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Quasicontinuous Functions, Domains, and Extended Calculus

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Abstract. One of the aims of domain theory is the construction of an embedding of a given structure or data type as the maximal or "ideal" elements of an enveloping domain of "approximations," sometimes called a domain environment. Typically the goal is to provide a computational model or framework for recursive and algorithmic reasoning about the original structure. In this paper we consider the function space of (natural equivalence classes of) quasicontinuous functions from a locally compact space X into L, an n-fold product of the extended reals $[-\infty, \infty]$ (more generally, into a bicontinuous lattice). We show that the domain of all "approximate maps" that assign to each point of X an order interval of L is a domain environment for the quasicontinuous function space. We rely upon the theory of domain environments to introduce an interesting and useful function space topology on the quasicontinuous function space. We then apply this machinery to define an extended differential calculus in the quasicontinuous function space, and draw connections with viscosity solutions of Hamiltonian equations. The theory depends heavily on topological properties of quasicontinuous functions that have been recently uncovered that involve dense sets of points of continuity and sections of closed relations and USCO maps. These and other basic results about quasicontinuous functions are surveyed and presented in the early sections.

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1. Introduction

Recall that a function $f : X \to Y$ between topological spaces is *quasicontinuous* if the inverse image of every open set is quasi-open, that is, has dense interior. Although such maps have been considered for some time [11], there has been a recent revival of interest in their topological study (e.g. the works of Borsík [2],[3]) and in their study in a variety of applications such as selection theorems for set-valued maps [4], [7], the dynamics of quasicontinuous functions under iteration [7], and viscosity solutions of certain partial differential equations [19], [20].

The primary purpose of this paper is to introduce the tools of domain theory to the study of quasicontinuous function spaces and point toward applications in nonsmooth analysis. An important aim of computational domain theory is to develop computationally useful mathematical models of data types (e.g., Booleans, integers, reals, and the higher types derived from them) that incorporate both the data types and computationally realizable approximations. These models can be useful for providing a theoretical computational framework for studying computational issues and questions, for investigating and developing a theory of computability, and in some cases for suggesting computational algorithms or approaches. The mathematical models considered here are *continuous domains*, a special class of partially ordered sets that are

- *directed complete*: each directed set has a supremum, and
- *continuous*: each element is a directed supremum of its ("finitary") approximations,

where x (*finitarily*) *approximates* y, written $x \ll y$, if $y \leq \sup D$, D a directed set, implies $x \leq d$ for some $d \in D$. The intuition is that directed sets represent partial states of knowledge or stages in a computation, and $x \ll y$ if any computation of y reveals x at some stage of the computation.

For some of the more important classes of quasicontinuous function spaces, we shall see that domain theory provides a nice and natural approach for their study. In addition, domain theory suggests a useful function space topology for these classes of functions, a function space topology that is built up from one that has been well-studied in domain theory. This approach provides a significant advantage for the study of quasicontinuous functions, since this function space topology has no clear counterpart for general sets of quasicontinuous functions, yet has many useful properties, as we shall see. As an application of the quasicontinuous function space and its domain setting, we close with a brief foray into a generalized differential calculus that employs quasicontinuous functions and suggest connections with viscosity solutions of partial differential equations.

For another approach to differential calculus and analysis via domain theory we refer the reader to recent work of Abbas Edalat, see for example [9], [8], and related works. There are certainly significant overlaps (and significant differences) between his notion and theory of a domain derivative and the generalized derivative treated in this paper. Keye Martin has introduced an alternative approach via the "informatic derivative," which again seeks a domain-based approach to differentiation and generalizations thereof [13],[14].

The following second and third sections of the paper survey and extend a variety of recent developments concerning topological aspects of quasicontinuous functions. In particular, we review and develop in our own framework the work of Crannell, Frantz, and LeMasurier [7] and others regarding equivalence classes of quasicontinuous functions arising from their graph closures and related selection theorems. In the third section we derive the important equivalence between classes of quasicontinuous maps and minimal USCO maps.

The fourth section develops an approach to quasicontinuous functions from the viewpoint of domain theory. We show how quasicontinuous function classes arise naturally in this context and can be fruitfully treated using basic ideas and results of domain theory. In particular domain theory suggests a useful function space topology for the quasicontinuous function space, which we explore in Sections 5 and 6, and this is a main motivation for introducing domain theory.

In the seventh section we consider extensions of the differential calculus to certain classes of quasicontinuous functions and in the eighth section explore Samborski's ideas and results concerning the use of quasicontinuous functions as viscosity solutions of certain partial differential equations, particularly those arising in Hamiliton-Jacobi theory [19], [20]. Indeed it was his work that was a major inspiration for this paper, and significant portions of this paper constitute a survey and amplification of his work, with the twist of a domain-theoretic perspective.

Much of the material in the latter portions of the paper are drawn from the dissertation of the first author [6].

2. Quasicontinuous Functions

A function $f : X \to Y$ is *quasicontinuous at* x if for any open set V containing $f(x)$ and any U open containing x, there exists a nonempty open set $W \subseteq U$ such that $f(W) \subseteq V$. It is *quasicontinuous* if it is quasicontinuous at every point. For an overview of the theory of quasicontinuous functions together with a rather extensive bibliography, we recommend the survey article of T. Neubrunn [17].

Call a set *quasi-open* (or *semi-open*) if it is contained in the closure of its interior. Then $f : X \to Y$ is quasicontinuous if and only if the inverse of every open set is quasi-open. It then follows that $g \circ f$ is quasicontinuous whenever g is continuous and f is quasicontinuous.

Some basic examples of quasicontinuous functions are the doubling function

$$D : [0,1) \to [0,1) \text{ defined by } D(x) = 2x \pmod 1,$$

the floor function from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\},$$

and the extended $\sin(1/x)$ function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sin(\frac{1}{x}) & \text{otherwise.} \end{cases}$$

The doubling map is a basic example in the study of the dynamics of real-valued functions, and indicates why there is interest in a general theory of the dynamics of quasicontinuous functions.

2.1. Graph closures.

Definition 2.1. *Let X, Y be topological spaces and $f : X \to Y$ a function. As usual, we identify f with its graph, $f = \{(x, y) \in X \times Y : y = f(x)\}$, and define the* graph closure $\overline{f}$ *as the closure $cl(f)$ of f in $X \times Y$. We define*

$$\overline{f}(x) := \{y \in Y : (x, y) \in \overline{f}\}, \qquad \overline{f}(A) := \bigcup_{x \in A} \overline{f}(x).$$

The same construction of the graph closure $\overline{h}$, closure taken in $X \times Y$, extends to partial functions $h : X \rightharpoonup Y$, functions defined from a subset D of X into Y. Note that the domain $\pi_X(\overline{h})$ of the relation $\overline{h}$ may be strictly larger than that of h. For partial functions h, $\overline{h}(x)$ and $\overline{h}(A)$ are defined by the same formulas, but now the possibility exists that they may be empty. Finally if $f : X \to Y$ is a function and $D \subseteq X$, then we denote by $f \restriction D$ the partial function that arises by restricting f to D and by $\overline{f \restriction D}$ the closure of this partial function in $X \times Y$.

If $f : X \to Y$ is continuous and Y is Hausdorff, then the graph closure of f is again f. Several elementary facts about continuous functions extend to quasicontinuous functions if we work with graph closures instead of the quasicontinuous functions themselves. For example, the fact that a continuous function on a dense subset has at most one continuous extension to the whole space becomes

Lemma 2.2. *If $f : X \to Y$ is quasicontinuous and $D \subseteq X$ is dense, then $\overline{f} = \overline{f \restriction D}$.*

Proof. It suffices to show that $f \subseteq \overline{f \restriction D}$, and then take closures. Let $y = f(x)$ and let $x \in U$, $y \in V$, where U is open in X and V is open in Y. By quasicontinuity, there exists W open in X such that $W \subseteq U$ and $f(W) \subseteq V$. There exists $w \in D \cap W$, and then $(w, f(w)) \in U \times V$. It follows that $f \subseteq \overline{f \restriction D}$. $\square$

Our focus will be more on equivalence classes determined by graph closures of quasicontinuous functions than on the individual functions themselves (see [7]).

Definition 2.3. *Two (arbitrary) functions $f, g : X \to Y$ are said to be* closed graph equivalent *or simply* equivalent *if $\overline{f} = \overline{g}$. We write $f \sim g$ if $\overline{f} = \overline{g}$, and denote the equivalence class of f by $[f]$.*

Corollary 2.4. *If $f, g : X \to Y$ agree on a dense subset of X and f is quasicontinuous, then $\overline{f} \subseteq \overline{g}$. Hence $f \sim g$ if both are quasicontinuous.*

Proof. We have from Lemma 2.2 for some dense subset D that

$$\overline{f} = \overline{f \restriction D} = \overline{g \restriction D} \subseteq \overline{g},$$

and dually if g is also quasicontinuous. □

There is also a converse of sorts to the preceding corollary, but first we need a small lemma.

Lemma 2.5. *For a partial function $f : X \rightharpoonup Y$, suppose that $f(U) \subseteq V$ for some U open in X. Then $\overline{f}(U) \subseteq V^-$.*

Proof. The inclusion $f \subseteq U \times V \cup U^c \times Y$, where $U^c = X \setminus U$, implies

$$\begin{aligned} \overline{f} &\subseteq U^- \times V^- \cup U^c \times Y \\ &= U \times V^- \cup U^c \times Y, \end{aligned}$$

where the last equality follows from $(U^- \setminus U) \times V^- \subseteq U^c \times Y$. It follows that if $u \in U$, $(u, w) \in \overline{f}$, then $(u, w) \in U \times V^-$, hence $w \in V^-$, and thus $\overline{f}(U) \subseteq V^-$. □

Proposition 2.6. *Suppose that $f, g : X \to Y$, f is quasicontinuous, Y is regular, and $\overline{g} \subseteq \overline{f}$. Then g is quasicontinuous.*

Proof. Let $y = g(x) \in O$ and $x \in U$, where O is open in Y and U is open in X. Pick V open in Y such that $y \in V \subseteq V^- \subseteq O$. Since $\overline{g} \subseteq \overline{f}$, we have $(x, y) \in \overline{f}$, and hence $(u, f(u)) \in U \times V$ for some $u \in U$. By quasicontinuity of f, there exists some nonempty open set $W \subseteq U$ such that $f(W) \subseteq V$. By Lemma 2.5 $\overline{f}(W) \subseteq V^- \subseteq O$. Then $g(W) \subseteq \overline{g}(W) \subseteq \overline{f}(W) \subseteq O$. □

Note 1. *Observe that the preceding proposition shows that the equivalence class $[f]$ of a quasicontinuous function consists of quasicontinuous functions only.*

2.2. **Points of continuity.** In this section we consider sets of points of continuity for quasicontinuous functions.

Note 2. *We denote the set of points of continuity of a function $f : X \to Y$ between two topological spaces by $C(f)$.*

Points of continuity enjoy a type of "extended continuity." (The following lemma slightly generalizes parts of [5, Theorem 3.1].)

Lemma 2.7. *Let D be a subspace of a topological space X, let $f : D \to Y$ be a function that is continuous at some $x \in D$.*

(i) *If $f(x) \in V$, an open subset of Y, then there exists U open in X containing x such that $\overline{f}(U) \subseteq V^-$, and in the case Y is regular, such that $\overline{f}(U) \subseteq V$.*
(ii) *If Y is Hausdorff, then $\overline{f}(x) = \{f(x)\}$.*

Proof. (i) Let $f(x) \in V$, an open set. Pick U open in X such that $x \in U$ and $f(U \cap D) \subseteq V$. Then by Lemma 2.5, $\overline{f}(U) \subseteq V^-$. If X is regular, then we may pick W open such that $y \in W \subset W^- \subseteq V$ and U such that $x \in U$, $\overline{f}(U) \subseteq W^- \subseteq V$.

