$(\mathcal{B}' \bigcup (G, \mathcal{I}'))$ is bounded.

Assume that X_F is bounded in $(G, \mathcal{I})$ for every $F \in \mathcal{I}$. Let $\{B_n : n \in \omega\}$ be a base for $\mathcal{I}$ such that $0 \in B_n,\ B_n \subset B_{n+1},\ n \in \omega$. Then we can choose inductively a sequence $(x_n)_{n\in\omega}$ in G such that the family $\{x_n + B_n : n \in \omega\}$ is disjoint and $(x_n + B_n) \setminus B'(x_n, 1) \neq \varnothing$ for every $n \in \omega$. For every $n \in \omega$, pick $y_n \in (x_n + B_n) \setminus B'(x_n, 1)$. Since the sequence $(y_n)_{n\in\omega}$ is going to infinity with respect to $\mathcal{I}$, we can choose inductively, using the arguments from example 4.2, an injective subsequence $(y_{n_k})_{k\in\omega}$ of $(y_n)_{n\in\omega}$ such that the ideal $\mathcal{I}''$ with the base

$$\{B_n + \underbrace{(Y + \cdots + Y)}_{n} : n \in \omega\}$$

is proper, where $Y = \{y_{n_k} : k \in \omega\}$ and the sequence $(x_{n_k})_{k\in\omega}$ is unbounded in $(G, \mathcal{I}'')$. Then we choose an ideal $\mathcal{I}'$ with countable base such that $\mathcal{I}'' \subseteq \mathcal{I}'$ and the set $\{x_{n_k} : k \in \omega\}$ is large in $(G, \mathcal{I}')$. Since Y is bounded in $(G, \mathcal{I}'')$, $\{x_{n_k} : k \in \omega\}$ is bounded in *env* $(\mathcal{B}' \bigcup (G, \mathcal{I}''))$. Since $\{x_{n_k} : k \in \omega\}$ is large in $\mathcal{I}'$ and $\mathcal{I}'' \subseteq \mathcal{I}'$, we conclude that *env* $(\mathcal{B}' \bigcup (G, \mathcal{I}'))$ is bounded.

To construct the ideal $\mathcal{J}$, we use CH to enumerate $\{\mathcal{B}_\lambda : \lambda < \omega_1\}$ the set of all monoball structure on G. Repeating the arguments from example 4.2, we get $\mathcal{J}$ such that, if $\mathcal{B}'_\lambda \nsubseteq (G, \mathcal{I})$, then *env* $(\mathcal{B}'_\lambda \bigcup (G, \mathcal{J}))$ is bounded. Then we apply Theorem 2.1.

Question 4.4. *Does there exist in ZFC (without additional set-theoretic assumption) an infinite group G and a proper group ideal $\mathcal{I}$ on G such that every unbounded subset of $(G, \mathcal{I})$ is large? Is $(G, \mathcal{I})$ maximal?*

Question 4.5. *Assume CH. Does every countable Abelian group G admit a proper group ideal $\mathcal{I}$ such that every unbounded subset of $(G, \mathcal{I})$ is large?*

To answer Question 4.5 in affirmative, it suffices to prove the following statement. Let G be a countable Abelian group, $\mathcal{I}$ be a proper group ideal with countable base on $\mathcal{I}$, A be an unbounded subset of $(G, \mathcal{I})$. Then there exists a proper group ideal $\mathcal{I}'$ with countable base on G such that $\mathcal{I} \subseteq \mathcal{I}'$ and A is large in $(G, \mathcal{I}')$. We have proved (in Example 4.2) that this statement is true if G is a group of exponent 2 and this was crucial for the construction of the ideal $\mathcal{J}$. Unfortunately, this statement is not true in general as the following example shows.

