

# Baire spaces and hyperspace topologies revisited

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## ABSTRACT

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*It is the purpose of this paper to show how to use approach spaces to get a unified method of proving Baireness of various hyperspace topologies. This generalizes results spread in the literature including the general (proximal) hit-and-miss topologies, as well as various topologies generated by gap and excess functionals. It is also shown that the Vietoris hyperspace can be non-Baire even if the base space is a 2nd countable Hausdorff Baire space.*

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## 1. INTRODUCTION

A topological space is a *Baire space* [7] provided countable collections of dense open subsets have a dense intersection. Baireness is one of the weakest completeness properties, and yet, it has fundamental applications throughout mathematics. This is why there has been interest in investigating Baireness, along with other completeness properties, in the theory of hyperspace topologies which in turn have applications in various branches of mathematics on their own ([1], [12]). In the present paper we will continue in this research by exhibiting some common features of the plethora of studied hyperspaces as far as their Baireness is concerned. Indeed, following the unified exposition of hyperspace topologies introduced in [20], we prove Baireness of these general hyperspaces in

one theorem, thereby generalizing and extending several results about specific hyperspaces spread in the literature ([1],[14],[19],[23],[21],[3],[4],[9]).

In hyperspace theory, it is customary to assume that the base space is Hausdorff (or at least  $T_1$ ), since then singletons are closed, and the base space embeds into the hyperspace. It was observed that imposing separation axioms on the base space is frequently not necessary to obtain results on hypertopologies (see [6], [19], [24], [10]), which is the case throughout this paper as well.

At the end, an example is provided that shows the limitations of Baireness results for the Vietoris topology.

## 2. PRELIMINARIES

Throughout the paper  $\omega$  stands for the non-negative integers,  $\mathcal{P}(X)$  for the power set,  $CL(X)$  for the nonempty closed subsets of a topological space  $X$ , and  $E^c$  for the complement of  $E \subset X$  in  $X$ . The general description of the hyperspaces we will use was given in [20], here we just provide the definitions so the paper is self-contained, more detail can be found in [20], and the references therein.

Suppose that  $(X, \delta)$  is an *approach space* [13], i.e.  $X$  is a nonempty set and  $\delta : X \times \mathcal{P}(X) \rightarrow [0, \infty]$  is a so-called *distance* (on  $X$ ) having the following properties:

- (D1)  $\forall x \in X : \delta(x, \{x\}) = 0$
- (D2)  $\forall x \in X : \delta(x, \emptyset) = \infty$
- (D3)  $\forall x \in X \forall A, B \subset X : \delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}$
- (D4)  $\forall x \in X \forall A \subset X \forall \varepsilon > 0 : \delta(x, A) \leq \delta(x, B_\varepsilon(A)) + \varepsilon,$

where  $B_\varepsilon(A) = \{x \in X : \delta(x, A) \leq \varepsilon\}$ ; we will also use the notation  $S_\varepsilon(A) = \{x \in X : \delta(x, A) < \varepsilon\}$ . Every approach space  $(X, \delta)$  generates a topology  $\tau_\delta$  on  $X$  defined by the closure operator:  $\bar{A} = \{x \in X : \delta(x, A) = 0\}$ ,  $A \subset X$ .

The functional  $D : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow [0, \infty]$  will be called a *gap* provided:

- (G1)  $\forall A, B \subset X : D(A, B) \leq \inf_{a \in A} \delta(a, B)$
- (G2)  $\forall x \in X \forall A \subset X : D(\overline{\{x\}}, A) = \inf_{y \in \{x\}} \delta(y, A)$
- (G3)  $\forall A, B, C \subset X : D(A \cup B, C) = \min\{D(A, C), D(B, C)\}.$

In the sequel (unless otherwise stated)  $X$  will stand for an approach space  $(X, \delta)$  with a gap  $D$  (denoted also as  $(X, \delta, D)$ ).

*Remark 2.1.*

(1) Examples of distances:

- [13] Let  $X$  be a topological space. For  $x \in X$  and  $A \subset X$  define

$$\delta_t(x, A) = \begin{cases} 0, & \text{if } x \in \bar{A}, \\ \infty, & \text{if } x \notin \bar{A}. \end{cases}$$

Then  $\delta_t$  is a distance on  $X$ , and  $\tau_{\delta_t}$  coincides with the topology of  $X$ .