(ii) If $y \neq f(x)$, then there exists an open set V containing $f(x)$ and an open set W containing y such that $V \cap W = \varnothing$. By part (i) we may pick U open containing x such that $\overline{f}(U) \subseteq V^- \subseteq Y \setminus W$. Thus $y \notin \overline{f}(x)$. Assertion (ii) follows. □

Corollary 2.8. *If $f \sim g$ for $f, g : X \to Y$ and Y is regular Hausdorff, then $C(f) = C(g)$ and f and g agree on this set.*

Proof. Suppose that $x \in C(f)$. By Lemma 2.7(ii), $g(x) \in \overline{g}(x) = \overline{f}(x) = \{f(x)\}$, so $g(x) = f(x)$. Let V be an open set containing $g(x) = f(x)$. Then by Lemma 2.7(i), there exists an open set U containing x such that $g(U) \subseteq \overline{g}(U) = \overline{f}(U) \subseteq V$. Thus $x \in C(g)$. Since the argument is symmetric, $C(f) = C(g)$ and f and g agree on this set. □

Recall that a *Baire space* is one in which in which every countable intersection of dense open sets is dense. The next proposition shows that under rather general hypotheses $C(f)$ is large for quasicontinuous functions.

Proposition 2.9. *If X is a Baire space, Y is a metric space, and $f : X \to Y$ is quasicontinuous, then $C(f)$ is a dense G_δ-set.*

Proof. Recall that the oscillation of f at x is defined by

$$\mathrm{Osc}(f)(x) = \inf\{\mathrm{diam} f(U) : U \text{ open}, x \in U\}.$$

Then it is standard and straightforward to verify that (i) f is continuous at x if and only if $\mathrm{Osc}(f) = 0$, and (ii) for $0 < \varepsilon$, $O_\varepsilon := \{x : \mathrm{Osc}(f)(x) < \varepsilon\}$ is open. It follows easily from the quasicontinuity of f that each O_ε is dense. Thus $\bigcap_n O_{1/n}$ is a dense G_δ-set, and is precisely the set of points of continuity. □

Corollary 2.10. *For $f : X \to Y$, consider the following conditions:*

(1) *f is quasicontinuous;*
(2) $\overline{f} = \overline{f \restriction C(f)}$.

Then (2) *implies* (1) *and the converse holds if $C(f)$ is dense, in particular if X is Baire and Y is metric.*

Proof. (1)⇒(2): An immediate consequence of Lemma 2.2 and Proposition 2.9.

(2)⇒(1): Let $y = f(x)$, $x \in U$, $y \in V$, where U is open in X, V is open in Y. By hypothesis there exists $w \in C(f)$ such that $(w, f(w)) \in U \times V$. Since $w \in C(f)$, there exists an open set W containing w such that $f(W) \subset V$, and $W \cap U$ is the desired neighborhood to establish quasicontinuity at x. □

A *selection function* of the graph closure $\overline{f}$ is a function $\sigma : X \to Y$ whose graph is contained in $\overline{f}$, i.e., $\sigma(x) \in \overline{f}(x)$ for all $x \in X$. Note that the function σ is a selection function of $\overline{f}$ if and only if $\overline{\sigma} \subseteq \overline{f}$.

Theorem 2.11. *Let $f, g : X \to Y$ where f is quasicontinuous, $C(f)$ is dense, and Y is regular Hausdorff. The following are equivalent:*

(1) $g \sim f$.
(2) *g is quasicontinuous and agrees with f on a dense subset.*
(3) *g is a selection function for $\overline{f}$.*

Proof. (2)⇒(1): Corollary 2.4.

(1)⇒(3): Always g is a selection function for $\overline{g}$, hence for $\overline{f}$ if the graph closures are equal.

(3)⇒(2): Let $g : X \to Y$ be a selection function of $\overline{f}$. Then $\overline{g} \subseteq \overline{f}$ and by Proposition 2.6 g is quasicontinuous. By Lemma 2.7(ii), $g(x) \in \overline{f}(x) = \{f(x)\}$, so $g(x) = f(x)$ for all $x \in C(f)$, which is dense. □

3. USCO Maps

Let $F : X \rightrightarrows Y$ be a set-valued map (also called a multifunction). We say that F is *compact-valued* if $F(x)$ is a nonempty compact subset of Y for each $x \in X$, that F is *upper semicontinuous at* $x \in X$ (*usc at* x) if $F(x) \subseteq V$, V open in Y, implies there exists an open neighborhood U of x such that $F(U) = \bigcup_{u \in U} F(u) \subseteq V$, and that F is *upper semicontinuous* (usc) if it is upper semicontinuous at each $x \in X$. If F is both usc and compact-valued, then is is said to be a *USCO map.*

If $F : X \rightrightarrows Y$, we identify F with its graph $\{(x, y) : y \in F(x)\}$. Thus multifunctions can alternatively be viewed as relations. We again let $\overline{F}$ denote the closure in $X \times Y$.

Lemma 3.1. *Let $F : X \rightrightarrows Y$ be USCO and let $R \subseteq F$ be a closed subset. Then the projection $\pi_X(R)$, the domain of R, is closed in X. In particular if the domain of R is dense, it is all of X.*

Proof. Suppose that $x \notin \pi_X(R)$. Then for each $y \in F(x)$, there exist open sets P_y containing x and Q_y containing y such that $(P_y \times Q_y) \cap R = \varnothing$. Since $F(x)$ is compact, finitely many of the $\{Q_y : y \in F(x)\}$ cover $F(x)$. Let Q be their union and P the corresponding intersection of the finite subcollection of the $\{P_y\}$. Since F is USCO, there exists an open set U containing x such that $F(U) \subseteq Q$, and by intersecting with P if necessary, we may assume that $U \subseteq P$. Then

$$(U \times Y) \cap R \subseteq (U \times Y) \cap F \subseteq U \times F(U) \subseteq P \times Q.$$

It follows that $(U \times Y) \cap R \subseteq (P \times Q) \cap R = \varnothing$, and thus U is an open set containing x and missing $\pi_X(R)$. □

Proposition 3.2. *If $F : X \rightrightarrows Y$ is USCO, Y is Hausdorff, and $G \subseteq F$, then G is USCO if and only if $G = \overline{G}$ and the domain of G, $\pi_X(G)$, equals X.*

Proof. Assume that G is closed and the domain of G is X. Then the intersection $G \cap (\{x\} \times F(x))$ is the intersection of a closed set and a compact set, hence a compact set. Thus its projection under π_Y, which is $G(x)$, is also compact. Let

V be an open set containing $G(x)$. By Lemma 3.2, $\pi_X(G \cap (X \times V^c))$ is closed in X, and it follows from a straightforward verification that its complement U in X is an open set containing x and satisfying $F(U) \subseteq V$. Hence F is also usc.

Conversely assume that G is USCO. Then by definition the domain of G is X. Suppose $y \notin G(x)$. Using the Hausdorffness of Y and the compactness of $G(x)$, one finds disjoint open sets V and W such that $y \in V$ and $G(x) \subseteq W$. For U open containing x such that $G(U) \subseteq W$, we have $U \times V \cap G = \varnothing$, so the complement of G is open. $\square$

Remark 3.3. From the preceding proposition we see that the theory of USCO maps is a generalization to more general spaces of the theory of closed relations $R \subseteq X \times Y$ with $\pi_X(R) = X$ for the case of Y compact Hausdorff (since in this case $F = X \times Y$ is a USCO map that contains all closed relations). Given a fixed USCO map $F : X \rightrightarrows Y$, we freely view any closed relation with domain X contained in F as a USCO map and vice-versa, as convenient.

The theory of quasicontinuous functions provides useful and important techniques for constructing special *selections for USCO maps* F, functions σ such that $\sigma(x) \in F(x)$ for all x. The following is a very general recent result of Cao and Moors [4].

Theorem 3.4. *Let X be a Baire space, Y a regular Hausdorff space, and $F : X \rightrightarrows Y$ a USCO map. Then F admits a quasicontinuous selection.*

The theorem is actually more general, and holds for compact-valued multifunctions that are "upper Baire continuous": for each pair of open sets U, W with $x \in U$ and $F(x) \subseteq W$, there exist a nonempty open set $V \subseteq U$ and a residual (Recall that a set is "residual" if it contains a countable intersection of dense open subsets.) set $R \subseteq V$ such that $F(z) \subseteq W$ for all $z \in R$. This work generalizes earlier work of Matejdes [15], who introduced the notion of upper Baire continuous.

We call a USCO map *minimal* if, interpreted as a graph, it contains no strictly smaller USCO map. The next result establishes an equivalence between quasicontinuous selections and minimal selections of USCO maps (where a selection f is *minimal* if $\overline{f}$ is a minimal USCO map). Observe that a selection is minimal if and only if it is a selection function for some minimal USCO map.

Corollary 3.5. *Let $F : X \rightrightarrows Y$ be a USCO map, where X is a Baire space and Y is a metric space, and let $f : X \to Y$ be a selection function. Then the graph closure of f is a minimal USCO map iff f is quasicontinuous. Furthermore, any minimal USCO map is the graph closure of any selection function, and these are all quasicontinuous.*

Proof. Assume that $\overline{f}$ is a minimal USCO map. By Theorem 3.4, $\overline{f}$ has a quasicontinuous selection g. Then $\overline{g} \subseteq \overline{f}$, and thus the two are equal by minimality of $\overline{f}$ and Proposition 3.2. Then f is quasicontinuous by Proposition 2.6.

Conversely, suppose that f is a quasicontinuous selection function for F. Then $\overline{f}$ is a USCO map (Proposition 3.2). Let $H : X \rightrightarrows Y$ be a USCO map such that $H \subseteq \overline{f}$. If h is any selection function for H, then h is also a selection function for $\overline{f}$. By Theorem 2.11 $h \sim f$, and hence $\overline{h} = \overline{f}$. Thus $\overline{h} = H = \overline{f}$ and $\overline{f}$ is minimal.

The last assertion follows along the lines of the first paragraph of the proof. □

Not surprisingly, stronger conclusions are available if we strengthen the hypotheses.

Lemma 3.6. *Let $F : X \rightrightarrows \mathbb{R}$ be USCO, where X is a Baire space. Then F admits a lower semicontinuous quasicontinuous selection function.*

Proof. By Theorem 3.4 F admits a quasicontinuous selection $g : X \to \mathbb{R}$. Then $\overline{g}$ is USCO, and $\overline{g} \subseteq F$. Define $h : X \to \mathbb{R}$ by $h(x) = \inf \overline{g}(x)$. That h is a lower semicontinuous selection of $\overline{g}$ follows directly from the fact that $\overline{g}$ is USCO. From Corollary 3.5 $\overline{g}$ is a minimal USCO map and its selection h is quasicontinuous. □

The following is a theorem of Crannell, Frantz, and LeMasurier [7] , which builds on ideas of W. Miller and E. Akin [16].

Theorem 3.7. *Let $R \subseteq X \times Y$ be a closed relation, where $proj_X(R) = X$, X is a Baire space, and Y is compact metric. Then R admits a quasicontinuous Borel selection function $f : X \to Y$.*

Proof. Let K be the standard Cantor set in $\mathbb{R}$ and let $g : K \to Y$ be a continuous surjective map, which is possible since Y is compact metric. Treating R as a USCO map $R : X \rightrightarrows Y$ (see Remark 3.3), we see that $G := g^{-1} \circ R : X \to K \subset \mathbb{R}$ is a USCO map. By the preceding lemma G admits a lower semicontinuous, quasicontinuous selection σ, and $g \circ \sigma$ is a quasicontinuous, Borel selection for R. □

Note that the selection function is actually the composition of a lower semicontinuous, quasicontinuous real-valued function with a continuous function.

Corollary 3.8. *Let $F : X \rightrightarrows Y$ be USCO, where X is a Baire space and Y is* (*separable*) *metrizable. Then F admits a* (*Borel*) *quasicontinuous selection function that is continuous at a dense set of points.*

Proof. If Y is separable metrizable, we can embed Y in Z, a countable product of the interval $[0, 1]$. We can then extend the codomain of F from Y to Z; note that $F : X \rightrightarrows Z$ is still USCO. Treating F equivalently as a closed relation, we can obtain a quasicontinuous selection function f for F by Theorem 3.7, and the range of f is contained in Y, since it is a selection function for F. By Proposition 2.9, f is continuous at a dense G_δ-set of points.

