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A Survey on Wallman Bases

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Abstract. Wallman bases are frequently used in compactification processes of topological spaces. However, they are also related with quasi–uniform structures and they are useful to characterize some topological properties. We present a brief survey on the subject which supports these statements.

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1. Historical Background

Among compactification methods in topology, we bring out two of them, which are perhaps the most useful:

- Wallman's method of ultrafilters.
- Completion of totally bounded uniform spaces.

The main advantadge of the method of ultrafilters is its generality: it may be applied to any T_1-space. Wallman applied this method for the first time in 1938 ([17]) and he proved that any T_1–space X has a T_1–compactification ωX which coincides with the *Stone–Čech* compactification if X is normal.

The concept of uniform space was introduced by A. Weil, in 1937 ([18]) and he proved, among other things, that every uniform space $(X,\mathcal{U})$ has unique uniform completion (unique up to uniform equivalences). The notion of totally bounded uniform spaces was formally introduced by N. Bourbaki in 1940, although it was used implicitly by Weil in the well known characterization of compactness in uniform spaces:

> *A uniform space* $(X,\mathcal{U})$ *is compact (i.e., the topology* $\mathcal{T}_{\mathcal{U}}$ *induced by* $\mathcal{U}$ *is compact) iff* $(X,\mathcal{U})$ *is complete and satisfies an extra condition which is precisely total boundedness.*

Since total boundedness is not lost under uniform extensions, we conclude that the uniform completion of a totally bounded uniform T_1–space X is a compactification of X.

The method of Wallman can be applied in smaller families of closed sets and produce a compactification in exactly the same way. One of the requirements is that the family of complements of the given closed sets is a basis for the topology of the space.

2. Annular bases and quasi–uniformities

Every topological space $(X, \mathcal{T})$ has many bases, *i.e.*, subfamilies $\mathcal{B}$ of $\mathcal{T}$ such that every $U \in \mathcal{T}$ can be expressed as a union of some members of $\mathcal{B}$. Annular bases are not so numerous:

A basis $\mathcal{B}$ of $\mathcal{T}$ is <u>annular</u> if it satisfies two conditions:

i) $\varnothing \in \mathcal{B}$ and $X \in \mathcal{B}$.

ii) $B, B' \in \mathcal{B}$ implies that $B \cap B' \in \mathcal{B}$ and $B \cup B' \in \mathcal{B}$.

It is possible that $\mathcal{T}$ is the only annular basis of itself. A less strong condition is that $\mathcal{T}$ has a minimal annular basis, *i.e.*, an annular basis which is contained in every other annular basis. This happens in any locally compact 0–dimensional space.

A successful generalization of uniform spaces was found in 1960 ([1]). Only completely regular spaces admit uniform structures which produce their topology. However, <u>any</u> topological space admits compatible quasi–uniform structures. We briefly state the main definitions.

A quasi–uniformity $\mathcal{U}$ on a set X is a filter in $X \times X$ satisfying the following properties:

i) The diagonal $\triangle(X) = \{(x, x) \mid x \in X\}$ is contained in each member of $\mathcal{U}$, *i.e.*, each $U \in \mathcal{U}$ is a reflexive relation in X, and

ii) whenever $U \in \mathcal{U}$, there exists an element $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.

A connector on X is, by definition, a reflexive relation on X. Therefore, every element of a quasi–uniformity is a connector and the family of all connectors on X is the largest quasi–uniformity on X.

Each quasi–uniformity $\mathcal{U}$ on a set X determines a topology $\mathcal{T}_{\mathcal{U}}$ on X according to the following definition:

$*$) A subset A of X belongs to $\mathcal{T}_{\mathcal{U}}$ iff for every $x \in A$, we may find an element $U_x \in \mathcal{U}$ such that:

$$U_x(x) = \{y \in X \mid (x, y) \in U_x\} \subseteq A.$$

A quasi–uniformity $\mathcal{U}$ on a topological space $(X, \mathcal{T})$ is <u>compatible</u> (with $\mathcal{T}$) if $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$.

A connector U on a topological space $(X, \mathcal{T})$ is a <u>neighbornet</u> if for each $x \in X$, $U(x) = \{y \in X \mid (x, y) \in U\}$ is a $\mathcal{T}$–neighborhood of x. A sequence $U_1, U_2, \ldots$ of neighbornets of a topological space $(X, \mathcal{T})$ is <u>normal</u> if for every $n \in \mathbb{N}$, we have $U_{n+1}^2 = U_{n+1} \circ U_{n+1} \subseteq U_n$. A neighbornet U is <u>normal</u> if it belongs to a normal sequence of neighbornets. Clearly, if $\mathcal{U}$ is a compatible quasi–uniformity on a topological space $(X, \mathcal{T})$, each member $U \in \mathcal{U}$ is a normal neighbornet of $(X, \mathcal{T})$.

In an arbitrary topological space $(X, \mathcal{T})$ there exists at least one compatible quasi–uniformity, the Pervin quasi–uniformity (see [9]). Among all compatible quasi–uniformities on a topological space $(X, \mathcal{T})$, there exists one, the fine quasi–uniformity, which contains every other compatible quasi-uniformity: take all normal neighbornets on $(X, \mathcal{T})$. Spaces where the fine quasi–uniformity is the only compatible quasi-uniformity are characterized in [8]. Such spaces have only one annular basis, the topology itself.

A basis η for a quasi–uniformity $\mathcal{U}$ is simply a filterbase in $X \times X$ which generates $\mathcal{U}$. A quasi–uniformity basis η is transitive if each element $N \in \eta$ is transitive (*i.e.*, $N \circ N \subseteq N$) and symmetric if each $N \in \eta$ is symmetric (*i.e.*, $N = N^{-1}$). A connector A on a set X is totally bounded if there exists a finite cover $\{E_1, E_2, \ldots, E_s\}$ of X such that $E_i \times E_i \subseteq A$ for every $i = 1, 2, \ldots, s$. A quasi–uniformity $\mathcal{U}$ on X is totally bounded if each member of $\mathcal{U}$ is totally bounded. A quasi–uniformity $\mathcal{U}$ on X is transitive (resp., a uniformity) if $\mathcal{U}$ has a transitive (resp., a symmetric) basis. The Pervin quasi–uniformity on a topological space is both transitive and totally bounded.