- Let  $(X, \mathcal{U})$  be a uniform space. Then  $\mathcal{U}$  is generated by the family  $\mathcal{D}$  of uniform pseudo-metrics on  $X$  such that  $d \leq 1$  for all  $d \in \mathcal{D}$ , and  $d_1, d_2 \in \mathcal{D}$  implies  $\max\{d_1, d_2\} \in \mathcal{D}$ . Then

$$\delta_u(x, A) = \sup_{d \in \mathcal{D}} d(x, A), \quad x \in X, A \subset X$$

defines a (bounded) distance on  $X$  and  $\tau_{\delta_u}$  coincides with the topology induced by  $\mathcal{U}$  on  $X$ .

- Let  $(X, d)$  be a metric space. Then  $\delta_m(x, A) = d(x, A)$  is a distance on  $X$  and  $\tau_{\delta_m}$  is the topology generated by  $d$  on  $X$ .

(2) Examples of gaps:

- For an arbitrary approach space  $(X, \delta)$

$$D(A, B) = \inf_{a \in A} \delta(a, B)$$

is clearly a gap on  $X$ . This is how we are going to define the gap  $D_t$  (resp.  $D_m$ ) in topological (metric) spaces using the relevant distance  $\delta_t$  (resp.  $\delta_m$ ).

- Let  $(X, \mathcal{U})$  be a uniform space generated by the family  $\mathcal{D}$  of uniform pseudo-metrics on  $X$  bounded above by 1. For  $A, B \subset X$  define

$$D_u(A, B) = \sup_{d \in \mathcal{D}} \inf_{a \in A} d(a, B).$$

Then  $D_u$  is a gap on  $X$ .

(3) Excess functional: for all  $A, B \subset X$  define the *excess* of  $A$  over  $B$  by

$$e(A, B) = \sup_{a \in A} \delta(a, B).$$

The symbols  $e_t, e_u, e_m$  will stand for the excess in topological, uniform and metric spaces, respectively defined via  $\delta_t, \delta_u, \delta_m$ .

### 3. HYPERSPACE TOPOLOGIES

For  $E \subset X$  write  $E^- = \{A \in CL(X) : A \cap E \neq \emptyset\}$ ,  $E^+ = \{A \in CL(X) : A \subset E\}$  and  $E^{++} = \{A \in CL(X) : D(A, E^c) > 0\}$ .

In what follows  $\Delta_1 \subset CL(X)$  is arbitrary and  $\Delta_2 \subset CL(X)$  is such that

$$\forall \varepsilon > 0 \forall A \in \Delta_2 \Rightarrow S_\varepsilon(A) \text{ is open in } X.$$

Denote  $\mathfrak{D} = \mathfrak{D}(\Delta_1, \Delta_2) = \bigcup_{k \in \omega} (\Delta_1 \cup \{\emptyset\})^{k+1} \times (\Delta_2 \cup \{X\})^{k+1} \times (0, \infty)^{2k+2}$ . Whenever referring to some  $S, T \in \mathfrak{D}$ , we will assume that for some  $k, l \in \omega$

$$\begin{aligned} S &= (S_0, \dots, S_k; \tilde{S}_0, \dots, \tilde{S}_k; \varepsilon_0, \dots, \varepsilon_k; \tilde{\varepsilon}_0, \dots, \tilde{\varepsilon}_k) \\ T &= (T_0, \dots, T_l; \tilde{T}_0, \dots, \tilde{T}_l; \eta_0, \dots, \eta_l; \tilde{\eta}_0, \dots, \tilde{\eta}_l). \end{aligned}$$

For  $S \in \mathfrak{D}$  denote

$$M(S) = \bigcap_{i \leq k} (B_{\varepsilon_i}(S_i))^c \cap S_{\tilde{\varepsilon}_i}(\tilde{S}_i) \text{ and}$$

$$S^* = \bigcap_{i \leq k} \{A \in CL(X) : D(A, S_i) > \varepsilon_i \text{ and } e(A, \tilde{S}_i) < \tilde{\varepsilon}_i\}.$$