If Y is only metrizable, then F again admits a quasicontinuous selection function f by Theorem 3.4, which has a dense G_δ-set of points of continuity, again by Proposition 2.9. □

Remark 3.9. Recall that a regular Hausdorff space Y is called a *Stegall space* if whenever X is a Baire space and $F : X \rightrightarrows Y$ is a minimal USCO map, there exists a residual set D of X such that $F(x)$ is a singleton for all $x \in D$. It follows from the upper semicontinuity of F that any selection function f will be continuous at any point of D and then from Corollary 2.10 that f will be quasicontinuous.

Conversely, suppose that the regular Hausdorff space Y has the property that whenever X is a Baire space and $F : X \rightrightarrows Y$ is a minimal USCO map, then any selection function f has a residual set of points of continuity. Then by Lemma 2.7(ii) $\overline{f} = F$ (by minimality) has a residual set of points for which $F(x)$ is a singleton.

Putting together the previous remark with Corollary 3.5, we obtain

Corollary 3.10. *A regular Hausdorff space Y is a Stegall space if and only if $C(f)$ is residual for every quasicontinuous selection function f for any USCO map $F : X \rightrightarrows Y$ from a Baire space X.*

Let X be a space and Y be a compact metric space. If for some dense subset D of X, $f : D \to Y$ is a continuous map, then the closed relation $\overline{f}$ is called a *densely continuous form*. The relation $\overline{f}$ is contained in the USCO map $X \times Y$, and hence by Lemma 3.1 the projection $\pi_X(\overline{f})$ is a closed set containing D, hence equal to X. By Proposition 3.2 $\overline{f}$ is a USCO map. Clearly for any selection function h of $\overline{f}$, $\overline{h} \subseteq \overline{f}$. By Lemma 2.7(ii) any h must agree with f on D, and hence the reverse inclusion holds, thus $\overline{h} = \overline{f}$. It follows that $\overline{f}$ is a minimal USCO map.

Corollary 3.11. *Let X be a Baire space and Y a compact metric space. Then $F : X \rightrightarrows Y$ is a minimal USCO map iff it is a densely continuous form. The graph closure of a map $f : X \to Y$ is a densely continuous form if and only if f is quasicontinuous, and the correspondence $[f] \leftrightarrow \overline{f}$ is a one-to-one correspondence between the equivalence classes of quasicontinuous functions and the densely continuous forms (resp. the minimal USCO maps).*

Proof. By the preceding comments a densely defined form is a minimal USCO map. The converse follows from Corollary 3.8, since by minimality F will agree with $\overline{f \upharpoonright D}$, where f is a selection function continuous on a dense subset D. The remaining assertions follows from the first and Corollary 3.5. □

4. Domains

As pointed out in the introduction, a central goal of this paper is the development and study of a quasicontinuous function space from the perspective of domain theory. To this task we now turn.

4.1. **Basic domain theory.**

In this section we quickly recall basic notions concerning continuous domains (see [10]).

A nonempty subset D of a partially ordered set $(X, \leq)$ is directed if given $x, y \in D$, there exists $z \in D$ such that $x, y \leq z$. A *directed complete partially* *ordered set* or *dcpo* is a partially ordered set $(X, \leq)$ such that every directed subset of X has a least upper bound in X.

Let $x, y \in X$ where X is a dcpo. Then we say x *approximates* y, denoted by $x \ll y$, if for every directed set D with $y \leq \sup D$ we have $x \leq d$ for some $d \in D$. For $y \in X$ we define

$$\Downarrow y = \{x \in X : x \ll y\}.$$

Then we say a dcpo is *continuous* if

- $y = \sup \Downarrow y$ for all $y \in X$ and
- each $\Downarrow y$ is a directed set.

A *base* for a continuous dcpo is a set $B \subseteq X$ such that for all $x \in X$,

$$x = \sup\{\Downarrow x \cap B\},$$

and the supremum is taken over a directed set. A *continuous domain*, or *domain* for short, is a continuous dcpo and an ω-*continuous domain* is a domain with a countable base.

For a dcpo X, we can define the *Scott topology* as follows: A subset $O \subseteq X$ is *Scott-open* if

- O is an upper set, i.e., if $x \leq y$ and $x \in O$, then $y \in O$.
- O is inaccessible by least upper bounds of directed sets, i.e., if $\sup D \in O$ for a directed set D, then $d \in O$ for some $d \in D$.

One of the unusual features of the Scott topology is that it is only T_0, not Hausdorff, as long as the order on X is non-trivial. We will henceforth use freely the fact (see [10, Chapter II.1]) that in a continuous domain the Scott topology has a basis of open sets of the form

$$\Uparrow z := \{y \in X : z \ll y\}.$$

A function between dcpos X and Y is *Scott continuous* if it is monotone and preserves directed suprema. Equivalently a Scott continuous function is continuous with respect to the Scott topologies on X and Y.

Example 4.1. Consider the extended real numbers $\overline{\mathbb{R}} = [-\infty, \infty]$ equipped with the usual order. Then the Scott topology consists of $[-\infty, \infty]$ and all open right rays $(x, \infty]$. A function $f : X \to \overline{\mathbb{R}}$ from a topological space X is Scott continuous if and only if it is lower semicontinuous in the usual sense.

The upper sets of the form $\uparrow x := \{y : x \leq y\}$, sometimes called principal filters, form a subbasis for the closed sets of another topology, commonly called the *lower topology.* Its join with the Scott topology (the smallest topology containing both) gives the *Lawson topology.* For continuous domains, the Lawson topology is a Hausdorff topology. It is finer than the *interval topology,* which has as subbasis for the closed sets all closed order intervals $[a, b] = \{x : a \leq x \leq b\}$.

For any topology defined from the order of a partially ordered set L, one can define the *dual topology* that arises by reversing the order and defining the topology for that order, i.e., defining the topology on L^{op}. For example the dual Scott topology on $\overline{\mathbb{R}}$ consists of all open left rays $[-\infty, x)$, $-\infty < x$. The *biScott* *topology* is the join of the Scott topology and the dual Scott topology. On $\overline{\mathbb{R}}$ both it and the Lawson topology agree with usual topology of the extended reals.

4.2. **Bicontinuous lattices.** A partially ordered set is a *lattice* if any two points have a least upper bound and a greatest lower bound and a *complete lattice* if every subset has a least upper bound and a greatest lower bound. A *continuous lattice* is a continuous domain that is also a complete lattice.

Definition 4.2. *A complete lattice L is* linked bicontinuous*, or simply* bicontinuous *for short, if it satisfies:*

(1) *L and L^{op} are continuous domains;*
(2) *L is a complete lattice;*
(3) *The interval, biScott, Lawson, and dual Lawson topologies all agree.*

Remark 4.3. A variety of equivalent conditions for being a bicontinuous lattice appear in [10] Proposition VII-2.9, for example the following :

(i) $(L, \vee, \wedge)$ is a compact topological lattice with a basis of open sets that are sublattices. In this case the topology must be the biScott.
(ii) A complete distributive lattice L is bicontinuous if and only if it is completely distributive, that is, arbitrary joins distribute over arbitrary meets and vice-versa.

Note 3. *When we are working in the context of bicontinuous lattices we have a notion of approximation in both directions, so there is potential for confusion in the notation. We adopt the conventions*

- *$a \ll b$ means any directed sup exceeding b must have some member exceeding a;*
- *$\Uparrow a = \{b : a \ll b\}$;*
- *$a \gg b$ means any directed inf preceding b must have some member preceding a;*
- *$\Downarrow a = \{b : a \gg b\}$.*

In what follows we will primarily restrict our attention to those bicontinuous lattices L that are ω-continuous and these are called *ω-bicontinuous lattices*. This is equivalent to assuming that the biScott topology is metrizable, and hence equivalent to the dual L^{op} being ω-continuous (see [12, Proposition 7.1]).

For applications our focus will not be on general ω-bicontinuous lattices, but instead on the following example, and those less familiar with domain may basically restrict their attention to this example. However, even for this specific example, domain theory provides a convenient tool and framework for our considerations.

Primary Example. For $\overline{\mathbb{R}} = [-\infty, \infty]$, the extended reals, we form $\overline{\mathbb{R}}^n$ extended n-dimensional euclidian space. Observe that $\overline{\mathbb{R}}^n$ is a product of completely distributive lattices, hence completely distributive with respect to the coordinatewise order:

$$(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n) \Leftrightarrow \forall i, x_i \leq y_i,$$

and thus a bicontinuous lattice. We observe that the Scott open sets are the open sets $U = \uparrow U$, the open upper sets. The biScott topology is the usual product topology, which is metrizable, so $\overline{\mathbb{R}}^n$ is ω-continuous. The preceding observations remain valid for $\overline{\mathbb{R}}^{\mathbb{N}}$, a countable product of extended reals. The latter is convenient to keep in mind for generalizations, since any separable metrizable space can be embedded in it.

4.3. **Domain environments.** One of the aims of domain theory is to provide semantic or computational models for structures that include approximations to members of the structure. Often members of the structure are modeled as maximal "ideal" members of the model and elements below are thought of as approximations. This is often thought of as an "information ordering," the higher the element the more nearly it approximates ideal elements at the top.

Definition 4.4. *A* domain environment *for a topological space X is a homeomorphic embedding $X \hookrightarrow Max(D)$ onto the set of maximal points of a continuous domain D equipped with the relative Scott topology.*

Remark 4.5. A natural domain environment $\mathbb{L}$ for a bicontinuous lattice L (always endowed with the biScott=Lawson topology) consists of all nonempty order intervals

$$[u, v] := \{x \in L \mid u \leq x \leq v\},$$

where the order intervals are ordered by reverse inclusion, the "information order," and L embeds as the degenerate intervals.

Lemma 4.6. *Let L be a bicontinuous lattice, and $\mathbb{L}$ the set of all order intervals. Let $a_1, a_2, b_1, b_2 \in L$. The following are equivalent:*

(i) $[a_1, b_1] \ll [a_2, b_2]$;
(ii) $[a_2, b_2] \subseteq int[a_1, b_1]$, *the topological interior;*
(iii) $a_1 \ll a_2$ *in L, and* $b_1 \ll b_2$ *in* L^{op}, *written* $b_2 \gg b_1$.

Proof. A directed subset $D \subseteq \mathbb{L}$ of closed intervals has supremum (equal intersection) contained in $[a_2, b_2]$ if and only if the lower endpoints have directed supremum greater than or equal to a_2 and the upper endpoints have directed infimum less than or equal to b_2. From this observation the equivalence of (i) and (iii) readily follows. By [10, Proposition II-1.10] $a_1 \ll a_2$ if and only if $a_2 \in \text{int} \uparrow a_1$, where the interior is taken in the Scott topology. This statement and its dual yield the equivalence of (ii) and (iii). $\square$

A *bounded complete domain* is a domain that is also a complete (meet)-semilattice, a partially ordered set in which every nonempty subset has an infimum.

Theorem 4.7. *The set $\mathbb{L}$ is a bounded complete domain.*

Proof. Any nonempty family $A \subseteq \mathbb{L}$ has supremum the closed interval obtained by taking the infimum (resp. supremum) of all lower (resp. upper) endpoints of members of A for its lower (resp. upper) endpoint. Thus $\mathbb{L}$ is a complete semilattice. Directed suprema are formed in an analogous way, but now taking the supremum (resp. infimum) of the lower (resp. upper) endpoints. The fact that every element is a supremum of approximating elements follows readily from the preceding lemma. □

Theorem 4.8. *The map*

$$u \mapsto [u, u] : L \longrightarrow \mathbb{L}$$

is a homeomorphic embedding, hence a domain environment for $(L, biScott)$*, representing* L *as the degenerate intervals* $[u, u]$*.*

Proof. Consider the map $u \mapsto [u, u] : L \longrightarrow \mathbb{L}$. We want to show that the map is one-to-one, continuous, open and its image is the set of maximal elements of $\mathbb{L}$.

