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Symmetric Bombay topology

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Abstract. The subject of hyperspace topologies on closed or closed and compact subsets of a topological space X began in the early part of the last century with the discoveries of Hausdorff metric and Vietoris hit-and-miss topology. In course of time, several hyperspace topologies were discovered either for solving some problems in Applied or Pure Mathematics or as natural generalizations of the existing ones. Each hyperspace topology can be split into a lower and an upper part. In the upper part the original set inclusion of Vietoris was generalized to proximal set inclusion. Then the topologization of the Wijsman topology led to the upper Bombay topology which involves two proximities. In all these developments the lower topology, involving intersection of finitely many open sets, was generalized to locally finite families but *intersection* was left unchanged. Recently the authors studied symmetric proximal topology in which proximity was used for the first time in the lower part replacing *intersection* with its generalization: *nearness*. In this paper we use two proximities also in the lower part and we obtain the lower Bombay hypertopology. Consequently, a new hypertopology arises in a natural way: the symmetric Bombay topology which is the join of a lower and an upper Bombay topology.

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1. Introduction and preliminaries.

Given a topological space, a topological vector space or a Banach space X, frequently it is necessary to study a family of closed or compact (convex) subsets of X, called a **hyperset of** X, in (a) Optimization (b) Measure Theory (c) Function space topologies (each function $f\colon X \to Y$, as a graph, is a subset of $X \times Y$) (d) Geometric Functional Analysis (e) Image Processing (f) Convex Analysis etc. So there is a need to put an appropriate topology on the hyperset

*We regret to announce the sad demise of our friend and collaborator Enrico.

and so we construct an **hyperspace**. Two early discoveries were the Hausdorff metric topology (1914) [12] [earlier studied by Pompeiu (1905)] when X is a metric space and the Vietoris hit-and-miss topology (1922) [26] when X is a T_1 space. Since then many hyperspace topologies have been studied (see [20]). All hyperspace topologies have a lower and an upper part. A typical member of the lower hyperspace topology consists of members which **hit** a finite or a locally finite family of open sets. The authors showed that all upper hyperspace topologies known until last year can be expressed with the use of two proximities ([7]) and called the resulting upper hyperspace topology, the **upper Bombay topology**. In this project the lower part was left unchanged using the **hit** sets. Recently the authors radically changed the lower part by replacing the *hit* sets by **near sets** and thus getting *symmetric hyperspaces* [8]. In this paper we generalize hyperspace topologies by using two proximities in the lower part obtaining the **lower Bombay topology**. Combining the lower and the upper Bombay topologies, we have the **symmetric Bombay topology**.

Henceforth, (X, τ), or X, denotes a T_1 space. For any $E \subset X$, clE, $intE$ and E^c stand for the closure, interior and complement of E in X, respectively. A binary relation δ on the power set of X is a *basic proximity* iff

(i) $A\delta B$ implies $B\delta A$;
(ii) $A\delta(B \cup C)$ implies $A\delta B$ or $A\delta C$;
(iii) $A\delta B$ implies $A \neq \varnothing$, $B \neq \varnothing$;
(iv) $A \cap B \neq \varnothing$ implies $A\delta B$.

A basic proximity δ is a *LO-proximity* iff it satisfies

(LO) $A\delta B$ and $b\delta C$ for every $b \in B$ together imply $A\delta C$.

A basic proximity δ is an *R-proximity* iff it satisfies

(R) $x\underline{\delta}A$ (where $\underline{\delta}$ means the negation of δ) implies there exists $E \subset X$ such that $x\underline{\delta}E$ and $E^c\underline{\delta}A$.

Moreover, a proximity δ which is both LO and R is called a *LR-proximity*.

A basic proximity δ is an *EF-proximity* iff it satisfies

(EF) $A\underline{\delta}B$ implies there exists $E \subset X$ such that $A\underline{\delta}E$ and $E^c\underline{\delta}B$.

Note that each EF-proximity is a LR-proximity, but, in general, the converse does not hold.

If δ is a LO-proximity, then for each $A \subset X$, we denote $A^\delta = \{x \in X : x\delta A\}$. Then $\tau(\delta)$ is the topology on X induced by the Kuratowski closure operator $A \to A^\delta$. The proximity δ is *compatible* with the topology τ iff $\tau = \tau(\delta)$ (see [10], [21], [25] or [28]).