Example 4.6. Let $G = \bigoplus_{\alpha<\omega} G_\alpha$ be a direct sum of cyclic groups G_α of order 4. Let $\mathcal{I}$ be the ideal of finite subsets of G. Put $A = 2G = \{2g : g \in G\}$ and note that A is unbounded in $(G, \mathcal{I})$. Assume that there exists a proper group ideal $\mathcal{I}'$ on G such that A is large in $(G, \mathcal{I}')$. Choose $F \in \mathcal{I}'$ such that $G = F + A$. Then $2G = 2F$ so A is bounded subset of $(G, \mathcal{I}')$. Since A is large in $(G, \mathcal{I}')$, we conclude that $(G, \mathcal{I}')$ is bounded, whence a contradiction.

Question 4.7. *Does there exist in ZFC a countable group G and a proper group ideal $\mathcal{I}$ on G such that every small subset of $(G, \mathcal{I})$ is bounded?*

Question 4.8. *Does there exist in ZFC a countable group G and a proper group ideal $\mathcal{I}$ on G such that every piecewise large subset of $(G, \mathcal{I})$ is large?*

In view of Theorem 3.3 and Example 4.2, under CH the answers to the Questions 4.7 and 4.8 are positive.

Question 4.9. *Let G be an infinite group, $\mathcal{I}$ be a proper group ideal on G such that every small subset of $(G, \mathcal{I})$ is bounded. Is every piecewise large subset of $(G, \mathcal{I})$ large?*

Question 4.10. *Let G be an infinite group, $\mathcal{I}$ be a proper group ideal on G such that every piecewise large subset of $(G, \mathcal{I})$ is large. Is every small subset of $(G, \mathcal{I})$ bounded?*

To answer Question 4.9 positively, it suffices to give the affirmative answer to the following question.

Question 4.11. *Let G be an infinite group, $\mathcal{I}$ be a proper group ideal on G, A be a piecewise large subset of $(G, \mathcal{I})$. Does there exist a small subset S of $(G, \mathcal{I})$ such that $G = AS$?*

We show that, if G is Abelian and $\mathcal{I}$ has a countable base, the Question 4.11 has the positive answer. We use the following auxiliary statement.

Let G be an infinite group, $\mathcal{I}$ be a proper group ideal on G. Let L be a subset of G such that, for every $F \in \mathcal{I}$, there exists $x \in L$ such that $xF \subseteq L$. Then, for every $F \in \mathcal{I}$, there exists $x \in L$ such that $xF \subseteq L$ and $xF \bigcap F = \varnothing$.

To prove this statement, we may suppose that $e \in F$, $F = F^{-1}$. Choose $H \in \mathcal{I}$ such that $F^4 \subset H$ and pick $h \in H \setminus F^4$. By assumption, there exists

$x \in L$ such that $xHF \subseteq L$. Assume that $xF \bigcap F \neq \varnothing$ and $xhF \bigcap F \neq \varnothing$. Then $x \in F^2$, $h \in x^{-1}F^2$, so $h \in F^4$. Hence, either $xF \bigcap F = \varnothing$ or $xhF \bigcap F = \varnothing$.

Now assume that G is Abelian and $\mathcal{I}$ has a countable base. Let A be a piecewise large subset of $(G, \mathcal{I})$. Let $\{B_n : n \in \omega\}$ be a base for $\mathcal{I}$ such that $e \in B_0$, $B_n \subset B_{n+1}$ for each $n \in \omega$. Since A is piecewise large, there exists $C \in \mathcal{I}$ such that, for every $F \in \mathcal{I}$, there exists $x \in AC$ such that $xF \subseteq AC$. Using the auxiliary statement with $L = AC$, we can choose inductively a sequence $(x_n)_{n\in\omega}$ in AC such that $x_nB_n \subseteq AC$ and the family $\{x_nB_n : n \in \omega\}$ is disjoint. Put $X = \{x_n : n \in \omega\}$ and note that $x_mB_n \nsubseteq X$ for every $m \geq n$. It follows that X is small. Since $x_nB_n \subseteq AC$ for every $n \in \omega$ and $\bigcup_{n\in\omega} B_n = G$, we have $G = X^{-1}AC$. Since G is Abelian, $G = A(X^{-1}C)$. Put $S = X^{-1}C$. Since X^{-1} is small and C is bounded, S is small.