If $x, y \in L$, $x \neq y$, then $[x, x] = \{x\} \neq \{y\} = [y, y]$, and so the map is one-to-one.

Let $U = \Uparrow [a, b]$ be a basic Scott-open set in $\mathbb{L}$, $[x, x] \in U$. By Lemma 4.6, $x \in \Uparrow a \cap \Downarrow b$, an open set in L. Let c be in this open set. Then $a \ll c$, and $b \gg c$. Using Lemma 4.6 we can conclude that $[a, b] \ll [c, c]$, and so $[c, c] \in U$.

To see that the embedding is an open map onto its image, it suffices to show that images of a subbasis of open sets are again open. Using Lemma 4.6, we see that the image of $\Uparrow a$ (resp. $\Downarrow b$) is the intersection of the maximal elements with $\Uparrow [a, \top]$ (resp. $\Uparrow [\bot, b]$), where $\top$ (resp. $\bot$) is the top (resp. bottom) element of L. □

5. Function Spaces

Our goal in this section is to define and study a natural domain environment for the equivalence classes of quasicontinuous functions that we introduced earlier.

5.1. **Approximate functions.**

Intuitively an "approximate" or "fuzzy" function is one for which we have incomplete information. One way of modelling such functions is to assume that we know $f(x)$ only up to an interval of values.

Definition 5.1. *An* approximate function f *from a topological space* X *into a bicontinuous lattice* L *is a function* $f : X \longrightarrow \mathbb{L}$*. The approximate function* f *is Scott-continuous if it is continuous into the Scott topology of* $\mathbb{L}$*.*

Since each $f(x)$ is an order interval, we can write $f(x)$ as $f(x) = [f^\wedge(x), f^\vee(x)]$, where $f^\wedge, f^\vee : X \longrightarrow L$. In this case we write the interval function $f = [f^\wedge, f^\vee]$.

Theorem 5.2. *Let* $f : X \to \mathbb{L}$ *be an approximate function,* $f = [f^\wedge, f^\vee]$*. The following are equivalent:*

1. *The approximate function* f *is Scott-continuous.*
2. *Viewed as a multifunction,* $f : X \rightrightarrows L$ *is upper semicontinuous, and hence a USCO map.*

(3) *$f^\wedge : X \to L$ is Scott-continuous (also called lower semicontinuous) and $f^\vee$ is dually Scott-continuous (or upper semicontinuous).*

In particular, as a relation f is a closed subset of $X \times L$.

Proof. (1)⇒(2): Suppose that f is Scott-continuous. We first observe that the Lawson topology on the bicontinous, hence continuous, lattice L is compact Hausdorff and each order interval $[a, b]$ is closed, hence compact (see, for example, [10, Chapter III.1]). Let $x \in X$, and let U be open in L and contain $f(x)$. We observe that $\tilde{U} := \{\xi \in \mathbb{L} : \xi \subseteq U\}$ is a Scott-open set in $\mathbb{L}$ since it is closed under subsets (hence an upper set) and any directed intersection (equal supremum) of closed, hence compact, order intervals with intersection a member of $\tilde{U}$, hence contained in U, must have some member contained in U. By Scott-continuity of f, there exists some open set W containing x such that $f(W) \subseteq \tilde{U}$, that is, $f(w) \subseteq U$ for each $w \in W$.

(2)⇒(3): Let $f(x) = [f^\wedge(x), f^\vee(x)]$ for each x. Let $x \in X$ and let $z \ll f^\wedge(x)$. Then $\Uparrow z$ is a basic Scott-open set containing $f^\wedge(x)$ and hence contains $[f^\wedge(x), f^\vee(x)]$. Therefore there exists an open set W containing x such that $f(w) \subseteq \Uparrow z$ for each $w \in W$. It follows that $z \ll f^\wedge(w)$ for each $w \in W$, and hence that $f^\wedge$ is Scott-continuous.

(3)⇒(1): Let $f(x) = [f^\wedge(x), f^\vee(x)]$ and let $\Uparrow [c, d]$ be a basic Scott-open set containing $f(x)$ in $\mathbb{L}$. Then $c \ll f^\wedge(x)$ by Lemma 4.6. By Scott-continuity of $f^\wedge$, there exists W_1 open containing x such that $f^\wedge(W_1) \subseteq \Uparrow c$. Similarly there exists W_2 open containing x such that $f^\vee(W_2) \subseteq \Downarrow d$, and then by Lemma 4.6 $f(w) \in \Uparrow [c, d]$ for all $w \in W = W_1 \cap W_2$.

The last assertion follows from Proposition 3.2. □

Note 4. *In light of the preceding, we henceforth refer to Scott-continuous approximate functions as* usc approximate functions*.*

5.2. **The domain of approximate functions.** In this subsection X denotes a locally compact Hausdorff space, L a bicontinuous lattice, and $\mathbb{L}$ its domain environment of closed order intervals.

Proposition 5.3. *The set of all usc approximate functions from a locally compact space X to a bicontinuous lattice L ordered by the pointwise order is a bounded complete domain $[X \longrightarrow \mathbb{L}]$, called* the domain of approximate functions*.*

Proof. From Theorem 4.7 we know that $\mathbb{L}$ is a bounded complete domain, and this makes the set of usc approximate functions to be one. See [10, Proposition II-4.6]. □

We have additionally the space of lower semicontinuous functions, denoted by $(\mathrm{LSC}(X, L), \leq)$ and the space of upper semicontinuous functions denoted by $(\mathrm{USC}(X, L), \leq_{op} = \geq)$, where the order for both of them is the pointwise order. These are each bounded complete domains, again by [10, Proposition II-4.6]. We define

$$\widehat{\mathbb{L}}_X = \{(f, g) \in \mathrm{LSC}(X, L) \times \mathrm{USC}(X, L) : f \leq g\}.$$

For LSC$(X, L) \times$ USC(X, L) we consider the order given by

$$(f_1, g_1) \leq (f_2, g_2) \Leftrightarrow f_1 \leq f_2 \text{ in LSC}(X, L) \text{ and}$$
$$g_1 \leq_{op} g_2 \text{ in USC}(X, L).$$

Proposition 5.4. *The set $\widehat{\mathbb{L}}_X$ is a Scott closed bounded complete subdomain of LSC$(X, L) \times$ USC(X, L), and it is homeomorphic to the domain of approximate functions, $[X \to \mathbb{L}]$ under the identification $(f, g) \leftrightarrow [f, g]$.*

Proof. The set $\widehat{\mathbb{L}}_X \subseteq$ LSC$(X, L) \times$ USC(X, L) is closed under directed sups and arbitrary infs, so, by [10] Theorem I-2.6, is a Scott closed bounded complete subdomain of the domain LSC$(X, L) \times$ USC(X, L).

For the second part of the Proposition let $O : [X \to \mathbb{L}] \to \widehat{\mathbb{L}}_X$ be defined by $O([f, g]) = (f, g)$ for any $[f, g] \in [X \to \mathbb{L}]$. Since $[f, g] \in [X \to \mathbb{L}]$ then $f \in$ LSC(X, L), $g \in$ USC(X, L) and $f \leq g$, which makes our application well defined. If $(f, g) \in \widehat{\mathbb{L}}_X$ then it is clear that $[f, g] \in [X \to \mathbb{L}]$, so O is surjective, and it is immediate that it is injective. One sees directly that this one-to-one correspondence is an order isomorphism, hence a homeomorphism for the Scott and Lawson topologies.

The inclusion of $\widehat{\mathbb{L}}_X$ into LSC$(X, L) \times$ USC(X, L) preserves directed sups and arbitrary nonempty infs, so is continuous for the Lawson topologies. Since both are compact T_2 in the Lawson topology, it follows that the Lawson topology of $\widehat{\mathbb{L}}_X$ agrees with the relative Lawson topology from LSC$(X, L) \times$ USC(X, L) (see Chapter III.1 of [10] for these facts about the Lawson topology). Using the compactness, one sees that if A is a Scott-closed subset of $\widehat{\mathbb{L}}_X$, then $\downarrow A$ is Lawson-compact, hence Scott-closed in LSC$(X, L) \times$ USC(X, L), and thus is Scott-closed. Since $A = \downarrow A \cap \widehat{\mathbb{L}}_X$, we conclude that the Scott topology on $\widehat{\mathbb{L}}_X$ agrees with the relative Scott topology. □

Remark 5.5. Proposition 5.4 is important because it allows us to study the topology of the domain of usc approximate functions in terms of the function spaces LSC(X, L) and USC(X, L). The latter function spaces have been objects of serious investigation in the theory of domains (see, for example, Section II.4 of [10]) and much is already understood about them. In particular it is important to note that in light of the previous proposition we have a net $F_\alpha = [f_\alpha, g_\alpha] \to F = [f, g]$ in the Scott (resp. Lawson topology) of the domain of approximate functions if and only if $f_\alpha \to f$ in the Scott (resp. Lawson) topology of LSC(X, L) and $g_\alpha \to g$ in the Scott (resp. Lawson) topology of USC(X, L). (Note that the convergence need not be directed convergence, only convergence in the respective topologies.)

Theorem 5.6. *The function $E : [X \to \mathbb{L}] \times X \to \mathbb{L}$ defined by $E(f, x) = f(x)$ is continuous, where we assume that $[X \to \mathbb{L}]$ and $\mathbb{L}$ are equipped with the Scott topology. Hence E satisfies the following joint continuity condition:*
If $E(f, x) = f(x) = [f^\wedge(x), f^\vee(x)] \subseteq W$, where W is open in L, then there exist Scott-open sets U_1 in LSC(X, L) and U_2 in USC(X, L) containing $f^\wedge$ and

$f^\vee$ *resp. and* V *open containing* x *such that if* $\gamma \in U_1$, $\delta \in U_2$, $\gamma \leq \delta$ *and* $y \in V$, *then* $[\gamma(y), \delta(y)] \subseteq W$.

Proof. By [10, Proposition II-4.10(7)] the map E is continuous if $[X \to \mathbb{L}]$ is endowed with the standard Isbell function space topology, and by [10, Proposition II-4.6] the Isbell and Scott topology agree since $\mathbb{L}$ is a bounded complete domain. Thus the first assertion follows.

We have $[f^\wedge(x), f^\vee(x)] = \bigcap\{[a,b] : a \ll \alpha(x),\, b \gg \beta(x)\}$. Since the intersection on the right is a directed intersection of compact subsets, it follows that $[f^\wedge(x), f^\vee(x)] \subseteq \Uparrow a \cap \Downarrow b \subseteq \uparrow a \cap \downarrow b = [a,b] \subseteq W$ for some $a \ll f^\wedge(x)$, $b \gg f^\vee(x)$. Then $[a,b] \ll [f^\wedge(x), f^\vee(x)]$, i.e., $[f^\wedge(x), f^\vee(x)] \in \Uparrow [a,b] \subseteq W$. Since $\Uparrow [a,b]$ is Scott-open in $\mathbb{L}$, the joint continuity condition follows from the previous paragraph and the equivalence of Proposition 5.4. □

Corollary 5.7. *The multifunction* $E : [X \to \mathbb{L}] \times X \rightrightarrows L$ *defined by* $E(f,x) = f(x)$ *is a USCO map.*

Proof. This follows directly from Theorem 5.6, Lemma 4.6, and the equivalence of Proposition 5.4, since the order intervals of L are compact. □

Definition 5.8. *We define the* extended compact-open topology *on* $[X \to \mathbb{L}]$ *as the topology that has a subbasis of open sets of the form*

$$N(K,U) := \{F \in [X \to \mathbb{L}] : F(K) = \bigcup_{x \in K} F(x) \subseteq U\},$$

where K *is compact in* X *and* U *is open in* L.

Note 5. *Note that when restricted to the continuous functions from* X *to* L *the extended compact-open topology is the compact-open topology.*

Proposition 5.9. *The extended compact-open topology on* $[X \to \mathbb{L}]$ *is equal to the Scott topology.*

Proof. It follows easily from Corollary 5.7 that the subbasic open sets $N(K,U)$ are open in the Scott topology of $[X \to \mathbb{L}]$.