An important fact observed by Romaguera and Sánchez Granero is the following:

There exists a bijection between the family of all transitive and totally bounded compatible quasi–uniformities on a topological space $(X, \mathcal{T})$ and the family of all annular bases of $\mathcal{T}$. This bijection, which respects inclusions, is described in [10]. Of course, the quasi-uniformity corresponding to $\mathcal{T}$ is the Pervin quasi–uniformity.

3. Wallman bases

A Wallman basis $\mathcal{B}$ on $(X, \mathcal{T})$ is an annular basis of $\mathcal{T}$ satisfying the extra condition:

WB) If $x \in B \in \mathcal{B}$, there exists a closed set H such that $x \in H \subseteq B$ and such that $X - H \in \mathcal{B}$.

Whenever we restrict ourselves to a definite basis $\mathcal{B}$ of a topological space $(X, \mathcal{T})$, by a cobasic set we simply mean a closed set $H \subseteq X$ such that $X - H \in \mathcal{B}$. $C(\mathcal{B})$ denotes the family of all cobasic sets with respect to $\mathcal{B}$. So, condition WB) is equivalent to say that every basic set $B \in \mathcal{B}$ is a union of cobasic sets $H \in C(\mathcal{B})$.

A quasi–uniformity $\mathcal{U}$ on a set X is said to be point symmetric if for each $U \in \mathcal{U}$ and each $x \in X$, there exists a symmetric connector $V_x \in \mathcal{U}$ such that $V_x(x) \subseteq U(x)$ (*i.e.*, $(x, y) \in V_x$ implies $(x, y) \in U$).

In the correspondence between transitive totally bounded compatible quasi–uniformities of a topological space $(X, \mathcal{T})$ and the family of annular bases of $\mathcal{T}$, if we restrict ourselves to point–symmetric quasi–uniformities, we have a bijection with the family of all Wallman bases of $(X, \mathcal{T})$ (see [10]).

Not every topological space $(\mathrm{X}, \mathcal{T})$ admits Wallman bases. However, a very mild restriction on $\mathcal{T}$ guarantees their existence:

Theorem 3.1. *A topological space* $(\mathrm{X}, \mathcal{T})$ *admits Wallman bases iff* $\mathcal{T}$ *itself is a Wallman basis, i.e., iff every open set* $\mathrm{U} \in \mathcal{T}$ *is a union of closed sets.*

The condition in the Theorem above is equivalent to the following property:

R_0) If $x, y \in \mathrm{X}$ and $y \in C\ell(\{x\})$, then $C\ell(\{y\}) = C\ell(\{x\})$.

Spaces satisfying condition R_0) are simply called R_0–spaces. It is obvious that every T_1–space (*i.e.*, spaces where every finite set is closed) is R_0. In fact, the condition T_1 is equivalent to the conditions R_0 and:

T_0) If $x, y \in \mathrm{X}$ and $y \in C\ell(\{x\})$ and $x \in C\ell(\{y\})$, then $x = y$.

A Wallman basis $\mathcal{B}$ on $(\mathrm{X}, \mathcal{T})$ is <u>regular</u> if whenever $x \in \mathrm{B} \in \mathcal{B}$, there exist elements $\mathrm{D} \in \mathcal{B}$ and $\mathrm{H} \in \mathrm{C}(\mathcal{B})$ such that $x \in \mathrm{D} \subseteq \mathrm{H} \subseteq \mathrm{B}$.

A quasi–uniformity $\mathcal{U}$ on X is <u>locally symmetric</u> if for each $\mathrm{U} \in \mathcal{U}$ and $x \in \mathrm{X}$, there exists a symmetric element $\mathrm{V}_x \in \mathcal{U}$ such that $\mathrm{V}_x^2(x) \subseteq \mathrm{U}(x)$.

Going back to the correspondence Theorem, we obtain:

Theorem 3.2. *The quasi–uniformity* $\mathcal{U}_{\mathcal{B}}$ *which corresponds to a Wallman basis* $\mathcal{B}$ *of* $(\mathrm{X}, \mathcal{T})$ *is locally symmetric iff* $\mathcal{B}$ *is regular.*

<u>Normality</u> is perhaps the most widely used property in General Topology; it may understood in at least four different ways: *a*) Topological spaces, *b*) open covers, *c*) connectors and *d*) Wallman bases.

Of course, the first is the most important: a topological space is <u>normal</u> if for every pair H K of disjoint closed sets, there exist disjoint open sets $\mathrm{U_H}$, $\mathrm{U_K}$ such that $\mathrm{H} \subseteq \mathrm{U_H}$ and $\mathrm{K} \subseteq \mathrm{U_K}$. The last one is closely related to this: a Wallman basis $\mathcal{B}$ of a topological space $(\mathrm{X}, \mathcal{T})$ is <u>normal</u> if for every pair H, K of disjoint cobasic sets, there exist disjoint basic sets $\mathrm{B_H}, \mathrm{B_K} \in \mathcal{B}$ such that $\mathrm{H} \subseteq \mathrm{B_H}$ and $\mathrm{K} \subseteq \mathrm{B_K}$. Combining these two definitions, we have:

> An R_0–space $(\mathrm{X}, \mathcal{T})$ is normal iff the Wallman basis $\mathcal{T}$ is normal.

However a space with a normal Wallman basis may not be normal. In fact, every locally compact Hausdorff space $(\mathrm{X}, \mathcal{T})$ has a normal Wallman basis: define $\mathcal{B}$ as the family of open sets B such that $C\ell(\mathrm{B})$ or $\mathrm{X} - \mathrm{B}$ is compact. More generally, if $(\mathrm{X}, \mathcal{T})$ is a Hausdorff space and the family $\mathcal{B}$ of open sets $\mathrm{V} \in \mathcal{T}$ with compact boundary is a basis of $\mathcal{T}$, then $\mathcal{B}$ is a normal Wallman basis of $(\mathrm{X}, \mathcal{T})$. These spaces were called <u>peripherally compact</u> by Gordon T. Whyburn and it is quite clear that every locally compact Hausdorff space is peripherally compact but the converse may not be true. There are many examples in the literature of locally compact Hausdorff spaces which are not normal (see [12], Ex. 106 and [2], Ex. 2, p.239). On other side, every completely regular space $(\mathrm{X}, \mathcal{T})$ has a normal Wallman basis, the collection of <u>cozero sets</u>: $\mathrm{V} \subseteq \mathrm{X}$ is a <u>cozero set</u> if there exists a continuous function:

$$f : (\mathrm{X}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_d) \quad (\mathcal{T}_d \text{ usual topology of } \mathbb{R})$$

such that $V = f^{-1}(\mathbb{R} - \{0\}) = \{x \in X \mid f(x) \neq 0\}$.