It should be mentioned that our consideration of maximal balleans on group was motivated by maximal topologies on groups. A topological space X without isolated points is called maximal if X has an isolated point in every stronger topology. Every infinite group has a plenty of left invariant topologies that are maximal, among them there are even regular topologies [10]. On the other hand [5], every maximal topological group has a countable open Abelian subgroup of exponent 2. Under MA, the examples of maximal topological groups were constructed in [5], but the existence of a maximal topological group implies P-point in $\omega^\star$ [8], so maximal topological groups can not be constructed in ZFC without additional set-theoretic assumptions.

5. Maximality and irresolvability

A topological space X without isolated points is called irresolvable if X can not be partitioned into two dense subsets. For resolvability of topological spaces and topological groups see the surveys [1, 2, 9].

By analogy, a proper ballean $\mathcal{B} = (X, P, B)$ is called *irresolvable* if X can not be partitioned into two large subsets. For resolvability of balleans see [16].

Let $\mathcal{B} = (X, P, B)$ be a proper ballean. We say that an ultrafilter φ on X is going to infinity in $\mathcal{B}$ if every member of φ is unbounded.

Theorem 5.1. *Let $\mathcal{B} = (X, P, B)$ be a proper ballean. Then the following statements are equivalent:*

(i) *there exists only one ultrafilter on X going to infinity in $\mathcal{B}$;*
(ii) *there exists an ultrafilter φ on X such that $\mathcal{B} = \mathcal{B}(X, \varphi)$;*
(iii) *$\mathcal{B}$ is maximal and irresolvable.*

Proof. (i)⇒(ii). Let φ be the ultrafilter on X going to infinity in $\mathcal{B}$. For every $\alpha \in P$, we put $Y_\alpha = \{x \in X : B(x, \alpha) = \{x\}\}$ and show that $Y_\alpha \in \varphi$. Suppose, otherwise, that $X \setminus Y_\alpha \in \varphi$. For every $x \in X \setminus Y_\alpha$, we take an element $f(x) \in B(x, \alpha)$ such that $f(x) \neq x$. For every $x \in Y_\alpha$, we put $f(x) = x$. Thus, we get the mapping $f : X \to X$. Since $\{x \in X : f(x) = x\} \notin \varphi$, we have $f^\beta(\varphi) \notin \varphi$, where f^β is the Stone-Čech extension of f and X is endowed with the discrete topology. Clearly, the ultrafilter $f^\beta(\varphi)$ is going to infinity in $\mathcal{B}$, a contradiction. Hence, the family $\{Y_\alpha : \alpha \in P\}$ is a base for some filter ψ on X and $\psi \subseteq \varphi$. If φ' is an ultrafilter on X and $\psi \subseteq \varphi'$, then φ' is going to infinity in $\mathcal{B}$, so $\varphi = \psi$. If $X \setminus Y_\alpha$ is unbounded in $\mathcal{B}$, then there exists an ultrafilter η on X going to infinity in $\mathcal{B}$ such that $X \setminus Y_\alpha \in \eta$. Hence, $X \setminus Y_\alpha$ is bounded for every $\alpha \in P$. We show that $\mathcal{B} = \mathcal{B}(X, \varphi)$. If $\alpha \in P$ and $x \in Y_\alpha$, then $B_\varphi(x, Y_\alpha) = B(x, \alpha)$. If $\alpha \in X \setminus Y_\alpha$, then we choose $\beta \in P$ such that $X \setminus Y_\alpha \subseteq B(x, \beta)$ for every $x \in X \setminus Y_\alpha$, so $B_\varphi(x, Y_\alpha) \subseteq B(x, \beta)$. Hence, $\mathcal{B}(X, \varphi) \subseteq \mathcal{B}$. Clearly, $\mathcal{B} \subseteq \mathcal{B}(X, \varphi)$ and we have $\mathcal{B} = \mathcal{B}(X, \varphi)$.

(ii)⇒(iii). By [16, Proposition 1], $\mathcal{B}(X, \varphi)$ is irresolvable, maximality of $\mathcal{B}(X, \varphi)$ follows from Example 2.3.