The Scott topology of the function space $\mathrm{LSC}(X,L)$ is equal to the Isbell topology [10, Proposition II-4.6], which in turn is equal to the the compact-open topology from X into L_σ, L equipped with the Scott-topology [10, Lemma II-4.2(i)]. For $f = [f^\wedge, f^\vee]$, K compact in X, and W open in L_σ, we have $f^\wedge(K) \subseteq W$ if and only if $f(K) = \cup_{x \in K}[f^\wedge(x), f^\vee(x)] \subseteq W$, since $W = \uparrow W$. Thus under the correspondence of Proposition 5.4, $N(K,W)$ corresponds to $\widehat{\mathbb{L}}_X \cap (\{g \in \mathrm{LSC}(X,L) : g(K) \subseteq W\} \times \mathrm{USC}(X,L))$. Clearly a dual argument is valid for $\mathrm{USC}(X,L)$. It then follows from the equivalence of Proposition 5.4 that the Scott topology of $[X \to \mathbb{L}]$ is contained in the extended compact-open topology. □

5.3. **Maximal approximate functions.** We turn to a common construction in domain theory and its basic properties (see, for example, [10, Exercise II-3.19]).

Definition 5.10. *For any function $f : X \to L$, we define*

$$f_*(x) := \sup\{\inf f(U) : x \in U,\ U \text{ is open}\}$$

and

$$f^*(x) := \inf\{\sup f(U) : x \in U,\ U \text{ is open}\}.$$

Lemma 5.11. *Let D be a dense subset of X, let $f : D \to L$, and set*

$$f_*(x) := \sup\{\inf f(U \cap D) : x \in U,\ U \text{ is open}\}.$$

Then $f_* : X \to L \in LSC(X, L)$, and satisfies the following:

(i) *$f_*(x) \leq f(x)$ for all $x \in D$,*
(ii) *for $x \in D$, $f_*(x) = f(x) \Leftrightarrow f$ is lower semicontinuous at x;*
(iii) *if $g : X \to L$, $g \leq f$ on D, and g is lower semicontinuous at $x \in X$, then $g(x) \leq f_*(x)$.*

Proof. Let $x \in X$ and let V be a Scott-open set containing $f_*(x)$. Pick $z \in V$ such that $z \ll f_*(x) = \sup\{\inf f(U \cap D) : x \in U, \text{open}\}$. Since this is a directed sup, $z \leq \inf f(U \cap D)$ for some U open, $x \in U$. If follows that $z \leq \inf f(U \cap D) \leq f_*(w)$ for all $w \in U$, i.e., $f_*(U) \subseteq \uparrow z \subseteq V$. Hence $f_* \in \mathrm{LSC}(X, L)$.

Since for $x \in D$, $\inf f(U \cap D) \leq x$ for each U open containing x, property (i) follows.

Since for any $x \in X$, f_* is lower semicontinous at x and $f_* \leq f$ on D by (i), it follows immediately that f is lower semicontinuous at any $x \in D$ where $f(x) = f_*(x)$. Conversely suppose that f is lower semicontinuous at $x \in D$ and let $z \ll f(x)$. Then there exists U open containing x such that $f(U \cap D) \subseteq \uparrow z$, and hence $z \leq f_*(x)$. Since $f(x) = \sup\{z : z \ll f(x)\}$, it follows that $f(x) \leq f_*(x)$ and hence from (i) $f(x) = f_*(x)$.

(iii) If g is lower semicontinuous at $x \in X$, then

$$\begin{aligned} g(x) &\leq \sup\{\inf g(U \cap D) : x \in U \text{ open}\} \\ &\leq \sup\{\inf f(U \cap D) : x \in U \text{ open}\} = f_*(x). \end{aligned}$$

□

The next proposition follows in a straightforward fashion from the preceding lemma.

Proposition 5.12. *Let $f : X \to L$ be a function, and f_*, f^* be defined as in Definition 5.10. The following are true:*

(i) *$f_* \leq f \leq f^*$;*
(ii) *f_* is lower semicontinuous and f^* is upper semicontinuous;*
(iii) *f is lower semicontinuous if and only if $f = f_*$;*
(iv) *f upper semicontinuous if and only if $f = f^*$;*
(v) *f is continuous if and only if $f = f^* = f_*$;*
(vi) *f_* is the largest lower semicontinuous function such that $f_* \leq f$;*
(vii) *f^* is the smallest upper semicontinuous function such that $f \leq f^*$.*

For any continuous function $f : X \to L$, the approximate function $F = [f, f]$ is clearly maximal in the domain of approximate functions. There are, however, additional maximal elements.

Proposition 5.13. *The maximal elements in the domain $[X \to \mathbb{L}]$ of approximate functions have the form $f(x) = [\alpha(x), \beta(x)]$, where $\alpha^* = \beta$ and $\beta_* = \alpha$. These include the continuous functions.*

Proof. We know that the elements of the domain $[X \to \mathbb{L}]$ are usc approximate functions, which, by Theorem 5.2, means that α is lower semicontinuous, and β is upper semicontinuous. That is, $\alpha_* = \alpha \leq \alpha^*$ and $\beta_* \leq \beta = \beta^*$.

Let f be a maximal element of the domain $[X \to \mathbb{L}]$. Since $\alpha \leq \beta$ we have that $\alpha^* \leq \beta^* = \beta$. Thus $[\alpha(x), \alpha^*(x)] \subseteq [\alpha(x), \beta(x)]$, which means that $[\alpha(x), \beta(x)] \leq [\alpha(x), \alpha^*(x)]$. If f is maximal then we must have $[\alpha(x), \beta(x)] = [\alpha(x), \alpha^*(x)]$, and that gives us $\alpha^* = \beta$. A similar proof yields that $\alpha = \beta_*$ if f is maximal in the domain.

Now suppose that $f = [\alpha, \beta]$ is such that $\alpha^* = \beta$ and $\beta_* = \alpha$. Suppose that $f \leq g = [\alpha_1, \beta_1]$. Then $\alpha \leq \alpha_1 \leq \beta_1 \leq \beta$ implies $\beta = \alpha^* \leq \alpha_1^* \leq \beta_1^* = \beta_1 \leq \beta$, so $\alpha_1^* = \beta_1$ and $\beta = \beta_1$. Similarly $\alpha_1 = (\beta_1)_*$ and $\alpha = \alpha_1$. That means $f = g$, so f is maximal in the domain. $\square$

6. Quasicontinuous Function Spaces

We assume in this section as a standing hypothesis that X is a locally compact Hausdorff space and L is a bicontinuous lattice, although we will often require even stronger hypotheses than this. We return to our consideration of quasicontinuous functions and their graph closure equivalence classes. The domain-theoretic setting allows us to define a useful function space topology on these classes.

Lemma 6.1. *If $f : X \to L$ is quasicontinuous, then $F := [f_*, f^*]$ is maximal in the domain of approximate functions, and every maximal element arises in this way.*

Proof. Suppose that $F \leq G = [\alpha, \beta]$. If it were the case that $\alpha \leq f \leq \beta$, then by Proposition 5.12, $\alpha \leq f_* \leq f^* \leq \beta$, yielding $G \leq F$, so $F = G$, implying that F is maximal. Thus it suffices to show that $f \subseteq G = \overline{G}$, where the last equality follows from Theorem 5.2.

Let $(x, f(x)) \in U \times V$, a basic open set in $X \times L$. Pick $b, c \in L$ such that $f(x) \in \Uparrow b \cap \Downarrow c \subseteq V$. By quasicontinuity there exists a nonempty open set $W \subseteq U$ such that $f(W) \subseteq \Uparrow b \cap \Downarrow c$. It follows that

$$b \leq f_*(w) \leq \alpha(w) \leq \beta(w) \leq f^*(w) \leq c$$

for all $w \in W$. In particular $G \cap (U \times V) \neq \varnothing$. It follows that $(x, f(x)) \in \overline{G} = G$ and hence $\alpha \leq f \leq \beta$.

Conversely let $G = [\alpha, \beta]$ be a maximal approximate function and let $f : X \to L$ be a quasicontinuous selection function (Theorem 3.4). By Proposition 5.12, $\alpha \leq f_* \leq f^* \leq \beta$, so by maximality $G = [f_*, f^*]$. $\square$

The next proposition extends the equivalences for two quasicontinuous functions to be closed graph equivalent.

Proposition 6.2. *Let $f, g : X \to L$ be quasicontinuous functions such that $C(f)$ is dense. The following are equivalent:*

(1) *f, g agree on a dense set.*
(2) *$f^* = g^*$, $f_* = g_*$.*
(3) *$f \sim g$.*

In particular, these all hold for L ω-bicontinuous.

Proof. Items (1) and (3) are equivalent by Theorem 2.11. Assume (3). Set $F = [f_*, f^*]$, a closed relation (Theorem 5.2). Thus

$$g \subseteq \overline{g} = \overline{f} \subseteq F.$$

It follows that $f_* \leq g \leq f^*$, and hence that $[f_*, f^*] \leq [g_*, g^*]$. Interchanging f and g yields item (2).

Conversely assume (2). Then f_* and f^* agree with f on the dense set $C(f)$ by Lemma 5.11, and thus so do g_* and g^* by hypothesis. Since $g_* \leq g \leq g^*$, g also agrees with f on $C(f)$. Hence $f \sim g$ by Theorem 2.11.

In the case the L is ω-bicontinuous, it is separable metrizable, so by Proposition 2.9 quasicontinuous functions have a dense G_δ-set of points of continuity. □

Note 6. *For L ω-bicontinuous, we denote by $Q(X, L)$ the space of equivalence classes of quasicontinuous functions, and denote the class of f by $[f]$. Note that a continuous function has a singleton equivalence class. We define for $[f] \in Q(X, L)$,*

$$[f](x) = [f_*(x), f^*(x)] \text{ and } [f](A) = \bigcup_{x \in A} [f](x).$$

Note that in light of Proposition 6.2, these definitions are well-defined.

Theorem 6.3. *For L an ω-bicontinuous lattice, the association $[f] \longleftrightarrow [f_*, f^*]$ is a one-to-one correspondence between the classes of quasicontinuous functions, $Q(X, L)$, and the maximal elements of the domain $[X \to \mathbb{L}]$ of approximate maps from X to L.*

Proof. The theorem follows readily from Lemma 6.1 and Proposition 6.2. □

Theorem 6.3 suggests a natural topology for the quasicontinuous equivalence classes, namely the Scott topology, on the domain of approximate functions restricted to the quasicontinuous equivalence classes, which we identify with the maximal approximate functions.

Definition 6.4. *Let X be a locally compact Hausdorff space, let L be an ω-bicontinuous lattice, and let the function space $[X \to L]$ be equipped with the Scott topology. The topology on $Q(X, L)$, the set of classes of quasicontinuous functions, that makes the injection $[f] \leftrightarrow [f_*, f^*]$ of Theorem 6.3 a topological embedding is called the* quasiorder topology *or* qo-topology *for short.*

Remark 6.5. The qo-topology is defined in such a way that the domain of approximate functions forms a domain environment for the quasicontinuous equivalence classes. By Proposition 5.4 the Scott topology in $[X \to L]$ agrees with the one arising from simultaneous Scott-convergence in the LSC and USC variables. By Proposition 5.9 this Scott topology on the space of approximate functions is equal to the extended compact-open topology, so the qo-topology may also be considered to be the restriction of the extended compact-open topology.

Theorem 6.6. *The Lawson and Scott topologies agree on the set of maximal elements of the domain of approximate functions* $[X \to \mathbb{L}]$*, and hence this topology is completely regular and Hausdorff. If we restrict to the case that* X *is locally compact and separable metrizable and* L *is* ω*-continuous, then we may identify this space with* $Q(X, L)$ *with the qo-topology, which makes the space a Polish space. In particular, we may restrict function space convergence to sequences in studying continuity, closedness, compactness, etc.*

Proof. To show that the Lawson and Scott relative topologies agree on the maximal elements of a domain D it suffices to show that for any $p \in D$, $\uparrow p \cap \mathrm{Max}D = A \cap \mathrm{Max}D$ for some Scott-closed set A [10, Definition V-6.1]. In the case of a bounded complete domain $\mathbb{L}$ this is always satisfied, since $\uparrow p$ is closed in the compact Hausdorff Lawson topology, hence compact, and thus $A := \downarrow(\uparrow p)$ is Scott-closed and satisfies $\uparrow p \cap \mathrm{Max}D = A \cap \mathrm{Max}D$. Since any subspace of the compact Hausdorff space $\mathbb{L}$ is completely regular and Hausdorff, the first assertion is satisfied.