For  $U_0, \dots, U_n \in \tau_\delta \setminus \{\emptyset\}$  and  $S \in \mathfrak{D}$  denote

$$(U_0, \dots, U_n)_S = \bigcap_{i \leq n} U_i^- \cap S^*,$$

$$[U_0, \dots, U_n]_S = \prod_{i \leq n} (U_i \cap M(S)) \times \prod_{i > n} M(S).$$

It is easy to see that the collections

$$\mathcal{B}^* = \{(U_0, \dots, U_n)_S : U_0, \dots, U_n \in \tau_\delta \setminus \{\emptyset\}, S \in \mathfrak{D}, n \in \omega\},$$

$$\mathcal{B} = \{[U_0, \dots, U_n]_S : U_0, \dots, U_n \in \tau_\delta \setminus \{\emptyset\}, S \in \mathfrak{D}, n \in \omega\}$$

form a base for topologies on  $CL(X)$  and  $X^\omega$ , respectively; denote them by  $\tau^*$ , and  $\tau$ , respectively.

*Remark 3.1.* Note that  $\tau$  is a “pinched-cube” topology as defined in [3], [23]; indeed, we just need to take  $\Delta = \{M(S)^c : S \in \mathfrak{D}\}$ .

*Remark 3.2.*

- Let  $(X, \tau)$  be a topological space. Let  $\Delta_1 = \Delta$  and  $\Delta_2 = \{X\}$ . Then for  $B \in \Delta$  and  $\varepsilon, \eta > 0$ ,  $\{A \in CL(X) : D_t(A, B) > \varepsilon\} = (B^c)^+$  and  $\{A \in CL(X) : e_t(A, X) < \eta\} = CL(X)$ . Thus  $\tau^* = \tau^+$  is the general *hit-and-miss topology* on  $CL(X)$  (see [17], [1], [8], [19], [3]). Choosing  $\Delta = CL(X)$  we get the most studied hit-and-miss topology, the so-called *Viectoris topology*  $\tau_V$  (cf. [15], [5], [1]); a typical base element for  $\tau_V$  is

$$(U_0, \dots, U_n) = \{A \in CL(X) : A \subseteq \bigcup_{i \leq n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for all } i \leq n\}.$$

- Let  $(X, \mathcal{U})$  be a uniform space. Let  $\Delta_1 = \Delta$  and  $\Delta_2 = \{X\}$ . Then for  $B \in \Delta$  and  $\varepsilon, \eta > 0$ ,  $\{A \in CL(X) : D_u(A, B) > \varepsilon\} = (B^c)^{++}$  and  $\{A \in CL(X) : e_u(A, X) < \eta\} = CL(X)$ . Thus  $\tau^* = \tau^{++}$  is the *proximal hit-and-miss topology* on  $CL(X)$  (see [1], [19], [3]).
- Let  $(X, d)$  be a metric space. Let  $\Delta_1, \Delta_2 \subset CL(X)$  be such that  $\Delta_1$  contains the singletons. Then  $\tau^*$  coincides with the weak hypertopology  $\tau_{weak}$  generated by gap and excess functionals (see [2], [9], [20]).

As we indicated in the Introduction, we do not assume any separation axiom on  $X$ , however, the following property seems to be necessary: we will say that  $(X, \delta, D)$  has *property (P)* provided

$$\forall x \in X \forall k \in \omega \forall A_0, \dots, A_k \in CL(X) \forall \varepsilon_0, \dots, \varepsilon_k > 0 \exists y \in \overline{\{x\}} : \\ \delta(x, A_i) > \varepsilon_i \implies D(\overline{\{y\}}, A_i) > \varepsilon_i \text{ for all } i \leq k.$$

This property is satisfied in uniform and metric spaces, and a topological space  $X$  has property  $(P)$  iff  $X$  is *weakly- $R_0$*  [24], i.e. for all open  $U \subseteq X$  and  $x \in U$  there is a  $y \in \overline{\{x\}}$  with  $\overline{\{y\}} \subseteq U$  iff each nonempty difference of open sets contains a nonempty closed set.