A remarkable property of normal Wallman basis is the existence of scales between any pair of disjoint cobasic sets:

Definition 3.3. *Let* A, B *be disjoint subsets of a topological space* $(X, \mathcal{T})$ *and let* $\mathcal{D} = \left\{ \frac{m}{2^n} \mid m, n \in \mathbb{N} \right\}$ *be the family of dyadic rationals in the open unit interval* $(0,1)$. *A scale between* A *and* B *is a map* $\varphi : \mathcal{D} \to \mathcal{P}(X)$ *such that whenever* $d_1, d_2 \in \mathcal{D}$, $d_1 < d_2$, *we have* $C\ell_X(\varphi(d_1)) \subseteq Int_X(\varphi(d_2))$.

Every scale $\varphi : \mathcal{D} \to \mathcal{P}(X)$ determines a continuous map $f : X \to [0,1]$ from X to the closed unit interval such that:

$$f^{-1}(0) = \cap \{\varphi(r) \mid r \in \mathcal{D}\}$$

and

$$f^{-1}(1) = X - \cup \{\varphi(r) \mid r \in \mathcal{D}\}.$$

The map f is defined exactly as in the proof of Urysohn's Lemma:

$$f(x) = \begin{cases} 0 & \text{if } x \in \cap\{\varphi(r) \mid r \in \mathcal{D}\} \\ \sup\{r \in \mathcal{D} \mid x \notin \varphi(r)\} & \text{otherwise} \end{cases}.$$

If $\mathcal{B}$ is a normal Wallman basis of a topological space and if H, K are disjoint cobasic sets, apply the normality of $\mathcal{B}$ and obtain two disjoint basic sets B_H, B_K such that $H \subseteq B_H$ and $K \subseteq B_K$. Proceed to define the scale φ by induction on the power of two in the denominator of a dyadic rational $\frac{m}{2^k} \in \mathcal{D}$, starting with $\varphi\left(\frac{1}{2}\right) = B_H$. Inductively, if φ has already been defined in:

$$\mathcal{D}_n = \left\{ \frac{m}{2^k} \in \mathcal{D} \mid k \leq n \right\}$$

and in such a way that for each $d \in \mathcal{D}_n$ we have $\varphi(d) \in \mathcal{B}$ and $H \subseteq \varphi(d) \subseteq F_d$ for some cobasic set F_d contained in $X - K$ and such that whenever $d_1, d_2 \in \mathcal{D}_n$, $d_1 < d_2$, there exists a cobasic set F such that:

$$\varphi(d_1) \subseteq F \subseteq \varphi(d_2), \tag{3.1}$$

we proceed to define φ on $\mathcal{D}_{n+1}$: let $\frac{m}{2^{n+1}} \in \mathcal{D}_{n+1}$, where m is odd and $1 < m < 2^{n+1} - 1$. Then $d_1 = \frac{m-1}{2^{n+1}}$ and $d_2 = \frac{m+1}{2^{n+1}}$ both belong to $\mathcal{D}_n$. By the induction hypothesis, there exists a cobasic set F such that (3.1) holds. Then F and $X - \varphi(d_2)$ are disjoint cobasic sets. Since $\mathcal{B}$ is normal, there exist disjoint basic sets B, B′ such that $F \subseteq B$ and $X - \varphi(d_2) \subseteq B'$. Define $\varphi\left(\frac{m}{2^{n+1}}\right) = B$. If $m = 1$, we find disjoint basic sets B, B′ containing H and $X - \varphi\left(\frac{1}{2^n}\right)$ and define $\varphi\left(\frac{1}{2^n}\right) = B$. If $m = 2^{n+1} - 1$, we find disjoint basic sets B, B′ containing $F_{\frac{2^n-1}{2^n}}$ and K and define $\varphi\left(\dfrac{2^{n+1}-1}{2^{n+1}}\right) = B$. This completes the inductive construction.

A very important consequence of this result, is that every topological space which admits a normal Wallman basis is completely regular. Hence we have the following characterization:

Theorem 3.4. *A topological space* $(X, \mathcal{T})$ *is completely regular iff* $\mathcal{T}$ *admits a normal Wallman basis. In fact, every completely regular infinite* T_2*–space of weight* α *has a normal Wallman basis of cardinality* α*.*

Completely regular Hausdorff spaces, and only them, admit Hausdorff compactifications (see [3], 3.30.3).

Every Wallman basis of a T_1–space $(X, \mathcal{T})$ yields a T_1–compactification of $(X, \mathcal{T})$. To construct such a compactification, we define the concept of Wallman ultrafilter:

Sea $\mathcal{B}$ be a Wallman basis of X and let $C(\mathcal{B})$ be the family of cobasic sets. A non-empty subfamily ξ of $C(\mathcal{B})$ is a Wallman ultrafilter if ξ satisfies the following conditions:

1) Each element $H \in \xi$ is non-empty.
2) For every pair of elements $H, K \in \xi$, $H \cap K$ also belongs to ξ.
3) If $H \in \xi$ and $H \subseteq K \in C(\mathcal{B})$, then $K \in \xi$.
4) An element $K \in C(\mathcal{B})$ belongs to ξ iff $K \cap H \neq \varnothing$ for every $H \in \xi$.

Observe every point $p \in X$ determines a Wallman ultrafilter, namely $\xi_p = \{H \in C(\mathcal{B}) \mid p \in H\}$.

The collection of Wallman ultrafilters (respect to $\mathcal{B}$) is denoted as $X(\mathcal{B})$. There is a natural map $v : X \to X(\mathcal{B})$ which assigns to every $p \in X$ its fixed ultrafilter ξ_p. For every $A \subseteq X$ we define a subset A^* of $X(\mathcal{B})$ by means of the formula:

$$A^* = \{\xi \in X(\mathcal{B}) \mid \text{ for some } F \in \xi,\ F \subseteq A\}.$$

This operator $A \mapsto A^*$ respects inclusions and for every pair of subsets C, D of X, we have:

$$(C \cap D)^* = C^* \cap D^*.$$

The formula $(C \cup D)^* = C^* \cup D^*$ is also valid provided that C and D both belong to $\mathcal{B} \cup C(\mathcal{B})$ (see [3]). The family

$$\mathcal{B}^* = \{B^* \mid B \in \mathcal{B}\}$$

is an annular basis for a compact T_1–topology $\mathcal{T}^*$ of $X(\mathcal{B})$. The natural map

$$v : (X, \mathcal{T}) \to (X(\mathcal{B}), \mathcal{T}^*)$$

is then injective, continuous, open onto its range $v(X)$ and with $v(X)$ dense in $X(\mathcal{B})$. Therefore, the pair $(v, X(\mathcal{B}))$ is a T_1–compactification of X, called the Wallman compactification of X with respect to the basis $\mathcal{B}$.