The fact that $[X \to \mathbb{L}]$ is ω-continuous if X is locally compact and separable metrizable and L is ω-bicontinuous follows from the identification of Proposition 5.4 and a standard theorem that gives the cardinality of a basis for the function space $\mathrm{LSC}(X, L)$ from those of the domain and codomain [10, Corollary III-4.10]. The space of maximal points of the ω-continuous domain is Polish by [10, Theorem V-6.6]. The identification with $Q(X, L)$ and its topology comes from Theorem 6.3 and Definition 6.4. Since Polish spaces are metrizable the last assertion of the theorem follows. □

Proposition 6.7. *Let* L *be an* ω*-bicontinuous lattice, and let* $f, g : X \to L$ *be quasicontinuous maps such that* $[f] \neq [g]$*. Then there exist a nonempty open set* $U \subseteq X$ *and* $a, b \in L$*,* $b \not\leq a$ *such that for any* $x \in U$*,* $[f](x) \subseteq \Downarrow a$ *and* $[g](x) \subseteq \Uparrow b$ *(or vice-versa).*

Proof. Since L is separable metrizable, by Theorem 2.9 and Proposition 6.2 there exists $x \in C(f) \cap C(g)$ such that $f(x) \neq g(x)$, say $g(x) \not\leq f(x)$. Since L is bicontinuous, we can find $a, b \in L$ such that $b \not\leq a$ and $f(x) \in \Downarrow a$, $g(x) \in \Uparrow b$. Since f and g are continuous at x we have that $[f](x) = [f(x), f(x)]$ and $[g](x) = [g(x), g(x)]$. By Theorem 5.6 there exists an open set U containing x such that $[f](U) \subseteq \Downarrow a$ and $[g](U) \subseteq \Uparrow b$. □

The following gives some equivalent characterizations of convergence in $Q(X, L)$.

Proposition 6.8. *Let $([f_n])_n \subseteq Q(X, L)$ and $[f] \in Q(X, L)$, where we assume that X is locally compact and separable metrizable and L is ω-bicontinuous. The following are equivalent:*

(1) *$(f_n)_* \longrightarrow f_*$ in the Scott topology, and $(f_n)^* \longrightarrow f^*$ in the dual-Scott topology;*

(2) $f_* = \sup_n \left(\inf_{n\le m},\ (f_m)_*\right)_* = \underline{\lim}(f_n)_* f^* = \inf_n \left(\sup_{n\le m}(f_m)^*\right)^*$
$= \overline{\lim}(f_n)^*$;

(3) *$[(f_n)_*, (f_n)^*] \to [f_*, f^*]$ in the relative Scott topology of the set of maximal elements of the domain $[X \to \mathbb{L}]$;*

(4) *There exist an increasing sequence $(g_n)_n \subseteq LSC(X, L)$ and a decreasing sequence $(h_n)_n \subseteq USC(X, L)$ such that $f_* = \sup_n g_n$, $f^* = \inf_n h_n$ and $g_n \le (f_n)_* \le (f_n)^* \le h_n$, for each n.*

Proof. (1) $\Leftrightarrow$ (2). From the definition of Scott convergence we have that $(f_n)_* \longrightarrow f_*$ if and only if $f_* \le \underline{\lim}(f_n)_*$, and similarly for the dual Scott convergence. The only thing that must be proved is that

$$f_* = \sup_n \left(\inf_{n\le m}(f_m)_*\right)_*$$

and

$$f^* = \inf_n \left(\sup_{n\le m}(f_m)^*\right)^*,$$

where the inequalities follow from [10, Proposition III-3.12]. We have that

$$\left[\sup_n \left(\inf_{m\le n}(f_n)_*\right)_*, \inf_n \left(\sup_{m\le n}(f_n)^*\right)^*\right] \in [X \to \mathbb{L}],$$

and

$$[f_*, f^*] \le \left[\sup_n \left(\inf_{m\le n}(f_n)_*\right)_*, \inf_n \left(\sup_{m\le n}(f_n)^*\right)^*\right] \text{ in } [X \to \mathbb{L}].$$

Since $[f_*, f^*]$ is a maximal element of the domain $[X \to \mathbb{L}]$, we must have the equality of the two intervals, therefore the equalities we want.

(1) $\Leftrightarrow$ (3). This equivalence follows directly from Proposition 5.4.

(2) $\Leftrightarrow$ (4). Suppose that (2) is true. For each $n \ge 1$ let

$$g_n = \left(\inf_{n\le m}(f_m)_*\right)_* \text{ and } h_n = \left(\sup_{n\le m}(f_m)^*\right)^*.$$

It is clear that each g_n is lower semicontinuous and each h_n is upper semicontinuous. Since $n_1 \le n_2$ implies

$$\inf_{n_1\le m}(f_m)_* \le \inf_{n_2\le m}(f_m)_*$$

and

$$\sup_{n_1 \le m} (f_m)^* \ge \sup_{n_2 \le m} (f_m)^*,$$

we have $g_{n_1} \le g_{n_2}$ and $h_{n_1} \ge h_{n_2}$, which means $(g_n)_n$ is increasing and $(h_n)_n$ is decreasing. It is also clear that we have $g_n \le (f_n)_* \le (f_n)^(*) \le h_n$ for each $n > 0$.

For the other implication, let $(g_n)_n$ and $(h_n)_n$ like in (3). Since $(g_n)_n$ is increasing, we have $g_n \le g_m \le f_m$ for every $m \ge n$, which implies

$$g_n \le \inf_{n \le m} (f_m)_*,$$

and, since $g_n \in \mathrm{LSC}(X, L)$,

$$g_n \le \left(\inf_{n \le m} (f_m)_* \right)_* .$$

Therefore

$$f_* \le \sup_n \left(\inf_{n \le m} (f_m)_* \right)_* .$$

Similarly we get

$$f^* = \inf_n \left(\sup_{n \le m} (f_m)^* \right)^* = \overline{\lim}(f_n)^*,$$

and because $f \in Q(X, L)$, $[f_*, f^*]$ is a maximal element of the domain $[X \to \mathbb{L}]$, hence we have (2). $\square$

7. Generalized Derivatives

In this section we will restrict our attention to the bicontinuous lattices $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^n$. We make the standing assumption that X is a locally compact, locally convex subset of $\mathbb{R}^m$ and consider functions from X to $\overline{\mathbb{R}}$ or $\overline{\mathbb{R}}^n$.

We adopt what will be a convenient convention of identifying two quasicontinuous functions f, g if they belong to the same equivalence class, in much the same way that we identify two functions in measure theory if they differ on set of measure 0. Since in this section we are only considering functions from a locally compact subset X of $\mathbb{R}^n$ into $\overline{\mathbb{R}}^m$, this means that the two agree on their common set of points of continuity (Corollary 2.8), a dense G_δ-set (Proposition 2.9), and $f(x) \in \overline{g}(x)$ and vice-versa otherwise. Thus f is uniquely defined on $C(f)$ and is ambiguous up to $\overline{f}(x)$ otherwise (with no ambiguity for continuous functions, their class consisting of one element). Occasionally it will also be convenient to treat (equivalence classes of) quasicontinuous functions as maximal elements of the domain of approximate functions, or as minimal USCO maps $f : X \rightrightarrows \mathbb{R}^n$ (via the identification of the previous sections) such that each $f(x)$ is an order interval $[f_*(x), f^*(x)]$, where the latter is independent of the representative of the equivalence class (Proposition 6.2). Furthermore, we don't distinguish between points of $\overline{\mathbb{R}}^n$ and degenerate order intervals. In particular, singleton-valued USCO maps from X to $\overline{\mathbb{R}}^n$ are, for us, the same as continuous maps from X to $\overline{\mathbb{R}}^n$.

We work mostly with finite-valued functions. We denote by $Q(X, \mathbb{R})$ the members f of $Q(X, \overline{\mathbb{R}})$ with $[f_*(x), f^*(x)] \subseteq \mathbb{R}$ for all $x \in X$ and employ a similar convention for $Q(X, \mathbb{R}^n)$.

In this section we extend results of Samborski [19] using the machinery that we have developed in earlier sections.

We consider the partial derivative operator

$$\frac{\partial}{\partial x_k} : C^1(X, \mathbb{R}) \longrightarrow C^0(X, \mathbb{R}) \subseteq Q(X, \mathbb{R}) \tag{7.1}$$

and the gradient operator

$$\nabla : C^1(X, \mathbb{R}) \longrightarrow C^0(X, \mathbb{R}^m) \subseteq Q(X, \mathbb{R}^m), \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_m} \right). \tag{7.2}$$

Lemma 7.1. *Suppose $f_n \to f$ in $Q(X, \mathbb{R})$, $f_n \in C^1(X, \mathbb{R}) \subseteq Q(X, \mathbb{R})$, and $\nabla f_n \to F$ in $Q(X, \mathbb{R}^m)$. Then $f \in Q(X, \mathbb{R})$ is a locally Lipschitz function, and f_n converges to f in the compact-open topology.*

Proof. Let $x \in X$. Since $F(X) \subseteq \mathbb{R}^m$, we may pick $a, b \in \mathbb{R}^m$ such that $b \gg F^*(x)$ and $a \ll F_*(x)$. It follows from Theorem 5.6 that there exists $N > 0$ and U open containing x such that

$$\nabla f_n(u) \in \Downarrow b \cap \Uparrow a \text{ for each } u \in U \text{ and each } n \geq N. \tag{7.3}$$

Let $M > 0$ such that $[a, b] \subseteq B_M(0)$, the open ball in $\mathbb{R}^m$ around 0 of radius M. Then

$$\|\nabla f_n(u)\| \leq M \text{ for each } u \in U \text{ and each } n \geq N. \tag{7.4}$$

We can choose U such that U is also convex, so that we can apply the Mean Value Theorem for differentiable functions on $\mathbb{R}^m$. Therefore for each $n > N$ and each $u, v \in U$ there exists $0 < t_n < 1$ such that

$$f_n(u) - f_n(v) = \langle \nabla f_n(\xi_n), u - v \rangle ,$$

where $\xi_n = t_n u + (1 - t_n) v \in U$, and by (7.4) we get

$$\|f_n(u) - f_n(v)\| \leq \|\nabla f_n(\xi_n)\| \|u - v\| \leq M \|u - v\|,$$

for each $n \geq N$, which means that $\mathcal{F} = \{f_n|_U : n \geq N\}$ is an equicontinuous family of functions.

Using the same arguments that we used for ∇f_n to find (7.4), we can find U_0 open containing x, $N_0 > 0$, $M_0 > 0$ such that $\{f_n(y) : n \geq N_0\} \subseteq (-M_0, M_0) \subseteq \mathbb{R}$ for each $y \in V = U_0 \cap U$, which makes the closure of $\{f_n(y) : n \geq N_0\}$ compact in $\mathbb{R}$. Thus we are in the setting of Ascoli's Theorem [18], so we obtain a subsequence of $\{f_n|_V : n \geq N\}$, (f_{n_k}), which converges pointwise to a continuous function g, the convergence being uniform on each compact subset of V. Indeed since all f_n are M-Lipschitz on U, then g is M-Lipschitz on V also. Equivalently, we can say that $(f_{n_k}) \to g$ in the compact-open topology, so in the qo-topology (see Note 5 and Proposition 5.9).

The convergence $f_n \to f$ in $Q(X, \overline{\mathbb{R}})$ makes $f_{n_k} \to f$ in $Q(X, \overline{\mathbb{R}})$, and since $Q(X, \overline{\mathbb{R}})$ is Hausdorff, $f|_V = g|_V$ in $Q(X, \overline{\mathbb{R}})$, so f is a locally Lipschitz function. □

Example 7.2. Consider the absolute value function on the interval $X = [-1, 1]$. It admits an extended derivative $[g]$ that is the sign function, with either the value 1 or -1 at 0, i.e., $[g](0) = \{-1, 1\}$.