We will say that the family  $\mathfrak{D}$  is *weakly quasi-Urysohn* [20] provided for all  $\emptyset \neq (U_0, \dots, U_n)_S \in \mathcal{B}^*$  there is a  $T \in \mathfrak{D}$  such that  $\emptyset \neq (U_0, \dots, U_n)_T \subseteq (U_0, \dots, U_n)_S$  and

$$(*) \quad \forall E \text{ countable} : E \subseteq M(T) \implies \overline{E} \in S^*.$$

The translation of this property for the (proximal) hit-and-miss topologies is as follows: given a topological (uniform) space  $X$  (resp.  $(X, \mathcal{U})$ ), the family  $\Delta \subseteq CL(X)$  is said to be (*uniformly*) *weakly quasi-Urysohn* provided whenever  $S \in \Delta$  is disjoint to some nonempty open  $U_i \subseteq X$  ( $i \leq n$ ), there exists  $T \in \Delta$  such that  $U_i \cap T^c \neq \emptyset$  for all  $i \leq n$ ,  $S \subseteq T$ , and

$$\forall E \text{ countable} : (E \subseteq T^c \implies \overline{E} \subseteq S^c) \text{ (resp. } \overline{E} \subseteq U[S]^c \text{ for some } U \in \mathcal{U}).$$

We will say that  $X$  is (*weakly*) *quasi-regular* provided every nonempty open  $U \subseteq X$  has a nonempty open subset  $V$  such that  $\overline{V} \subseteq U$  (resp.  $\overline{E} \subseteq U$  for all countable  $E \subseteq V$ ). It is not hard to see that if  $X$  is weakly quasi-regular, then  $CL(X)$  is weakly quasi-Urysohn.

**Proposition 3.3** ([20, Lemma 3.1]). *labelZs2 Suppose that  $(X, \delta, D)$  has property  $(P)$ , and  $(U_0, \dots, U_n)_S, (V_0, \dots, V_m)_T \in \mathcal{B}^*$ . Then  $(U_0, \dots, U_n)_S \subseteq (V_0, \dots, V_m)_T$  implies  $M(S) \subseteq M(T)$ , and for all  $j \leq m$  there exists  $i \leq n$  with  $M(S) \cap U_i \subseteq M(T) \cap V_j$ .*

Although it would be possible to establish Baireness of  $(CL(X), \tau^*)$  using the so-called Banach-Mazur game, as done in [14], [19], or [3], we have chosen a different method, which makes the proofs more transparent and actually improves on results of the above mentioned papers.

We will need some auxiliary material: if  $Y, Z$  are topological spaces, the mapping  $f : Y \rightarrow Z$  is said to be *feebly continuous* [7] provided  $\text{int} f^{-1}(U) \neq \emptyset$  for each open  $U \subseteq Z$  with  $f^{-1}(U) \neq \emptyset$ ; further  $f$  is  *$\delta$ -open* [7] provided  $f(A)$  is somewhere dense in  $Z$  for every somewhere dense  $A$  of  $Y$ .

**Proposition 3.4** ([7, Theorem 4.7]). *If  $f$  is a feebly continuous  $\delta$ -open function from a Baire space onto a space  $Y$ , then  $Y$  is a Baire space.*

The following theorem generalizes [14, Theorem 3.8], [19, Theorem 4.1], and [3, Theorem 2.5]:

**Theorem 3.5.** *Suppose that  $X$  has property  $(P)$  and  $\mathfrak{D}$  is a weakly quasi-Urysohn family. If  $(X^\omega, \tau)$  is a Baire space, then so is  $(CL(X), \tau^*)$ .*

*Proof.* Denote  $S(X) = \{A \in CL(X) : A \text{ is separable}\}$ . Then  $S(X)$  is dense in  $(CL(X), \tau^*)$ , since even the set of closures of finite subsets of  $X$  is. Thus, it suffices to prove that  $(S(X), \tau^* \upharpoonright_{S(X)})$  is a Baire space. Define the mapping  $f : X^\omega \rightarrow S(X)$  via

$$f((x_k)_k) = \overline{\{x_k : k \in \omega\}}, \text{ where } (x_k)_k \in X^\omega.$$

We will show that  $f$  is a feebly continuous  $\delta$ -open function, so Proposition 3.4 will apply.