4. Topological and embedding properties

A general problem we may set ourselves is the following:

> *Given a topological property* P *and an embedding property* J, *find conditions on a Wallman basis* $\mathcal{B}$ *of* $(\mathrm{X}, \mathcal{T})$ *which insure that* $\left(\mathrm{X}(\mathcal{B}), \mathcal{T}^*\right)$ *has property* P *or that* X *is* J*–embedded in* $\mathrm{X}(\mathcal{B})$.

We give first some definitions:

Definition 4.1. *Two Wallman bases of* $(\mathrm{X}, \mathcal{T})$ *are* *<u>equivalent</u>* *if every pair of disjoint cobasic sets with respect to any one of the bases, are contained in disjoint cobasic sets with respect to the other.*

It is not difficult to prove that if $\mathcal{B}$ is a normal Wallman basis of a Tychonoff space X and if there exists a pair of non–compact disjoint cobasic sets, then there exists a normal Wallman basis $\mathcal{B}'$ for the same topology which is not equivalent to $\mathcal{B}$.

Definition 4.2. *An extension* Z *of a space* X *is* *<u>perfect</u>* *if whenever we have a separation in* X*:*

$$\mathrm{X} - \mathrm{K} = \mathrm{U} \cup \mathrm{V}, \quad \mathrm{K} \textit{ closed in } \mathrm{X}, \;\; \mathrm{U}, \mathrm{V}, \textit{ open and disjoint}$$

we also have a separation in Z*:*

$$\mathrm{X} - C\ell_{\mathrm{Z}}(\mathrm{K}) = \mathrm{U}_1 \cup \mathrm{V}_1, \quad \mathrm{U}_1 \supseteq \mathrm{U}, \;\; \mathrm{V}_1 \supseteq \mathrm{V}, \quad \mathrm{U}_1, \mathrm{V}_1, \textit{ open and disjoint in } \mathrm{Z}.$$

A simple characterization of perfect extensions can be given if we use the operator $e : \mathcal{T}_{\mathrm{X}} \to \mathrm{T}_{\mathrm{Z}}$ between the topologies of X and Z, where

$$e(\mathrm{U}) = \mathrm{Z} - C\ell_{\mathrm{Z}}(\mathrm{X} - \mathrm{U}).$$

Then Z is a perfect extension of X iff for every pair U, V of disjoint open sets in X, we have $e(\mathrm{U} \cup \mathrm{V}) = e(\mathrm{U}) \cup e(\mathrm{V})$.

Definition 4.3. *If* Z *is an extension of* X*, we say* X *is* *<u>locally connected</u>* *in* Z *if* Z *has a basis* $\mathcal{B}$ *such that for every* $\mathrm{B} \in \mathcal{B}$, $\mathrm{B} \cap \mathrm{X}$ *is a connected subset of* X.

It is easy to see that if X is locally connected in Z, then Z is a perfect extension of X.

It is well known that if X is a Tychonoff space, then the Stone–Čech compactification $\beta\mathrm{X}$ is a perfect extension of X and X is C^*–embedded in $\beta\mathrm{X}$.

Another important example of a perfect compactification is the Wallman compactification of a peripherally compact Hausdorff space X, where $\mathcal{B}$ is the family of open subsets of X with compact boundary.

We state a few classical results in this context:

Theorem 4.4. *Let* $\mathcal{B}$ *a Wallman basis of a* T_1*–space* $(\mathrm{X}, \mathcal{T})$. *Then* $\left(\mathrm{X}(\mathcal{B}), \mathcal{T}^*\right)$ *is a Hausdorff space iff* $\mathcal{B}$ *is normal.*

Theorem 4.5. *If $\mathcal{B}$ is a normal Wallman basis of a Tychonoff space $(\mathrm{X},\mathcal{T})$, then* X *is* C^**–embedded in* $\mathrm{X}(\mathcal{B})$ *iff* $\mathcal{B}$ *is equivalent to the cozero Wallman basis of* $(\mathrm{X},\mathcal{T})$.

The following results were proved by myself: ([5])

Theorem 4.6. *If $\mathcal{B}$ is a Wallman basis of a* T_1*–space* $(\mathrm{X},\mathcal{T})$*, then* $\mathrm{X}(\mathcal{B})$ *is a perfect compactification of* X *iff* $\mathcal{B}$ *satisfies the following condition:*

$*$) *If* $\mathrm{K}\subseteq\mathrm{B}$*, where* $\mathrm{K}\in\mathrm{C}(\mathcal{B})$ *and* $\mathrm{B}\in\mathcal{B}$ *and if* L *is an open set in* X *such that* $\mathrm{B}\cap Fr(\mathrm{L})=\varnothing$*, then there exists a basic set* $\mathrm{B_L}\in\mathcal{B}$ *such that* $\mathrm{K}\cap\mathrm{L}\subseteq\mathrm{B_L}\subseteq\mathrm{B}\cap\mathrm{L}$.

Condition $*$) is clearly satisfied if every clopen subset of a basic set is also a basic set.

Theorem 4.7. *If $\mathcal{B}$ is a Wallman basis of a* T_1*–space* $(\mathrm{X},\mathcal{T})$ *satisfying the property:*

$$\mathrm{B}\in\mathcal{B},\ \mathrm{E}\ \textit{component of}\ \mathrm{B}\Rightarrow\mathrm{E}\in\mathcal{B}, \tag{4.1}$$

then X *is locally connected in* $\mathrm{X}(\mathcal{B})$ *iff* $\mathcal{B}$ *satisfies the following condition:*

$**$) *If* $\mathrm{K}\subseteq\mathcal{B}$*, where* $\mathrm{K}\in\mathrm{C}(\mathcal{B})$ *and* $\mathrm{B}\in\mathcal{B}$*, then there exists a finite collection* $\mathrm{C}_1,\mathrm{C}_2,\ldots,\mathrm{C}_n$ *of connected subsets of* X *such that:*

$$\mathrm{K}\subseteq\bigcup_{i=1}^{n}\mathrm{C}_i\subseteq\mathrm{B}.$$

We say $\mathcal{B}$ is locally connected if it satisfies the property 4.1.