Recall that the *strong derivative* of a function $f : U \subseteq \mathbb{R} \to \mathbb{R}$ is given by

$$\lim_{\substack{u,v \to x \\ u \neq v}} \frac{f(u) - f(v)}{u - v},$$

if the limit exists.

Definition 7.3. *Let $U \subseteq \mathbb{R}^m$ be locally compact with dense interior, and let $f : U \to \overline{\mathbb{R}^n}$. and $x \in U$. We will say that f is* strongly differentiable *at x if there exists a linear operator $L : \mathbb{R}^m \to \mathbb{R}^n$ such that for all $u, v \in U$,*

$$f(u) - f(v) = L(u - v) + r(u, v) \text{ where } \lim_{\substack{u,v \to x \\ u \neq v}} \frac{\|r(u, v)\|}{\|u - v\|} = 0.$$

The operator L, if it exists, is unique and is called the strong derivative *at x and denoted $Df(x)$.*

Theorem 7.4. *Let $U \subset \mathbb{R}^m$ be locally compact, locally convex with dense interior, and $f_n \in C^1(U, \mathbb{R}) \subseteq Q(U, \mathbb{R})$ such that $f_n \to f$ in $Q(X, \mathbb{R})$ and $\nabla f_n \to G$ in $Q(U, \mathbb{R}^m)$. Then the strong derivative of f exists and is equal to G on a dense G_δ-set $D \subseteq X$. In particular $\nabla f = G$ on D so we can say that the gradient of f is given by*

$$\overline{\nabla} f = [(\nabla f)_*, (\nabla f)^*] = [G_*, G^*],$$

where

$$(\nabla f)_*(x) = \sup_{x \in U \text{ open}} \inf \{\nabla f(y) : y \in U \cap D\}, \tag{7.5}$$

and

$$(\nabla f)^*(x) = \inf_{x \in U \text{ open}} \sup \{\nabla f(y) : y \in U \cap D\}. \tag{7.6}$$

Furthermore, $[\overline{\nabla} f] = [G]$.

Proof. By the previous lemma f is a locally Lipschitz function. By Proposition 2.9, the set D of points of continuity of G is a dense G_δ-set.

Let $x \in \operatorname{int}(U) \cap C(G)$, and let $u, v \in U$. Since each f_i is differentiable and U is locally convex, we can apply the Mean Value theorem on $\mathbb{R}^m$ for each f_i for $u \neq v$ close to x. Therefore, there exists $\xi_i = (1 - t)u + tv$ for some $0 < t < 1$ such that

$$f_i(u) - f_i(v) = \langle \nabla f_i(\xi_i), u - v \rangle.$$

Then we have

$$\begin{aligned}\frac{|f(u)-f(v)-\langle G(x),u-v\rangle}{\|u-v\|} &\leq \frac{|f(u)-f(v)-(f_i(u)-f_i(v))}{\|u-v\|}\\ &+ \frac{|f_i(u)-f_i(v)-\langle\nabla f_i(\xi_i),u-v\rangle\,|}{\|u-v\|}\\ &+ \frac{|\,\langle\nabla f_i(\xi_i)-G(x),u-v\rangle\,|}{\|u-v\|}.\end{aligned}$$

The middle term of the right-hand side of the inequality is zero. Since $\xi_i \to x$ as $u, v \to x$, and since $\nabla f_n \to G$ in $Q(U, \mathbb{R}^m)$, by Corollary 5.7 for any $\varepsilon > 0$ there exists $N_1 > 0$ and V open and convex containing x such that $\|\nabla f_i(\xi_i) - G(x)\| < \frac{\varepsilon}{2}$ for $i \geq N_1$ and $u, v \in V$, $u \neq v$. Then we have

$$\begin{aligned}\frac{\|\,\langle\nabla f_i(\xi_i)-G(x),u-v\rangle\,\|}{\|u-v\|} &\leq \frac{\|\nabla f_i(\xi_i)-G(x)\|\|u-v\|}{\|u-v\|}\\ &= \|\nabla f_i(\xi_i)-G(x)\| < \tfrac{\varepsilon}{2}.\end{aligned}$$

By the previous lemma f_n converges to f in the compact-open topology, so in particular $f_n(y) \to f(y)$ for any $y \in U$. Thus there exists for any distinct $u, v \in V$, an $N_2 > 0$ such that

$$\frac{|f(u)-f(v)-(f_i(u)-f_i(v))|}{\|u-v\|} < \frac{\varepsilon}{2} \text{ for every } i \geq N_2.$$

Putting this all together, we conclude that

$$\frac{|f(u)-f(v)-\langle G(x),u-v\rangle\,|}{\|u-v\|} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Therefore

$$\lim_{\substack{u,v\to x\\ u\neq v}} \frac{f(u)-f(v)}{\|u-v\|} = \frac{\langle G(x),u-v\rangle}{\|u-v\|},$$

so the strong derivative of f exists for x a continuity point for G, and such points form a dense G_δ-set D.

For $x \in D$ we have also $\nabla f(x) = G(x)$, and since D is dense we can define

$$(\nabla f)_* = (x \to \nabla f(x) | x \in D \subseteq X)_*$$

and

$$(\nabla f)^* = (x \to \nabla f(x) | x \in D \subseteq X)^*.$$

Hence, by Lemma 5.11(iii), we have

$$G_* \leq (\nabla f)_* \text{ and } (\nabla f)^* \leq G^*,$$

so by minimality of G

$$\overline{\nabla} f = [(\nabla f)_*, (\nabla f)^*] = G \text{ in } Q(X, \overline{\mathbb{R}}),$$

and the theorem is proved. □

The preceding theorem easily extends to the case of general functions from $\mathbb{R}^m$ to $\mathbb{R}^n$. Let $D : C^1(U, \mathbb{R}^n) \to C^0(U, \mathbb{R}^{m\times n})$ be defined by $Df(x)$ is the Jacobian matrix of f at x, where U is a locally compact subset of $\mathbb{R}^m$ with dense interior. We consider the closure of the set $\{(f, Df) : f \in C^1(U, \mathbb{R}^n), Df \in C^0(U, \mathbb{R}^{m\times n}\}$ in $Q(X, \mathbb{R}^n) \times Q(X, \mathbb{R}^{m\times n})$, where the latter is endowed with the product of the qo-topologies.

Corollary 7.5. *(i) Closing up the differentiation operator $D : C^1(X, \mathbb{R}^n) \to C^0(X, \mathbb{R}^{m\times n})$ in $Q(X, \mathbb{R}^n) \times Q(X, \mathbb{R}^{m\times n})$ yields an extended operator $\overline{D}$. We denote the domain of the extended operator by $Q^1(X, \mathbb{R}^n)$ and call $\overline{D}f$ the* generalized derivative *of f for $f \in Q^1(X, \mathbb{R}^n)$.*
(ii) Each member $f \in Q^1(X, \mathbb{R}^n)$ is a locally Lipschitz map from X to $\mathbb{R}^n$ and is strongly differentiable at a dense subset of points of X. The image $\overline{D}f$ in $Q(X, \mathbb{R}^{m\times n})$ is the closure of the densely defined mapping on X sending x to the strong derivative $D_s f(x)$.

Proof. It follows from the preceding theorem that the theorem is true in each of the n-coordinate functions and hence true overall. □

We remark that the generalized derivatives considered in [9] are interval-valued, i.e., approximate functions in our sense. We consider only the special case of maximal approximate functions identified with quasicontinuous functions. Thus we only consider derivatives for which the intervals are degenerate (or single-valued) for a dense subset and the resulting function is continuous on this dense subset.

8. Hamiltonian Equations

In this section we recall ideas of Samborski [19], [20] for applying the theory of quaisicontinuous functions to the study of viscosity solutions of Hamiltonian equations.

Let X be a locally compact subset of $\mathbb{R}^n$ that has dense interior, and let $H : X \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function convex in the last argument. In this section we consider solutions of the Hamiltonian

$$H(x, y(x), \nabla y(x)) = h(x). \tag{8.1}$$

8.1. **Continuous Hamiltonians.** Recall that if $\alpha \in \mathrm{LSC}(X, \mathbb{R})$ and $\partial_-\alpha(x) = \{\zeta \in \mathbb{R}^m : \alpha(y) \geq \alpha(x) + \langle \zeta, y - x\rangle - \sigma\|y - x\|^2$, for some $\sigma > 0$ and y close enough to $x\}$ is the subgradient of α at x, then the subset $\partial_-\alpha(x) \neq \varnothing$ for x in a dense subset of X. The same is true for $\beta \in \mathrm{USC}(X, \mathbb{R})$ and its supergradient $\partial_+\beta$.

We shall need the following proposition.

Proposition 8.1. *Let $(u_n)_n \subseteq LSC(X, \overline{\mathbb{R}})$ and $U \in LSC(X, \overline{\mathbb{R}})$. Then $u_n \to U$ in $LSC_\lambda(X, \overline{\mathbb{R}})$, where λ denotes the Lawson topology, if and only if the following are true:*

1. *If $x_n \to x \in X$, then $U(x) \leq \liminf_n u_n(x_n)$;*
2. *For $x \in X$ there exists $z_n \to x$ such that $u_n(z_n) \to U(x)$.*

Proof. (⇒): Suppose $u_n \to U$ in $\mathrm{LSC}_\lambda(X, \bar{\mathbb{R}})$. Then (1) is a consequence of the Scott convergence and the continuity of the evaluation function $E : \mathrm{LSC}_\sigma(X, \bar{\mathbb{R}}) \times X \to \bar{\mathbb{R}}_\sigma$.

(2) For each n, set

$$\beta_n = \inf\{d(y, x) + d(u_n(y), u(x)) : y \in X\}.$$

We will prove that $\beta_n \to 0$.

Let $\varepsilon > 0$. Pick B open in $LSC_\sigma(X, \bar{\mathbb{R}})$ containing U and $0 < \delta < \frac{\varepsilon}{2}$ such that $E(B \times B_\delta(x)) \subseteq (U(x) - \frac{\varepsilon}{2}, \infty]$. Define $Q : X \to \bar{\mathbb{R}}$ by $Q(B_\delta(x)) = U(x) + \frac{\varepsilon}{2}$, $Q(y) = -\infty$ otherwise. Then $U \notin \uparrow Q$ since $U(x) < Q(x)$. Thus there exists N such that $u_n \notin \uparrow Q$ and $u_n \in B$ for $n \geq N$. Then for $n \geq N$,

$$U(x) - \frac{\varepsilon}{2} < u_n(z) < U(x) + \frac{\varepsilon}{2}$$

for some $z \in B_\delta(x)$. Thus

$$d(z, x) + d(u_n(z), U(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and hence $\beta_n < \varepsilon$ for $n \geq N$, thus $\beta_n \to 0$.

Now choose for each n, a point z_n such that

$$d(z_n, x) + d(u_n(z_n), U(x)) < \beta_n + \frac{1}{n}.$$

It follows that $z_n \to x$ and $u_n(z_n) \to U(x)$.

(⇐). Suppose $(u_n)_n \subseteq \mathrm{LSC}(X, \mathbb{R})$ and $U \in \mathrm{LSC}(X, \mathbb{R})$ such that we have (1) and (2). It is clear that (1) implies $u_n \to U$ in $\mathrm{LSC}_\sigma(X, \mathbb{R})$. Let $F \in \mathrm{LSC}(X, \mathbb{R})$ such that $U \in \mathrm{LSC}(X, \mathbb{R}) \setminus \uparrow F$, a basic open set in the $\lambda-$topology. Therefore $F \nleq U$, or equivalently, there exists $x \in X$ such that $F(x) \nleq U(x)$ in $\mathbb{R}$, which means $U(x) < F(x)$. Then there exists $a \in \mathbb{R}$ such that $U(x) < a < F(x)$. By (2) there exists $(z_n)_n \subseteq \mathbb{R}$ such that $z_n \to x$ and $u_n(z_n) \to U(x)$. Since $U(x) \in [-\infty, a) \subseteq \overline{\mathbb{R}}$ is open, there exists $N_1 > 0$ such that for every $n \geq N_1$ $u_n(z_n) \in [-\infty, a)$.