A T_1 space X admits a compatible LO-proximity. A space X has a compatible LR- (respectively, EF-) proximity iff it is T_3 (respectively, Tychonoff). Moreover, δ is a compatible LR-proximity iff

(LR) for each $x\underline{\delta}A$, there is a closed nbhd. W of x such that $W\underline{\delta}A$.

If $A\delta B$, then we say A is *δ-near* B; if $A\underline{\delta}B$ we say A is *δ-far from* B.

$A \ll_\delta B$ stands for $A\underline{\delta}B^c$ and A is said to be *strongly δ-contained in* B whereas $A\underline{\ll_\delta}B$ stands for its negation, i.e. $A\delta B^c$.

In the sequel η_1, η_2, η or γ_1, γ_2, γ denote (compatible) proximities on X.

We recall that η_2 is *coarser* than η_1 (or equivalently η_1 is *finer* than η_2), written $\eta_2 \leq \eta_1$, iff $A\underline{\eta}_2B$ implies $A\underline{\eta}_1B$.

The most important and well studied proximity is the Wallman or fine LO-proximity η_0 given by

$$A\eta_0 B \Leftrightarrow clA \cap clB \neq \varnothing.$$

The Wallman proximity η_0 is the finest compatible LO-proximity on a T_1 space X. We note that η_0 is a LR-proximity iff X is regular (see [13] Lemma 2). Moreover, η_0 is an EF-proximity iff X is normal (Urysohn's Lemma).

If X is a metric space with metric d, the metric proximity η is given by $A\eta B$ iff $D(A,B) = \inf\{d(a,b), a \in A, b \in B\} = 0$. Another useful proximity is the discrete proximity $\eta^\star$ given by

$$A\eta^\star B \Leftrightarrow A \cap B \neq \varnothing.$$

We note that $\eta^\star$ is the finest proximity on X, but is not a compatible one, unless (X,τ) is discrete.

We point out that in this paper $\eta^\star$ is the only proximity that might be non compatible.

U, V denote open sets. $CL(X)$ is the family of all nonempty closed subsets of X.

For any set E in X we use the notation:

$E^{++}_\gamma = \{F \in CL(X) : F \ll_\gamma E$ or equivalently $F\underline{\gamma}E^c\}$.
$E^{++}_{\gamma_0} = E^+ = \{F \in CL(X) : F \ll_{\gamma_0} E$ or equivalently $F \subset intE\}$.
$E^{++}_{\eta^*} = \{F \in CL(X) : F \ll_{\eta^*} E$ or equivalently $F \subset E\}$.
$E^-_\eta = \{F \in CL(X) : F\eta E\}$.
$E^-_{\eta^\star} = \{F \in CL(X) : F\eta^\star E$ or equivalently $F \cap E \neq \varnothing\}$.

Now, we define the **lower Bombay topology**.

Let η_1, η_2 be two proximities on a T_1 space X with $\eta_2 \leq \eta_1$.

A typical nbhd. of $A \in CL(X)$ in the **lower Bombay topology** $\sigma(\eta_2,\eta_1)^-$ consists of the sets of the form $U^-_{\eta_2}$ with $A\eta_1U$, i.e.

$$\{E \in CL(X) : E\eta_2U, \text{ where } U \text{ is open and } A\eta_1U\}.$$

We stress that in the description of the lower Bombay topology $\sigma(\eta_2,\eta_1)^-$ the order in which the coordinate proximities are written (η_2,η_1) emphasizes the fact that the first coordinate proximity η_2 is coarser than the second one η_1, i.e. $\eta_2 \leq \eta_1$. Furthermore, the proximity η_1 selects the open subsets U which delineate the nbhds of A (U fulfills the property $A\eta_1 U$), whereas η_2 describes the nbhd. labelled by U, namely $U^-_{\eta_2}$.

If $\eta_2 = \eta_1 = \eta$, then we have the **lower η-proximal topology** (cf. [8]) denoted by $\sigma(\eta^-) = \sigma(\eta, \eta)^-$.

As for the upper part we have two compatible LO-proximities γ_1, γ_2 with $\gamma_1 \leq \gamma_2$, and $\Delta \subset CL(X)$ a cobase, i.e. Δ is closed under finite unions and contains singletons.