Condition $**$) is satisfied if X is locally connected, Hausdorff and peripherally compact and X has only a finite number of components.

Before stating two more results, we need a definition:

Definition 4.8. *A Hausdorff compactification* Z *of a space* X *is of Wallman type if* X *has a Wallman basis* $\mathcal{B}$ *such that* Z *and* $\mathrm{X}(\mathcal{B})$ *are equivalent compactifications of* X.

We have then:

Theorem 4.9 ([14])**.** *If* Z *is a compact metrizable space and if* X *is dense in* Z*, then* Z *is a Wallman type compactification of* X.

Theorem 4.10 ([5])**.** *If* Z *is a compact and Hausdorff and if* X *is* G_δ*–dense in* Z *(i.e., every non-empty* G_δ *set in* Z *intersects* X*) then* Z *is a Wallman type compactification of* X.

Before going on, we need some more definitions:

Definition 4.11. *A topological space* $(\mathrm{X},\mathcal{T})$ *is* S*–metrizable if there exists a metric* d *on* X *inducing* $\mathcal{T}$ *and satisfying the following property:*

S) *For every* $\varepsilon>0$*, there exists a finite cover* $\mathrm{C}_1,\mathrm{C}_2,\ldots,\mathrm{C}_n$ *of* X *consisting of connected sets of diameter* $<\varepsilon$.

Every S–metrizable space is locally connected, separable and has only a finite number of connected components.

We characterize now S–metrizability:

Theorem 4.12. *The following three properties of a topological space* $(\mathrm{X}, \mathcal{T})$ *are equivalent:*

a) X *is* S*–metrizable.*
b) X *has a perfect locally connected metrizable compactification* Z *(*[4]*).*
c) $\mathcal{T}$ *has a countable normal locally connected Wallman basis consisting of open domains. (This equivalence is easily obtainable from results of* [14]*).*

Another topological concept which leads to many open problems is weak pseudocompactness:

Definition 4.13. *A Tychonoff space* $(\mathrm{X}, \mathcal{T})$ *is* <u>*weakly pseudocompact*</u> *if* $(\mathrm{X}, \mathcal{T})$ *has a Hausdorff compactification in which* X *is* G_δ*–dense.*

A non-compact locally compact T_2–Lindelöf space X cannot be weakly pseudocompact. Obviously, every Tychonoff pseudocompact space is weakly pseudocompact. The Hedgehog $\mathrm{J}(\alpha)$, where $\alpha \geq \aleph_1$ is an example of a weakly pseudocompact space which is not pseudocompact. As far as I know, it is an open problem if $\mathbb{R}^\alpha$ $(\alpha \geq \aleph_1)$ is weakly pseudocompact.

We have however, the following characterization:

Theorem 4.14 ([6])**.** *A Tychonoff space* $(\mathrm{X}, \mathcal{T})$ *is weakly pseudocompact iff* $\mathcal{T}$ *has a normal Wallman basis* $\mathcal{B}$ *such that every countable cover of* X *with elements of* $\mathcal{B}$ *has finite subcover.*

5. Cover uniformities and Wallman bases

There is a strong relation between normal Wallman bases and cover uniformities. The treatment of uniform spaces thru covers instead of connectors must be seen as an alternative, but the two treatments are equivalent.

Definition 5.1. *A* <u>*cover uniformity basis*</u> *on a set* X *is a non–empty family* $\mathcal{G}$ *of covers of* X *such that for every pair of covers* $\alpha, \beta \in \mathcal{G}$*, there exists a cover* $\gamma \in \mathcal{G}$ *which refines baricentrically each of the covers* α*,* β*, i.e., for every* $p \in \mathrm{X}$ *we may find elements* $\mathrm{A}_p \in \alpha$*,* $\mathrm{B}_p \in \beta$ *such that* $\mathrm{S_T}(p, \gamma) = \cup\{\mathrm{C} \mid p \in \mathrm{C} \in \gamma\} \subseteq \mathrm{A}_p \cap \mathrm{B}_p$*. A* <u>*cover uniformity*</u> *on* X *is simply a cover uniformity basis* $\mathcal{G}$ *on* X *with the additional property:*

$*)$ *If* $\alpha \in \mathcal{G}$ *and if* β *is a cover of* X *refined by* α*, then* $\beta \in \mathcal{G}$*.*

Each cover uniformity basis $\mathcal{G}$ *on* X *is contained in a unique smallest cover uniformity on* X*, namely* $\mathcal{G} \subseteq \mathcal{G}^+$ *where*

$$\mathcal{G}^+ = \{\beta \mid \beta \text{ is a cover of X and some cover } \alpha \in \mathcal{G} \text{ refines } \beta\}.$$

Two cover uniformity bases $\mathcal{G}_1$*,* $\mathcal{G}_2$ *on* X *are* equivalent *if* $\mathcal{G}_1^+ = \mathcal{G}_2^+$*. Each cover uniformity basis* $\mathcal{G}$ *on* X *determines a topology* $\mathcal{T}_\mathcal{G}$ *on* X *defined by:*

$$\mathrm{L} \in \mathcal{T}_\mathcal{G} \Leftrightarrow \forall x \in \mathrm{L},\ \exists\, \alpha_x \in \mathcal{G} \ \pitchfork\ \mathrm{S_T}(x, \alpha_x) \subseteq \mathrm{L}.$$

It is easy to see that two equivalent cover uniformity bases determine the same topology on X but the converse is not true in general.

A standard result (but not so obvious), states that for every cover uniformity basis $\mathcal{G}$ on a set X, $\mathcal{T}_\mathcal{G}$ is a completely regular topology on X. Besides, $\mathcal{T}_\mathcal{G}$ is T_1 (and hence, $\mathcal{T}_\mathcal{G}$ is a Tychonoff topology on X) iff for every $x \in \mathrm{X}$, we have $\cap \{\mathrm{S_T}(x, \alpha) \mid \alpha \in \mathcal{G}\} = \{x\}$.

In our terminology, a cover uniform space is a pair $(\mathrm{X}, \mathcal{G})$, where $\mathcal{G}$ is a cover uniformity basis on X. It is easy to see that for every subset $\mathrm{A} \subseteq \mathrm{X}$, $\mathcal{G}|_\mathrm{A} = \left\{\alpha|_\mathrm{A} \mid \alpha \in \mathcal{G}\right\}$ is a cover uniformity basis on A and $\left(\mathrm{A}, \mathcal{T}_{\mathcal{G}|_\mathrm{A}}\right)$ is a subspace of $(\mathrm{X}, \mathcal{T}_\mathcal{G})$. So, by a cover uniform subspace of $(\mathrm{X}, \mathcal{G})$, we simply mean a pair $\left(A, \mathcal{G}|_\mathrm{A}\right)$, where $\mathrm{A} \subseteq \mathrm{X}$.