Since F is lower semicontinuous and $F(x) \in (a, \infty]$, there exists an open $W \subseteq X$, $x \in W$ such that $F(W) \subseteq (a, \infty]$, and since $z_n \to x$ there exists $N_2 > 0$ such that $z_n \in W$ for any $n \geq N_2$. Then for every $n \geq N = \max(N_1, N_2)$ we have $F(z_n) \nleq u_n(z_n)$, which implies that for any $n \geq N$ $F \nleq u_n$, or, equivalently, $u_n \in LSC(X, \mathbb{R}) \setminus \uparrow F$. Therefore we have $u_n \to U$ in $LSC_\lambda(X, \bar{\mathbb{R}})$. □

Proposition 8.2. *Let $H : X \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuous. For $f \in Q(X, \mathbb{R})$ let $D_1 = \{x : \partial_- f_*(x) \neq \varnothing\}$ and $D_2 = \{x : \partial_+ f^*(x) \neq \varnothing\}$, which are known to be dense subsets of X. We define*

$$\mathcal{D}_- : Q(X, \mathbb{R}) \to LSC(X, \overline{\mathbb{R}})$$

by

$$\mathcal{D}_- f = \left(x \to \inf_{\substack{a \in \partial_- f_*(x) \\ \partial_- f_*(x) \neq \varnothing}} \{H(x, f_*(x), a) \mid x \in D_1 \subset X\} \right)_*, \tag{8.2}$$

and

$$\mathcal{D}_+ : Q(X,\mathbb{R}) \to USC(X,\overline{\mathbb{R}})$$

by

$$(8.3) \qquad \mathcal{D}_+ f = \left(x \to \sup_{\substack{b\in\partial_+ f^*(x)\\ \partial_+ f^*(x)\neq\varnothing}} \{H(x, f^*(x), b) \mid x \in D_2 \subseteq X\} \right)^* .$$

Let $\Delta \subseteq Q(X,\mathbb{R})$*,* $\Delta = \{f \in Q(X,\mathbb{R}) : \mathcal{D}f = [\mathcal{D}_- f, \mathcal{D}_+ f] \in Q(X,\mathbb{R})\}$*. Then* $\mathcal{D}$ *is a closed operator in* $Q(X,\mathbb{R})$ *with domain* Δ*.*

For proving this proposition we will need the next result.

Lemma 8.3. *Let* $U \subseteq \mathbb{R}^n$ *be locally compact,* $f : U \to \mathbb{R}$ *be lower semicontinuous,* $x \in U$ *such that* $\partial_- f(x) \neq \varnothing$ *and* $a \in \partial_- f(x)$*. Suppose also that* $(f_i) \in LSC(X,\mathbb{R})$ *is a sequence such that* $f_i \to f$ *in* $LSC_\lambda(X,\mathbb{R})$*. Then there exists* $x_i' \in U$*,* $\partial_- f_i(x_i') \neq \varnothing$ *and* $a_i' \in \partial_- f_i(x_i')$ *such that*

$$(8.4) \qquad x_i' \to x,\ f_i(x_i') \to f(x) \text{ and } a_i' \to a.$$

Proof. This is a particular case of Proposition 8.1 from [1], applied to lower semicontinuous functions. By our Proposition 8.1 from earlier in this section the Lawson convergence in $\mathrm{LSC}(X,\mathbb{R})$ is equivalent with the conditions assumed in Proposition 8.1 from [1] for the lower semicontinuous case. □

Proof. (Of Proposition 8.2) Let $(f_i) \subseteq \Delta$ such that $f_i \to f$, $f \in \Delta$, $\mathcal{D}f_i \to F$ in $Q(X,\mathbb{R})$. We will show that $F = \mathcal{D}f$ in $Q(X,\mathbb{R})$. Suppose $F \neq \mathcal{D}f$. By Proposition 6.7 there exist a nonempty open $U \subseteq X$, b_1, $b_2 \in \mathbb{R}$, $b_1 < b_2$ such that $\mathcal{D}f(x) \subseteq [-\infty, b_1)$ and $F(x) \subseteq (b_2, \infty]$ for any $x \in U$ or vice versa. Therefore in U we have $F_* > b_2$ and $\mathcal{D}_+ f < b_1$. Let $x \in U$. Then $F_*(x) > b_2$ and $\mathcal{D}_+ f(x) < b_1$. Using the continuity of the evaluation map $E : \mathrm{LSC}_\sigma(X,\mathbb{R}) \times X \to \mathbb{R}_\sigma$ we find $O \subseteq \mathrm{LSC}(X,\mathbb{R})$ open, $F_* \in O$ and $U_1 \subseteq X$ open, $x \in U_1$ such that $E(O \times U_1) \in (b_2, \infty]$. Since $\mathcal{D}f_i \to F$ in $Q(X,\mathbb{R})$ then $\mathcal{D}_- f_i \to F_*$ in $LSC_\sigma(X,\mathbb{R})$. Therefore there exists $N_1 > 0$ such that

$$\mathcal{D}_- f_i(y) > b_2, \text{ for each } i \geq N_1, y \in U_1.$$

For every $y \in U$ we have

$$\mathcal{D}_+ f(y) < b_1.$$

Let $W = U \cap U_1$. Thus

$$\mathcal{D}_- f_i(y) > b_2 \text{ and } \mathcal{D}_+ f(y) < b_1, \text{ each } i \geq N_1, y \in W,$$

which implies that for every $y, y' \in W$ and $n \geq N_1$ we have

$$\mathcal{D}_- f_i(y') - \mathcal{D}_+ f(y) > b_2 - b_1 = c.$$

Because $\mathcal{D}f \in Q(X,\mathbb{R})$ we have $\mathcal{D}_+ f(y) \geq \mathcal{D}_- f(y)$ for any $y \in W$, so we get

$$\mathcal{D}_- f_i(y') - \mathcal{D}_- f(y) > c, \text{ for each } y, y' \in W \text{ and } i \geq N_1.$$

Let $\mu = \inf \mathcal{D}_- f(W)$. It follows from the fact that $\inf(F_*(W)) = \inf(F(W))$ for any open set W and any function F and the definition of $\mathcal{D}_- f$ that

$$\inf\{H(x, f_*(x), a) : x \in W,\ a \in \partial_- f_*(x)\} = \mu.$$

Thus there exist $z \in W$, $a \in \partial_- f_*(z) \neq \varnothing$ such that

$$\mathcal{D}_- f_i(y') - H(z, f_*(z), a) > c/2, \text{ for every } y' \in W \text{ and } i \geq N_1.$$

From the definition of $\mathcal{D}_-(f_i)_*$, for any $x' \in W$ for which $\partial_-(f_i)_*(x') \neq \varnothing$, we have

$$H(x', (f_i)_*(x'), a') \geq \mathcal{D}_- f_i(x'),$$

for every $a' \in \partial_-(f_i)_*(x')$. Therefore, from the last two inequalities we conclude:

Statement 1.
For any $i \geq N_1$, there exists $(z, a) \in W \times \mathbb{R}^n$, $a \in \partial_- f_*(z) \neq \varnothing$ such that for every $(x', a') \in W \times \mathbb{R}^n$, $a' \in \partial_-(f_i)_*(x') \neq \varnothing$ we have

$$H(x', (f_i)_*(x'), a') - H(z, f_*(z), a) > c/2. \tag{8.5}$$

We now apply Lemma 8.3, knowing from Theorem 6.6 that the Lawson topology and the Scott topology agree on the set maximal elements of $\widehat{\mathbb{L}}$ (which are the ones whose coordinates come from quasicontinuous functions).

Statement 2.
For any $\varepsilon > 0$, there exists $N_2 > 0$ such that for every $i \geq N_2$, there exists $(x', a') \in W \times \mathbb{R}^n$ where $a' \in \partial_-(f_i)_*(x')$ with the property

$$\|z - x'\| < \varepsilon,\ |f_*(z) - (f_i)_*(x')| < \varepsilon,\ \|a - a'\| < \varepsilon. \tag{8.6}$$

The continuity of H implies that for any $\eta > 0$ there exists $\varepsilon > 0$ such that for any $i > 0$

$$\max\{\|z - x'\|, |f_*(z) - (f_i)_*(x')|, \|a - a'\|\} < \varepsilon$$

implies

$$|H(z, f_*(z), a) - H(x', (f_i)_*(x'), a')| < \eta.$$

Choosing $\eta < c/2$ we obtain an $\varepsilon = \varepsilon(\eta)$, and for this ε, using Statement 2, we can find an $N > 0$ for which there exists $(x', a') \in W \times \mathbb{R}^n$, where $a' \in \partial_-(f_i)_*(x')$ such that we have (8.6), which implies

$$|H(z, f_*(z), a) - H(x', (f_i)_*(x'), a')| < c/2. \tag{8.7}$$

We can observe that (8.7) is in contradiction with (8.5), and that means the operator $\mathcal{D}$ is closed. $\square$

Remark 8.4. In the begining of the proof of Proposition 8.2 we considered only one case of Proposition 6.7. For the other case the proof is similar to this one, only we work in $\mathrm{USC}(X, \mathbb{R})$, and we use the exact form of Proposition 8.1 from [1].

9. Viscosity Functions

Definition 9.1. *A function $\varphi : X \to \mathbb{R}$ is a* (discontinuous) viscosity solution *of $H(x, f, \nabla f) = g(x)$ if for any $x \in X$ such that $\partial_- \varphi_*(x) \neq \varnothing$, for any $a \in \partial_- \varphi_*(x)$ the inequality*

$$H_*(x, \varphi_*(x), a) \geq g_*(x) \tag{9.1}$$

is true, and for any $x \in X$ such that $\partial_+ \varphi^*(x) \neq \varnothing$, for any $b \in \partial_+ \varphi^*(x)$ the inequality

$$H^*(x, \varphi^*(x), b) \leq g^*(x) \tag{9.2}$$

is true. We will call such a function a viscosity function.

Remark 9.2. If $f \in Q(X, \mathbb{R})$, then either none or all representatives of the class of f are viscosity solutions of the equation (8.1).

Proposition 9.3. *Let $H : X \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $\mathcal{D}$ be the operator in $Q(X, \mathbb{R})$ with the domain Δ defined in Proposition 8.2, and $g \in Q(X, \mathbb{R})$.Then every solution $f \in \Delta$ of the equation $\mathcal{D}y = g$ is a viscosity solution of the equation*

$$H(x, y(x), \nabla y(x)) = g(x). \tag{9.3}$$

Proof. Let $f \in \Delta$ be such that $\mathcal{D}f = g$. Then we have

$$\mathcal{D}_- f = g_* \text{ and } \mathcal{D}_+ f = g^*.$$

By definition of $\mathcal{D}_-$ in Proposition 8.2, for any $x \in X$ such that $\delta_- f_*(x) \neq \varnothing$ we have that

$$g_*(x) = \mathcal{D}_- f(x) \leq \inf_{a \in \partial_- f_*(x)} H(x, f_*(x), a),$$

which implies that for any $a \in \partial_- f_*(x)$

$$H(x, f_*(x), a) \geq g_*(x).$$

Since H is continuous, $H = H_*$, which implies that inequality (9.1) from Definition 9.1 is true.

Similarly we can obtain inequality (9.2), therefore f is a viscosity solution. □

10. Future Directions

In regard to generalized derivatives, we would like to extend the definition of $Q^1(X, \mathbb{R}^n) \subseteq Q(X, \mathbb{R}^n)$ to include those quasicontinuous functions that have a strong derivative at a dense set of points. One would then like to work out in more detail the calculus of such functions.

We have only indicated an approach to connecting quasicontinuous functions with the study of Hamiltonian equations. Although Samborski [19] has carried out some work in this direction, it appears that much remains to be done. In particular, we would like to see if domain theoretic ideas can contribute to this investigation.

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