(iii)⇒(i). For every $\alpha \in P$, we put $Y_\alpha = \{x \in X : B(x, \alpha) = \{x\}\}$. Since $\mathcal{B}$ is irresolvable, the family $\{Y_\alpha : \alpha \in P\}$ is a base for some filter φ on X. Assume that there exists an unbounded subset Z of X such that $Y_\alpha \nsubseteq Z$ for every $\alpha \in P$. We put

$$B'(x, 1) = \begin{cases} \{x\}, & if \;\; x \notin Z; \\ Z, & if \;\; x \in Z; \end{cases}$$

and consider the monoball structure $\mathcal{B}' = (X, \{1\}, P')$. Clearly, $\mathcal{B}' \nsubseteq \mathcal{B}$. On the other hand, $B(Z, \alpha) \neq X$ for every $\alpha \in P$, so *env* $(\mathcal{B}' \bigcup \mathcal{B})$ is unbounded. By Theorem 2.1, $\mathcal{B}$ is not maximal. Hence, φ is an ultrafilter and the only ultrafilter on X going to infinity in $\mathcal{B}$. □

Let X be an infinite set, φ be a filter on X such that $\bigcap \varphi = \varnothing$. Then the proper ballean $\mathcal{B}(X, \varphi)$ is irresolvable. If φ is not ultrafilter, then $\mathcal{B}(X, \varphi)$ is not maximal. On the other hand, the maximal ballean $\mathcal{B}$ from Example 2.2 is resolvable by Proposition 1 from [16]. Thus, in contrast to topological situation, the classes of irresolvable and maximal balleans are not incident.

Let G be an infinite group and let $\mathcal{I}$ be a proper group ideal on G. Pick $F \in \mathcal{I}$ such that $e \in F$ and $|F| > 1$. Then $|gF| > 1$ for every $g \in G$ and, by Proposition 1 from [16], the ballean $(G, \mathcal{I})$ is resolvable. Thus, every proper uniformly left invariant ballean on a group is resolvable.

A filter φ on a group G is called *left invariant* if $gF \in \varphi$ for any $F \in \varphi$ and $g \in G$. If φ is a left invariant filter on G and $\bigcap \varphi = \varnothing$, then the ballean $\mathcal{B}(G, \varphi)$ is irresolvable and left invariant. By [4, Theorem 6.42], for every infinite group G, there exist $2^{2^{|G|}}$ left invariant filters, so G admits a plenty of irresolvable left invariant balleans.

Let $\mathcal{B} = (G, P, B)$ be an irresolvable left invariant ballean. For every $\alpha \in P$, we put $Y_\alpha = \{g \in G : B(g, \alpha) = \{g\}\}$ and note that the family $\{Y_\alpha : \alpha \in P\}$ is a base of some left invariant filter φ on G. Clearly, $\mathcal{B} \subseteq \mathcal{B}(G, \varphi)$, but the suspicion that always $\mathcal{B} = \mathcal{B}(G, \varphi)$ does not hold.

Example 5.2. Let $G = \mathbb{Z}$. For any $z \in \mathbb{Z}$ and $(m, n) \in \mathbb{N} \times \mathbb{N}$, we put

$$B(z, (m, n)) = \begin{cases} \{z\}, & if \;\; z \geq n; \\ \{y \in \mathbb{Z} : |y - z| \leq m\}, & if \;\; z < n; \end{cases}$$

and consider the ballean $\mathcal{B} = (\mathbb{Z}, \mathbb{N} \times \mathbb{N}, B)$. Clearly, $\mathcal{B}$ is irresolvable and left invariant. In this case, the filter φ has the base $\{F_n : n \in \omega\}$, where $F_n = \{z \in \mathbb{Z} : z \geq n\}$, but $\mathcal{B}(G, \varphi)$ is stronger than $\mathcal{B}$.

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O. I. Protasova (polla@unicyb.kiev.ua)
Department of Cybernetics, Kyiv National University, Volodimirska 64, Kiev 01033, UKRAINE.