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# On the Order Hereditary Closure Preserving Sum Theorem

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Abstract. The main purpose of this paper is to prove the following two theorems, an order hereditary closure preserving sum theorem and an hereditary theorem:

(1) If a topological property $\mathcal{P}$ satisfies $(\sum')$ and is closed hereditary, and if $\mathcal{V}$ is an order hereditary closure preserving open cover of $X$ and each $V \in \mathcal{V}$ is elementary and possesses $\mathcal{P}$, then $X$ possesses $\mathcal{P}$.
(2) Let a topological property $\mathcal{P}$ satisfy $(\sum')$ and $(\beta)$, and be closed hereditary. Let $X$ be a topological space which possesses $\mathcal{P}$. If every open subset $G$ of $X$ can be written as an order hereditary closure preserving (in $G$) collection of elementary sets, then every subset of $X$ possesses $\mathcal{P}$.

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## 1. Introduction

R. E. Hodel [1] obtained sum theorems and an hereditary theorem for topological spaces. S. P. Arya and M. K. Singal [1, 2] and G. Gao [4] have improved some of Hodel's sum theorems. We provide in this paper further improvements of these theorems.

A topological property $\mathcal{P}$ is said to be hereditary (closed hereditary, open hereditary) if when $\mathcal{P}$ is possessed by a topological space $X$, it is also shared by every subspace (closed subspace, open subspace) of $X$. It is well known that covering properties such as paracompactness, subparacompactness, countable paracompactness, pointwise paracompactness, $\theta$-refinement and collectionwise normality satisfy the following result which is denoted by $(\beta)$.

$(\beta)$ : If every open subset of a space $X$ has a property $\mathcal{P}$, then every subset of $X$ has the property $\mathcal{P}$.

Notice that $X$ is an open subspace of itself, thus ($\beta$) states that open hereditary implies hereditary.

Y.K atuta [6] introduced the notion of an order locally finite family of subsets of a topological space. Later G. Gao [4] also introduced the notion of an order hereditary closure preserving family of subsets of a topological space.
A family $\{A_\gamma : \gamma \in \Gamma\}$ of subsets of a topological space $X$ is called *hereditary closure preserving relative to a subspace* $A$ of $X$ if for any $\Gamma' \subset \Gamma$ and any $E_\gamma \subset A_\gamma$ the following is true for all points in $A$.

$$\overline{\bigcup_{\gamma\in\Gamma'} E_\gamma} = \bigcup \overline{E}_\gamma.$$

**Definition 1.1** (G. Gao [4]). *A family $\{A_\alpha : \alpha < \tau\}$ ($\alpha$ and $\tau$ are ordinal numbers) is defined to be order hereditary closure preserving if for every ordinal number $\beta < \tau$, the family $\{A_\alpha : \alpha < \beta\}$ is hereditary closure preserving relative to $A_\beta$.*

It is not difficult to see that the following implications are true for a family of subsets of a topological space. However, the converse implications are not true in general.

**Proposition 1.2.** *Given a family of subsets of a topological space, then*

$$\begin{array}{ccc} \text{locally finite} & \Rightarrow & \text{hereditary closure preserving} \\ \Downarrow & & \Downarrow \\ \sigma-\text{locally finite} & \Rightarrow & \sigma-\text{hereditary closure preserving} \\ \Downarrow & & \Downarrow \\ \text{order locally finite} & \Rightarrow & \text{order hereditary closure preserving} \end{array}$$

**Definition 1.3** (R. E. Hodel [5]). *Let $N$ be the set of all positive integers. An open subset $V$ of a topological space is called an elementary set if $V = \bigcup_{i=1}^{\infty} V_i$, where each $V_i$ is open and $\overline{V_i} \subset V$ for all $i \in N$.*

The following two lemmas show that each open $F_\sigma$ set in a normal space is exactly an elementary set.