A typical nbhd. of $A \in CL(X)$ in the **upper Bombay topology** $\sigma(\gamma_1, \gamma_2; \Delta)^+$ consists of the sets of the form $U^{++}_{\gamma_2}$, where $U^c \in \Delta$ and $A \ll_{\gamma_1} U$, i.e.

$$\{E \in CL(X) : E \ll_{\gamma_2} U, \text{ where } U^c \in \Delta \text{ and } A \ll_{\gamma_1} U\}.$$

Similarly, in the description of the upper Bombay topology $\sigma(\gamma_1, \gamma_2; \Delta)^+$, the order in which the coordinate proximities are written (γ_1, γ_2) stresses the fact that the first coordinate proximity γ_1 is coarser than the second one γ_2, i.e. $\gamma_1 \leq \gamma_2$. Furthermore, the proximity γ_1 and the cobase $\Delta \subset CL(X)$ together select the open subsets U which delineate the nbhds of A (U fulfills the property $A \ll_{\gamma_1} U$ and $U^c \in \Delta$), whereas γ_2 describes the nbhd. indexed by U, namely $U^{++}_{\gamma_2}$.

If $\gamma_2 = \gamma_1 = \gamma$, then we have the **upper γ-Δ-proximal topology** (cf. [8]) $\sigma(\gamma^+; \Delta) = \sigma(\gamma, \gamma; \Delta)^+$.

For futher details on proximities and hyperspaces see [7], [8], [25].

2. Basic results on lower Bombay topology.

Since we have already studied the upper Bombay topology in our previous paper, we investigate here the salient properties of the lower Bombay topology.

First, we recall that if (X, τ) is a T_1 space, then a topology τ' on $CL(X)$ is declared *admissible* if the map $i\colon (X, \tau) \to (CL(X), \tau')$, defined by $i(x) = \{x\}$, is an embedding.

We point out that all the classical hypertopologies, namely, lower Vietoris topology, upper Vietoris topology, Wijsman topology, Fell topology, Hausdorff topology etc. are admissible.

On the contrary, if the involved proximities η_1, η_2 are different from the discrete proximity $\eta^\star$, then the map $i\colon (X, \tau) \to (CL(X), \sigma(\eta_2, \eta_1)^-)$ is, in general, not even continuous as the following example shows.

Example 2.1. Let $X = [-1, 1]$ with the metric proximity η, η_0 the Wallman proximity. Let $A = \{0\}$, $A_n = \{\frac{1}{n}\}$ for all $n \in \mathbb{N}$. Then $\frac{1}{n}$ converges to 0 in X, but A_n does not converge to A in the topology $\sigma(\eta, \eta_0)^-$, because if $U = (-1, 0)$, then $0\eta_0 U$, but $A_n \underline{\eta} U$ for all n. Hence the map $i\colon (X, \tau) \to (CL(X), \sigma(\eta, \eta_0)^-)$, where $i(x) = \{x\}$, is not an embedding.

Lemma 2.2. *Let X be a T_1 space, η_1, η_2, LO-proximities on X with $\eta_2 \leq \eta_1$ and $\eta^\star$ the discrete proximity. The following results hold:*

(a) $\sigma(\eta^{\star -}) = \tau(V^-)$, *the lower Vietoris topology.*
(b) $\sigma(\eta_2, \eta_1)^- \subset \sigma(\eta_1^-) \cap \sigma(\eta_2^-)$.

Example 2.3. We give examples to show that, in general, $\sigma(\eta_2, \eta_1)^-$ is not comparable with the lower Vietoris topology $\tau(V^-)$.

(a) Let $X = \mathbb{R}$ with the usual metric and η the metric proximity, $A = [0, 1]$, $A_n = [0, 1 - \frac{1}{n}]$, $n \in \mathbb{N}$. Then $A_n \to A$ in $\tau(V^-)$, but does not converge in the topology $\sigma(\eta, \eta_0)^-$, since for $U = (1, 2)$ $A\eta_0 U$, but $A_n \underline{\eta} U$ for all n. Hence $\sigma(\eta, \eta_0)^- \not\subset \tau(V^-)$ (see also Example 2.1).
(b) Let l_2 denote the space of all square summable sequences. $X = B(\theta, 1) \cup \{(1 + \frac{1}{k})e_k : k \in \mathbb{N}\} \subset l_2$, endowed with the Alexandroff proximity η_a (i.e. $E\eta_a F$ iff $clE \cap clF \neq \varnothing$ or both clE, clF are non-compact) and the Wallman proximity η_0. Then $A_n = \{(1 + \frac{1}{k})e_k : k \geq n\}$ converges to $A = \{\theta\}$ in the topology $\sigma(\eta_a, \eta_0)^-$, but not in $\tau(V^-)$. Hence $\tau(V^-) \not\subset \sigma(\eta_a, \eta_0)^-$.