To see feeble continuity, take a  $\mathbf{U} = (U_0, \dots, U_n)_S \cap S(X)$  such that  $f^{-1}(\mathbf{U}) \neq \emptyset$ . Now if we take the  $T \in \mathfrak{D}$  from weak quasi-Urysohnness of  $\mathfrak{D}$  corresponding to  $(U_0, \dots, U_n)_S$ , then (\*) virtually claims that  $\emptyset \neq [U_0, \dots, U_n]_T \subset f^{-1}(\mathbf{U})$ .

To justify  $\delta$ -openness of  $f$ , take an  $\mathbf{A} \subset X^\omega$  which is dense in some  $\emptyset \neq \mathbf{V} = [V_0, \dots, V_m]_P \in \mathcal{B}$ . Then  $f(\mathbf{A})$  is dense in  $\mathbf{V}^* = (V_0, \dots, V_m)_P \cap S(X)$ . Indeed, if  $\mathbf{U}^* = (U_0, \dots, U_n)_S \cap S(X)$  is a nonempty open subset of  $\mathbf{V}^*$ , take the  $T \in \mathfrak{D}$  from weak quasi-Urysohnness of  $\mathfrak{D}$  corresponding to  $(U_0, \dots, U_n)_S$ . Then in view of Proposition ??,  $[U_0, \dots, U_n]_T$  is a nonempty open subset of  $\mathbf{V}$ . Consequently, we can find an  $(x_k)_k \in \mathbf{A} \cap [U_0, \dots, U_n]_T$ . Then  $E = \{x_k : k \in \omega\} \subset M(T)$ , so by (\*),  $f((x_k)_k) = \overline{E} \in S^*$ , and hence  $f((x_k)_k) \in f(\mathbf{A}) \cap \mathbf{U}^*$ .  $\square$

A collection  $\mathcal{P}$  of nonempty open subsets in a space  $X$  is a  $\pi$ -base, if every nonempty open subset of  $X$  contains at least one member of  $\mathcal{P}$ ; moreover,  $\mathcal{P}$  is a *countable-in-itself*  $\pi$ -base [22], provided each member of  $\mathcal{P}$  contains countably many members of  $\mathcal{P}$ .

**Corollary 3.6.** *Suppose that  $X$  is a Baire space with a countable-in-itself  $\pi$ -base and it has property (P). Suppose that  $\mathfrak{D}$  is a weakly quasi-Urysohn family. Then  $(CL(X), \tau^*)$  is a Baire space.*

*Proof.* Since  $\tau$  is a pinched-cube topology (see Remark 3.1), it follows by [23, Theorem 2.1], that  $(X^\omega, \tau)$  is a Baire space, so Theorem 3.5 applies.  $\square$

#### 4. APPLICATIONS

The following theorem improves [19, Corollary 4.2], and [3, Theorem 4.1, Theorem 5.1]:

**Theorem 4.1.** *Suppose that  $X$  is a weakly- $R_0$  (resp. uniform) space having a countable-in-itself  $\pi$ -base, which is also a Baire space. If  $\Delta \subset CL(X)$  is a (uniformly) weakly quasi-Urysohn subfamily, then  $(CL(X), \tau^+)$  (resp.  $(CL(X), \tau^{++})$ ) is a Baire space.*

The previous theorem yields the following corollary, which slightly generalizes [14, Corollary 3.9] (see also [4, Corollary 2.2]):

**Theorem 4.2.** *Let  $(X, \tau)$  be a weakly- $R_0$ , weakly quasi-regular Baire space having a countable-in-itself  $\pi$ -base. Then  $(CL(X), \tau_V)$  is a Baire space.*

In the next example we show that weak quasi-regularity of  $X$  is an essential condition in the Baireness results concerning the Vietoris topology. To do this we need to recall some basic facts about the *Banach-Mazur game*  $BM(X)$  (see [7] or [11]) played by players  $\alpha$  and  $\beta$  who take turns choosing sets from a fixed  $\pi$ -base  $\mathcal{P}$  in the topological space  $X$ :  $\beta$  starts by picking  $V_0$ , then  $\alpha$  responds by choosing some  $U_0 \subseteq V_0$ . Continuing like this, while each player picks a

nonempty open set contained in the previous choice of the opponent, one gets a run  $V_0, U_0, \dots, V_n, U_n, \dots$  of  $BM(X)$ , which is won by  $\alpha$ , if  $\bigcap_{i < \omega} U_i \neq \emptyset$ , otherwise,  $\beta$  wins. The key result about the Banach-Mazur game is that  $X$  is not a Baire space iff  $\beta$  has a winning tactic in  $BM(X)$ , i.e. there is a function  $t : \mathcal{P} \cup \{\emptyset\} \rightarrow \mathcal{P}$  such that if  $t(\emptyset) = V_0$ , and  $t(U_n) = V_{n+1}$  for each  $n < \omega$ , then  $\beta$  wins.