**Lemma 1.4.** *Every elementary set in a topological space is an open $F_\sigma$ set.*

*Proof.* Suppose the open subset $V$ of a topological space is an elementary set, then $V = \bigcup_{i=1}^{\infty} V_i$, $V_i$ is open and $\overline{V_i} \subset V$ for all $i \in N$. Hence $\bigcup_{i=1}^{\infty} \overline{V_i} \subset V$. On the other hand, $V_i \subset \overline{V_i}$ for all $i \in N$, so $V = \bigcup_{i=1}^{\infty} V_i \subset \bigcup_{i=1}^{\infty} \overline{V_i}$. Therefore, $V = \bigcup_{i=1}^{\infty} \overline{V_i}$, it follows that $V$ is an open $F_\sigma$ set. □

**Lemma 1.5.** *Every open $F_\sigma$ subset of a normal space is an elementary set.*

*Proof.* Let $V$ be an open $F_\sigma$ set of a normal space $X$, then $V = \bigcup_{i=1}^{\infty} W_i$, $W_i$ is closed and $W_i \subset V$ for all $i \in N$. By the normality of $X$, for each $W_i$ there exists an open set $V_i$ such that $W_i \subset V_i \subset \overline{V_i} \subset V$. Thus, $V = \bigcup_{i=1}^{\infty} W_i \subset \bigcup_{i=1}^{\infty} V_i$ and $\bigcup_{i=1}^{\infty} V_i \subset V$. That is $V = \bigcup_{i=1}^{\infty} V_i$ where each $V_i$ is open and $\overline{V_i} \subset V$ for all $i \in N$. Therefore $V$ is an elementary set. □

Notice that an open $F_\sigma$ set may fail to be an elementary set in non-normal spaces, as the following example shows.

**Example 1.6.** Let $X$ be the set $N$ of all positive integers with cofinite topology. Then $X$ is a $T_1$ space which is not a normal space. Take the set $V = N/\{1,2,3\}$, then $V$ is an open set. Furthermore, $V = \bigcup_{i=4}^{\infty}\{i\}$. Since $X$ is a $T_1$ space, each singleton $\{i\}$ is a closed subset, so that $V$ is an open $F_\sigma$ set. For any subset $S$ of $X$ we have

$$\overline{S} = \begin{cases} S & \text{if } S \text{ is finite,} \\ X & \text{if } S \text{ is infinite.} \end{cases}$$

Since every non-empty open subset $S$ of $X$ is infinite, for every open subset $S$ of $V$,

$$\overline{S} = X \not\subset V.$$

So $V$ is not an elementary set.

We say that a topological property $\mathcal{P}$ satisfies the Locally Finite Closed Sum Theorem if the following is satisfied and denote it by $(\sum)$.

$(\sum)$ : Let $\{F_\alpha : \alpha \in A\}$ be a locally finite closed cover of a topological space $X$ and let each $F_\alpha$ possess a property $\mathcal{P}$, then $X$ possesses the property $\mathcal{P}$.

We say that a topological property $\mathcal{P}$ satisfies the Hereditary Closure Preserving Closed Sum Theorem if the following is satisfied and denote it by $(\sum')$.

$(\sum')$ : Let $\{F_\alpha : \alpha \in A\}$ be an hereditary closure preserving closed cover of a topological space $X$ and let each $F_\alpha$ possess a property $\mathcal{P}$, then $X$ possesses the property $\mathcal{P}$.

Observe from Proposition 1.2 that $(\sum') \Rightarrow (\sum)$.
For example, if the topological property $\mathcal{P}$ is one of paracompactness, subparacompactness, pointwise paracompactness, meso-compactness, $\theta$-refinement, weak $\theta$-refinement and ortho-compactness, then the property $\mathcal{P}$ satisfies $(\sum)$. If the topological property $\mathcal{P}$ is either paracompactness or $T_1$ meso-compactness, then the property $\mathcal{P}$ satisfies $(\sum')$.

## 2. A Sum Theorem

In this section, we assume that the topological property $\mathcal{P}$ satisfies $(\sum')$ (hence $(\sum)$) and is closed hereditary.