The correspondence between the concepts of cover uniform space and uniform space is very simple:

Given a cover uniform space $(\mathrm{X}, \mathcal{B})$ we construct a filter basis $\mathcal{F}_\mathcal{B}$ of symmetric connectors of X, namely, each $\alpha \in \mathcal{B}$ determines the symmetric connector:

$$\mathrm{E}(\alpha) = \cup \{\mathrm{L} \times \mathrm{L} \mid \mathrm{L} \in \alpha\}.$$

The filter $\mathcal{F}_\mathcal{B}^+ = \{\mathrm{T} \subseteq \mathrm{X} \times \mathrm{X} \mid \mathrm{E}(\alpha) \subseteq \mathrm{T} \text{ for some } \alpha \in \mathcal{B}\}$ is a uniformity on X and the topologies $\mathcal{T}_\mathcal{B}$, $\mathcal{T}_{\mathcal{F}_\mathcal{B}}$ coincide. Conversely, given a symmetric quasi–uniformity basis $\mathcal{F}$ on X, each $\mathrm{F} \in \mathcal{F}$ determines an indexed cover:

$$\alpha_\mathrm{F} = \{\mathrm{F}(x) \mid x \in \mathrm{X}\}.$$

The family

$$\mathcal{B} = \{\alpha_\mathrm{F} \mid \mathrm{F} \in \mathcal{F}\}$$

is a cover uniformity basis: let $\alpha_{\mathrm{F}_1}, \alpha_{\mathrm{F}_2} \in \mathcal{B}$ and let $\mathrm{G} \in \mathcal{F}$ be such that $\mathrm{G}^2 \subseteq \mathrm{F}_1 \cap \mathrm{F}_2$. Then

$$\alpha_\mathrm{G}^\triangle = \{\mathrm{S_T}(x, \alpha_\mathrm{G}) \mid x \in \mathrm{X}\}$$

refines both covers α_{F_1}, α_{F_2}: in fact, if $z \in \mathrm{S_T}(x, \alpha_\mathrm{G})$ and $\mathrm{G}(y)$ contains both points x, z, then $(y, z), (y, x) \in \mathrm{G}$ and so $(x, y) \in \mathrm{G}^{-1} = \mathrm{G}$ and $(x, z) \in \mathrm{G}^2 \subseteq \mathrm{F}_1 \cap \mathrm{F}_2$ and $z \in \mathrm{F}_1(x) \cap \mathrm{F}_2(x)$. In this case we have also the same topologies $\mathcal{T}_\mathcal{B} = \mathcal{T}_\mathcal{F}$. □

A map $\varphi : (\mathrm{X}, \mathcal{G}) \to (\mathrm{Y}, \mathcal{H})$ between cover uniform spaces is uniformly continuous if for every $\varepsilon \in \mathcal{H}$ we may find a cover $\delta \in \mathcal{G}$ such that δ refines $\varphi^{-1}(\varepsilon) = \left\{\varphi^{-1}(\mathrm{E}) \mid \mathrm{E} \in \varepsilon\right\}$. φ is a unimorphism if φ es bijective and both maps $\varphi : (\mathrm{X}, \mathcal{G}) \to (\mathrm{Y}, \mathcal{H})$, $\varphi^{-1} : (\mathrm{Y}, \mathcal{H}) \to (\mathrm{X}, \mathcal{G})$ are uniformly continuous. $\varphi : (\mathrm{X}, \mathcal{G}) \to (\mathrm{Y}, \mathcal{H})$ is a unimorphic embedding if φ is a unimorphism from $\varphi : (\mathrm{X}, \mathcal{G}) \to \left(\varphi(\mathrm{X}), \mathcal{H}|_{\varphi(\mathrm{X})}\right)$ and if $\varphi(\mathrm{X})$ is dense in Y. It is obvious that if $\mathcal{G}_1$, $\mathcal{G}_2$ are cover uniformity bases on X, then the identity map $j : (\mathrm{X}, \mathcal{G}_1) \to (\mathrm{X}, \mathcal{G}_2)$ is a unimorphism iff $\mathcal{G}_1$ and $\mathcal{G}_2$ are equivalent.

We have several important concepts in uniform space theory:

Definition 5.2.

a) *A filter* $\mathcal{F}$ *on a cover uniform space* $(\mathrm{X}, \mathcal{G})$ *is* <u>*Cauchy*</u> *if* $\mathcal{F} \cap \alpha \neq \varnothing$ *for every* $\alpha \in \mathcal{G}$.

b) *A cover uniform space* $(\mathrm{X}, \mathcal{G})$ *is* <u>*complete*</u> *if every Cauchy filter on* $(\mathrm{X}, \mathcal{G})$ *is convergent (with respect to the topology* $\mathcal{T}_{\mathcal{G}}$*).*

c) *A cover uniform space* $(\mathrm{X}, \mathcal{G})$ *is* <u>*totally bounded*</u> *if every cover* $\alpha \in \mathcal{G}$ *has a finite subcover for* X.

d) *A cover uniform space* $(\mathrm{Y}, \mathcal{H})$ *is a* <u>*cover completion*</u> *of a cover uniform space* $(\mathrm{X}, \mathcal{U})$ *if* $(\mathrm{Y}, \mathcal{H})$ *is complete and if there exists a unimorphic embedding* $\varphi : (\mathrm{X}, \mathcal{U}) \to (\mathrm{Y}, \mathcal{H})$.

The easiest example of a completion is the metric completion: Let (X, d) be a metric space and let $\left(\widetilde{\mathrm{X}}, \widetilde{d}\,\right)$ be a metric completion of (X, d). For each $\varepsilon > 0$ define

$$\alpha_\varepsilon = \left\{ \mathrm{V}_\varepsilon^d(x) \mid x \in \mathrm{X} \right\};$$
$$\widetilde{\alpha}_\varepsilon = \left\{ \mathrm{V}_\varepsilon^{\widetilde{d}}(x) \mid x \in \mathrm{X} \right\}.$$

Then $\mathcal{G}_d = \{\alpha_\varepsilon \mid \varepsilon > 0\}$ is a cover uniformity basis on X, $\mathcal{G}_{\widetilde{d}} = \{\widetilde{\alpha}_\varepsilon \mid \varepsilon > 0\}$ is a cover uniformity basis on $\widetilde{\mathrm{X}}$ and $\left(\mathrm{X}, \mathcal{G}_{\widetilde{d}}\right)$ is a cover completion of $(\mathrm{X}, \mathcal{G}_d)$.