**Example 4.3.** There exists a Hausdorff, Baire, 2nd countable space such that  $(CL(X), \tau_V)$  is not Baire.

*Proof.* Let  $\mathcal{E}$  be the Euclidean topology on  $\mathbb{R}$ , and  $B \subset \mathbb{R}$  be a Bernstein set [7], i.e. such that both  $B$  and  $\mathbb{R} \setminus B$  meet every uncountable  $G_\delta$  subset of  $\mathbb{R}$ . Define a finer than  $\mathcal{E}$  topology  $\mathcal{E}_B$  on  $\mathbb{R}$  as follows:

$$\mathcal{E}_B = \{U \cup V \cap B : U, V \in \mathcal{E}\}.$$

Then  $X = (\mathbb{R}, \mathcal{E}_B)$  is clearly Hausdorff and 2nd countable.

In order to prove that  $(CL(X), \tau_V)$  is  $\beta$ -favorable, we will play the Banach-Mazur game using the following  $\pi$ -base for the hyperspace:

$$\mathcal{P}_V = \{(I_0 \cap B, \dots, I_n \cap B) : I_0, \dots, I_n \text{ are pairwise disjoint open intervals}\}.$$

Define a tactic  $t_V$  for  $\beta$  in  $BM(CL(X))$  as follows: put  $t_V(\emptyset) = B^+$ ; moreover, if  $\mathbf{U} = (I_0 \cap B, \dots, I_n \cap B) \in \mathcal{P}_V$ , for each  $i \leq n$ , choose disjoint bounded open intervals  $J_i^0, J_i^1$  so that  $\overline{J_i^0 \cup J_i^1}^\mathcal{E} \subseteq I_i$ , and put

$$t_V(\mathbf{U}) = (J_0^0 \cap B, J_0^1 \cap B, \dots, J_n^0 \cap B, J_n^1 \cap B).$$

Let  $B^+, \mathbf{U}_0, t_V(\mathbf{U}_0), \dots, \mathbf{U}_k, t_V(\mathbf{U}_k), \dots$  be a run of  $BM(CL(X))$  compatible with  $t_V$ . Suppose that there is some  $A \in \bigcap_k \mathbf{U}_k$ . Then  $A$  is a  $\mathcal{E}_B$ -closed subset of  $B$ , and since the  $\mathcal{E}$ -, and  $\mathcal{E}_B$ -closure of subsets of  $B$  coincide,  $A$  would be a  $\mathcal{E}$ -closed, and in view of the definition of  $t_V$ , also dense-in-itself subset of  $B$ , which contradicts the definition of the Bernstein set. Consequently,  $\bigcap_k \mathbf{U}_k = \emptyset$ , and  $\beta$  wins the run.  $\square$

*Remark 4.4.* It follows by [18, Example 2.7], that the previous example is not quasi-regular, but it is not even weakly quasi-regular by Theorem 4.2. It is also an example showing that quasi-regularity cannot be removed in the key result of [4, Theorem 2.1], stating that if  $X$  is a quasi-regular space such that  $X^\omega$  is a Baire space, then  $(CL(X), \tau_V)$  is a Baire space, since by a result of Oxtoby [16] products of 2nd countable Baire spaces are Baire.

Our last result generalizes [9, Theorem 4.2] and [20, Corollary 6.1]:

**Theorem 4.5.** Let  $(X, d)$  be a separable metric Baire space, and  $\Delta_1, \Delta_2 \subset CL(X)$  be such that  $\Delta_1$  contains the singletons. Then  $(CL(X), \tau_{weak})$  is a Baire space.

*Proof.* It follows from Corollary 3.6, since  $\mathfrak{D}(\Delta_1, \Delta_2)$  is weakly quasi-Urysohn by [20, Remark 3.1(iii)].  $\square$

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