**Theorem 2.1.** *Let $\mathcal{V} = \{V_\alpha : \alpha < \tau\}$ be an order hereditary closure preserving open cover of a topological space $X$, and let each $V_\alpha$ be an elementary set which possesses a topological property $\mathcal{P}$. Then $X$ possesses the topological property $\mathcal{P}$.*

*Proof.* Since each $V_\alpha$ is an elementary set and possesses the property $\mathcal{P}$,

$$V_\alpha = \bigcup_{i=1}^{\infty} V_{\alpha,i}, \qquad \overline{V_{\alpha,i}} \subset V_\alpha, \quad \alpha < \tau, i \in N, \tag{2.1}$$

where each $V_{\alpha,i}$ is an open set. Then the closed set $\overline{V_{\alpha,i}}$ possesses the property $\mathcal{P}$ by closed hereditary.
For each $i \in N$, let

$$\mathcal{V}_i = \{V_{\alpha,i} : \alpha < \tau\}.$$

For each $\alpha < \tau$, let

$$F_{0,i} = \overline{V_{0,i}}, \qquad F_{\alpha,i} = \overline{V_{\alpha,i}} - \bigcup_{\beta<\alpha} V_\beta, \quad 0 < \alpha < \tau. \tag{2.2}$$

Then each closed set $F_{\alpha,i}$ possesses the property $\mathcal{P}$. And we claim that the family $\{F_{\alpha,i} : \alpha < \tau\}$ is an hereditary closure preserving collection.
Without loss of generality, for each $\alpha < \tau$, let $A_{\alpha,i} \subset F_{\alpha,i}$, we need to prove

$$\overline{\bigcup_{\alpha<\tau} A_{\alpha,i}} = \bigcup_{\alpha<\tau} \overline{A}_{\alpha,i}.$$

Obviously, it is enough to prove

$$\overline{\bigcup_{\alpha<\tau} A_{\alpha,i}} \subset \bigcup_{\alpha<\tau} \overline{A}_{\alpha,i}. \tag{2.3}$$

Suppose $x \in \overline{\bigcup_{\alpha<\tau} A_{\alpha,i}}$, since $\mathcal{V}$ is a cover of $X$, we may assume $x \in V_{\beta_0}$. Now the inequality (2.3) can be expressed in another way:

$$\left(\overline{\bigcup_{\alpha<\beta_0} A_{\alpha,i}}\right) \cup \overline{A}_{\beta_0,i} \cup \left(\overline{\bigcup_{\beta_0<\alpha<\tau} A_{\alpha,i}}\right) \subset \left(\bigcup_{\alpha<\beta_0} \overline{A}_{\alpha,i}\right) \cup \overline{A}_{\beta_0,i} \cup \left(\bigcup_{\beta_0<\alpha<\tau} \overline{A}_{\alpha,i}\right). \tag{2.4}$$

According to (2.2), $V_{\beta_0} \cap F_{\alpha,i} = \varnothing$, $\beta_0 < \alpha < \tau$. So $X - V_{\beta_0} \supset \bigcup_{\beta_0<\alpha<\tau} F_{\alpha,i}$. Since $X - V_{\beta_0}$ is a closed set, then $X - V_{\beta_0} \supset \overline{\bigcup_{\beta_0<\alpha<\tau} F_{\alpha,i}}$, that is $x \notin \overline{\bigcup_{\beta_0<\alpha<\tau} F_{\alpha,i}}$. Therefore $x \notin \overline{\bigcup_{\beta_0<\alpha<\tau} A_{\alpha,i}}$. If $x \in A_{\beta_0,i}$, the inequality (2.4) is satisfied. We may assume $x \in \overline{\bigcup_{\alpha<\beta_0} A_{\alpha,i}}$. Since $\mathcal{V}$ is order hereditary closure preserving, $\{V_\alpha : \alpha < \beta_0\}$ is hereditary closure preserving at every point of $V_{\beta_0}$. Notice that $x \in V_{\beta_0}$, thus $x \in \bigcup_{\alpha<\beta_0} \overline{A}_{\alpha,i}$. So the inequality (2.2) is proved.