We state without proof a few important theorems on cover uniform space theory. (See [3], Chap. 7).

Theorem 5.3. *Let* $\left(\mathrm{A}, \mathcal{G}|_{\mathrm{A}}\right)$ *be a dense subspace of a cover uniform space* $(\mathrm{X}, \mathcal{G})$ *and let* $\varphi : \left(\mathrm{A}, \mathcal{G}|_{\mathrm{A}}\right) \to (\mathrm{Y}, \mathcal{H})$ *be a uniformly continuous map into a complete* T_2 *cover uniform space* $(\mathrm{Y}, \mathcal{H})$. *Then* φ *has unique continuous extension* $\widetilde{\varphi} : (\mathrm{X}, \mathcal{T}_{\mathcal{G}}) \to (\mathrm{Y}, \mathcal{T}_{\mathcal{H}})$ *and this unique extension is uniformly continuous as a map from* $(\mathrm{X}, \mathcal{G})$ *to* $(\mathrm{Y}, \mathcal{H})$.

Theorem 5.4. *Every Hausdorff cover uniform space* $(\mathrm{X}, \mathcal{G})$ *has a cover completion* $\left(\widetilde{\mathrm{X}}, \widetilde{\mathcal{G}}\right)$ *and every other cover completion* $(\mathrm{Y}, \mathcal{H})$ *is unimorphic to* $\left(\widetilde{\mathrm{X}}, \widetilde{\mathcal{G}}\right)$.

Theorem 5.5. *A cover uniform space* $(\mathrm{X}, \mathcal{G})$ *is totally bounded iff every ultrafilter* $\mathcal{F}$ *on* X *is Cauchy.*

Theorem 5.6. *Let* $\left(\mathrm{A}, \mathcal{G}|_{\mathrm{A}}\right)$ *be a dense subspace of a cover uniform space* $(\mathrm{X}, \mathcal{G})$. *Then* $\left(\mathrm{A}, \mathcal{G}|_{\mathrm{A}}\right)$ *is totally bounded iff* $(\mathrm{X}, \mathcal{G})$ *is totally bounded. Also, every subspace of a totally bounded cover uniform space is totally bounded.*

Since every adherence point of an ultrafilter in a topological space is a convergence point and since a topological space is compact iff every ultrafilter converges, we obtain as a corollary the result of a A. Weil mentioned in the introduction:

Theorem 5.7. *The topology* $\mathcal{T}_{\mathcal{G}}$ *of a cover uniform space* $(\mathrm{X}, \mathcal{G})$ *is compact iff* $(\mathrm{X}, \mathcal{G})$ *is complete and totally bounded.*

Normal Wallman bases of Tychonoff spaces determine totally bounded cover uniform spaces:

Theorem 5.8 ([3])**.** *Let* $\mathcal{B}$ *be a normal Wallman basis of a Tychonoff space* $(\mathrm{X}, \mathcal{T})$*. Then the family* $\mathcal{U}(\mathcal{B})$ *of finite covers of* X *consisting of elements of* $\mathcal{B}$ *is a totally bounded cover uniformity basis on* X *and* $\mathcal{T} = \mathcal{T}_{\mathcal{U}(\mathcal{B})}$*. The cover completion* $\left(\widetilde{\mathrm{X}}, \widetilde{\mathcal{U}}(\mathcal{B})\right)$ *is a Hausdorff compactification of* $(\mathrm{X}, \mathcal{T})$ *which is equivalent to the Wallman compactification* $\left(\mathrm{X}(\mathcal{B}), \mathcal{T}^*\right)$*.*

Theorem 5.9 ([3])**.** *Let* X, Y *be a Tychonoff spaces with respective normal Wallman bases* $\mathcal{B}_{\mathrm{X}}$*,* $\mathcal{B}_{\mathrm{Y}}$*. A map* $\varphi : (\mathrm{X}, \mathcal{U}(\mathcal{B}_{\mathrm{X}})) \to (\mathrm{Y}, \mathcal{U}(\mathcal{B}_{\mathrm{Y}}))$ *is uniformly continuous iff* φ *satisfies the following condition:*

$*$) *whenever* $\mathrm{H}, \mathrm{K} \in \mathrm{C}(\mathcal{B}_{\mathrm{Y}})$ *are disjoint, there exist disjoint cobasic sets* $\mathrm{H}_1, \mathrm{K}_1 \in \mathrm{C}(\mathcal{B}_{\mathrm{X}})$ *such that* $\varphi^{-1}(\mathrm{H}) \subseteq \mathrm{H}_1$ *and* $\varphi^{-1}(\mathrm{K}) \subseteq \mathrm{K}_1$*.*

As a corollary of this last theorem, we obtain the well known universal property of the Stone–Čech compactification of a Tychonoff space X:

Theorem 5.10. *Let* $\varphi : \mathrm{X} \to \mathrm{Y}$ *be a continuous map from a Tychonoff space* X *into a compact Hausdorff space* Y*. Then* φ *has a continuous extension* $\varphi_1 : \beta\mathrm{X} \to \mathrm{Y}$*.*

Proof. Apply last theorem taking as $\mathcal{B}_{\mathrm{X}}$, $\mathcal{B}_{\mathrm{Y}}$ the cozero bases of X, Y, respectively. Apply then the extension theorem of uniformly continuous maps into complete spaces. □

Take a look of [3] for further applications of these theorems.

6. The Wallman type compactification problem

Every annular basis $\mathcal{B}$ of a compact Hausdorff space Z is a normal Wallman basis of Z. If X is a dense subspace of Z, then $\mathcal{B}|\mathrm{X} = \{\mathrm{B} \cap \mathrm{X} \mid \mathrm{B} \in \mathcal{B}\}$ is a Wallman basis of X but $\mathcal{B}|\mathrm{X}$ may not be normal. Assuming $\mathcal{B}|\mathrm{X}$ is normal, we wonder under what conditions Z and $\mathrm{X}(\mathcal{B}|\mathrm{X})$ are equivalent compactifications of X. The answer is quite simple:

Z and $\mathrm{X}(\mathcal{B}|\mathrm{X})$ are equivalent compactifications of X iff Z is the only member of $\mathcal{B}$ containing X (see [13]). If this happens, we say then that $\mathcal{B}$ has the trace property respect to X.