Let $F_i = \bigcup_{\alpha<\tau} F_{\alpha,i}$, then $F_i$ possesses the property $\mathcal{P}$ by applying $(\sum')$, for all

$i \in N$.
For each $i \in N$, let

$$\mathcal{V}_i^* = \bigcup_{\alpha<\tau} \{V_{\alpha,i}\},$$

then $\mathcal{V}_i^* \subset F_i$ by the well order property. Hence $\{\mathcal{V}_i^*\}$ and $\{F_i\}$ are open covers and closed covers of the space $X$ respectively.
Finally, let

$$H_1 = F_1, \qquad H_i = F_i - \bigcup_{j=1}^{i-1} \mathcal{V}_j^*, \quad i = 2, 3, ...$$

then $\{H_i\}$ is a locally finite closed cover of $X$ and each $H_i$ possesses the property $\mathcal{P}$. It follows from $(\sum)$ that $X$ possesses the property $\mathcal{P}$. □

Apply Proposition 1.2 to Theorem 2.1, we can obtain the following two corollaries.

**Corollary 2.2** (S. P. Arya and M. K. Singal [2])**.** *Let $\mathcal{V}$ be a $\sigma$-hereditary closure preserving cover of a topological space $X$ and each $V \in \mathcal{V}$ be an elementary set which possesses a topological property $\mathcal{P}$, then $X$ possesses the property $\mathcal{P}$.*

**Corollary 2.3** (R. E. Hodel [5])**.** *Let $\mathcal{V}$ be a $\sigma$-locally finite cover of a topological space $X$ and each $V \in \mathcal{V}$ be an elementary set which possesses a topological property $\mathcal{P}$, then $X$ possesses the property $\mathcal{P}$.*

## 3. Two Hereditary Theorems

We assume that the topological property $\mathcal{P}$ in this section satisfies $(\sum')$ (hence $(\sum)$), $(\beta)$ and is closed hereditary.

**Theorem 3.1.** *Let $X$ be a topological space which possesses a topological property $\mathcal{P}$. If every open subset $G$ of $X$ can be written as an order hereditary closure preserving (in $G$) collection of elementary sets, then every subset of $X$ possesses the property $\mathcal{P}$.*

*Proof.* Let $\mathcal{V} = \{V_\alpha : \alpha < \tau\}$ be order hereditary closure preserving at every point of $G$, and let $\mathcal{V}^* = \bigcup_{\alpha<\tau} V_\alpha = G$, where each $V_\alpha, \alpha < \tau$ is an elementary subset of $X$. We may assume

$$V_\alpha = \bigcup_{i=1}^{\infty} V_{\alpha,i}, \quad \overline{V_{\alpha,i}} \subset V_\alpha, \quad \alpha < \tau$$

where each $V_{\alpha,i}$ is an open set. Let

$$F_{\alpha,1} = \overline{V_{\alpha,1}}, \quad F_{\alpha,i} = \overline{V_{\alpha,i}} - \bigcup_{j<i} V_{\alpha,j}, \quad i = 2, 3, ...$$

Then $\{F_{\alpha,i}\}$ is a locally finite cover of $V_\alpha$, so it is an hereditary closure preserving cover of $V_\alpha$. Since each $F_{\alpha,i}$ is a closed subset of $X$ and the property $\mathcal{P}$ is closed hereditary, then each $F_{\alpha,i}$ possesses the property $\mathcal{P}$. According to ($\sum'$), each subspace $V_\alpha$ possesses the property $\mathcal{P}$. Apply Theorem 2.1 to the subspace $G$, then $G$ possesses the property $\mathcal{P}$. Since ($\beta$) holds, then every subset of $X$ possesses the property $\mathcal{P}$. □

As Lemma 1.4 and Lemma 1.5 give that open $F_\sigma$ sets are equivalent to elementary sets in a normal space, we attain the following theorem immediately.

**Theorem 3.2.** *Let a normal space $X$ possess a topological property $\mathcal{P}$. If every open subset $G$ of $X$ can be written as an order hereditary closure preserving (in $G$) collection of open $F_\sigma$ sets, then every subset of $X$ possesses the property $\mathcal{P}$.*

**Definition 3.3** (C. H. Dowker [3])**.** *A normal space $X$ is totally normal if every open subset $G$ of $X$ can be written as a locally finite (in $G$) collection of open $F_\sigma$ sets of $X$.*

Finally, Theorem 3.2 and Proposition 1.2 imply the following corollary.

**Corollary 3.4** (R. E. Hodel [5])**.** *Let $X$ be a totally normal space and $X$ have a topological property $\mathcal{P}$, then every subset of $X$ has the property $\mathcal{P}$.*

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