If X is G_δ dense in Z, then the cozero basis of Z has the trace property respect to X. If $\mathcal{B}$ consists of open domains (*i.e.*, sets which coincide with the interior of their closures) then we have a better result: Z has the trace property respect to any dense subspace of Z. This suggests the following definition:

Definition 6.1. *A compact* T_2*–space* Z *is* <u>*regular Wallman*</u> *if* Z *is a Wallman type compactification of each of its dense subspaces.*

As we saw before, if the compact Hausdorff space Z has an annular basis consisting of open domains, then Z is regular Wallman. This happens in compact metric spaces and, more generally, in arbitrary products of compact metrizable spaces (see [14]). Even better, the Stone–Čech compactification of any metrizable space has an annular basis consisting of open domains and hence every metrizable space has a normal Wallman basis consisting of open domains. (See [7]).

Consider the following two questions:

Q_1. Is every compact T_2–space regular Wallman?

Q_2. Is every Hausdorff compactification of a Tychonoff space X of Wallman type?

R. C. Solomon ([11]) answered in the negative the first question proving that some closed subspace of the cube I^k, where $k = (2^{\mathfrak{c}})^+$, is not regular Wallman. As far as the second question is concerned, A. K. Steiner and E. F. Steiner reduced Q_2 to an equivalent problem ([15], 1977):

Q_2'. Is every T_2–compactification of a discrete space X of Wallman type?

In the same year (1977), ([16]) Ul'janov answered Q_2 in the negative exhibiting a compactification of the discrete space with $2^{\mathfrak{c}}$ elements which is not of Wallman type. He also proved that the continuum hypothesis is equivalent to the fact that every Hausdorff compactifications of the set of natural numbers is of Wallman type.

Observe that if a Tychonoff space X has a normal Wallman basis $\mathcal{B}$ consisting of open domains, then $\mathcal{B}^*$ is an annular basis of $X(\mathcal{B})$ consisting of open domains and hence $X(\mathcal{B})$ is regular Wallman.

We finish this section with two questions which, as far as I know, still remain open:

Q_3. Is every Hausdorff Wallman type compactification of a metrizable space regular Wallman?

Q_4. Is every normal Wallman basis of a Tychonoff space X equivalent to a subfamily of the cozero basis of X?

7. A topological dream

I conclude this brief survey on Wallman bases with a topological dream:

> *"If* P *is any topological property, there exists a list of conditions on annular basis of a space* X *which are equivalent to property* P*".*

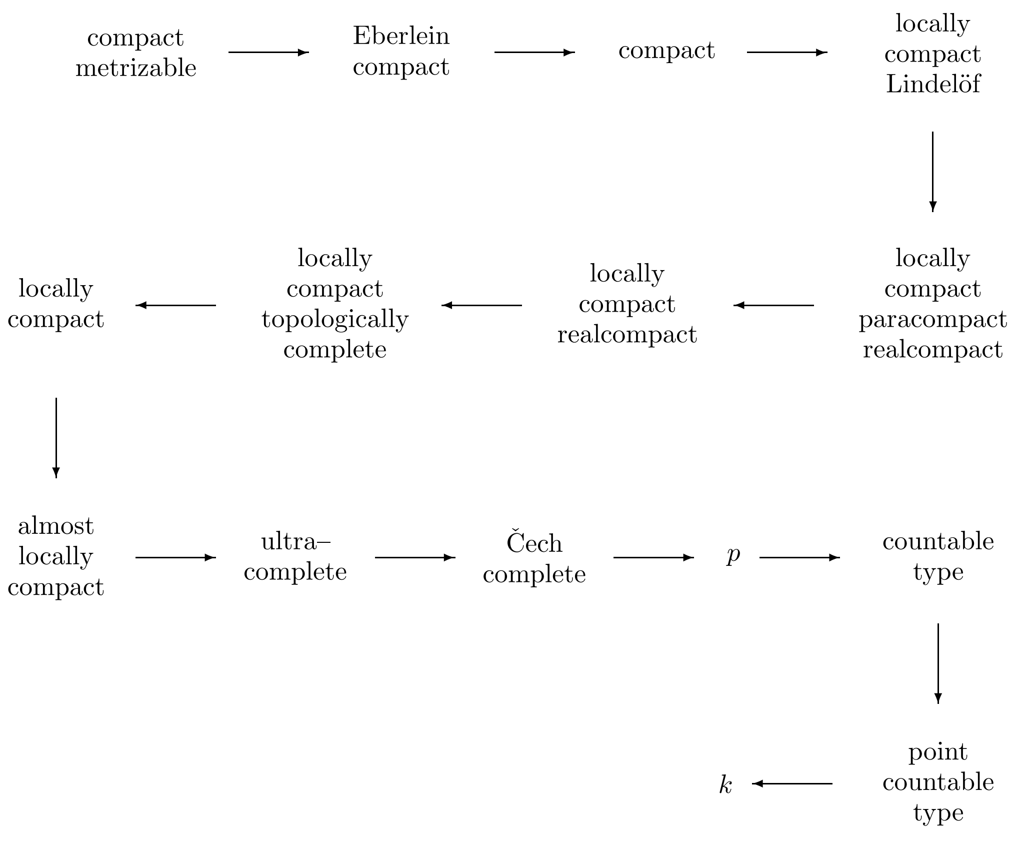
We have some examples where this dream becomes true:

1) A topological space X is compact and pseudo–metrizable iff X has a countable normal Wallman basis such that every cobasic set is pseudocompact.

2) A Hausdorff space X is compact iff X has an annular basis such that every cobasic set is compact.
3) A Hausdorff space X is locally compact iff X has a normal Wallman basis such that every basic set has compact closure or compact complement.
4) A Tychonoff space X is almost compact (*i.e.*, it admits only one compatible uniformity) iff X has a normal Wallman basis $\mathcal{B}$ such that in every pair of disjoint cobasic sets, at least one them is compact. By a previous remark, a Tychonoff space X is almost compact iff it admits only one (except for equivalence) normal Wallman basis.

We have previously characterized S–metrizable and weakly pseudo–compact spaces specifying the existence of a normal Wallman basis with some properties.

Consider the following long list of topological properties of Tychonoff spaces. (I do not include any definition):



Can you characterize each of these properties in a similar way?

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