Theorem 2.4. *Let (X, τ) be a T_1 regular space and η_1, η_2 LO-proximities on X. If η_2 is also a compatible LR-proximity on X, then $\tau(V^-) \subset \sigma(\eta_2, \eta_1)^-$.*

Proof. Suppose that the net (A_λ) of closed sets converges to a closed set A in the topology $\sigma(\eta_2, \eta_1)^-$. If $A \in U^-$, where $U \in \tau$, then there is an $a \in A \cap U$. Since η_2 is a compatible LR-proximity, there is an open set V such that $a \in V$ and $clV \underline{\eta}_2 U^c$ by (LR) axiom. Therefore, $a \in V \subset clV \subset U$ and $clV \underline{\eta}_2 U^c$. We claim that eventually A_λ intersects U. For if not, then frequently $A_\lambda \subset U^c$ and so frequently $A_\lambda \underline{\eta}_2 V$; a contradiction. □

Corollary 2.5. *If η is the metric proximity on a metric space (X, d), then*

(a) $\tau(V^-) \subset \sigma(\eta, \eta_0)^- \subset \sigma(\eta^-)$.
(b) $\tau(V^-) \subset \sigma(\eta, \eta_0)^- \subset \sigma(\eta_0^-)$.

Remark 2.6. Example 2.3 (b) points out that the assumption η_2 is a compatible LR-proximity on X cannot be dropped in Theorem 2.4.

Example 2.3 (a) shows that the inclusion in Theorem 2.4 might be strict even in nice spaces. We note that the base space X is one of the *best* possible spaces and the sets involved are also compact.

Theorem 2.7. *If η is the metric proximity on a metric space (X, d), then the following are equivalent:*

(a) *X is a UC space;*
(b) $\sigma(\eta^-) \subset \sigma(\eta, \eta_0)^-$;
(c) $\sigma(\eta^-) = \sigma(\eta, \eta_0)^-$.

Proof. We need prove only $(b) \Rightarrow (a)$. Suppose X is not a UC space. Then, there are two disjoint closed sets of distinct points $A = \{a_n : n \in \mathbb{N}\}$, $B = \{b_n : n \in \mathbb{N}\}$ such that $d(a_{n+1}, b_{n+1}) < d(a_n, b_n) \to 0$. Then, $A_n = \{a_k : k \leq n\}$ converges to A in the topology $\sigma(\eta, \eta_0)^-$, but not in $\sigma(\eta^-)$. In fact, for each natural number n choose $0 < \varepsilon_n < (\frac{1}{4})^n\, d(a_n, b_n)$ and for $n \neq m$, $S_d(b_n, \varepsilon_n) \cap S_d(b_m, \varepsilon_m) = \varnothing$ (where $S_d(x, r)$ is the open sphere centered at x with radius r). Let $U = \bigcup_{n\in\mathbb{N}} S_d(b_n, \varepsilon_n)$. Then, U is open and $A\eta U$ (in fact $B \subset U$ and $d(a_n, b_n) \to 0$). But, $A_n \underline{\eta} U$ for each $n \in \mathbb{N}$ (in fact, it is easy to check that for each $n \in \mathbb{N}$, $0 < \varepsilon_n < \inf\{d(a_k, u) : a_k \in A_n \text{ and } u \in U\}$). □

Let $(X,\mathcal{U})$ be a T_2 uniform space. We recall that for $U \in \mathcal{U}$ $U(x) = \{y \in X : (x, y) \in U\}$ and for any subset E of X, $U(E) = \bigcup_{e\in E} U(e)$. Moreover, E is declared *U-discrete* if $U(e) \cap E = \{e\}$ for each $e \in E$.

Definition 2.8 (cf. [12] or [1])**.** *Let $(X,\mathcal{U})$ be a T_2 uniform space.*

(a) *A typical nbhd. of $A \in CL(X)$ in the* **lower Hausdorff-Bourbaki** *or* **lower H-B topology** *$\tau(H^-)$ consists of the sets of the form*

$$\{B \in CL(X) : A \subset U(B)\}, \text{ where } U \in \mathcal{U}.$$

(b) *A typical nbhd. of $A \in CL(X)$ in the* **upper Hausdorff-Bourbaki** *or* **upper H-B topology** *$\tau(H^+)$ consists of the sets of the form*

$$\{B \in CL(X) : B \subset U(A)\}, \text{ where } U \in \mathcal{U}.$$

Note that the lower H-B uniform topology $\tau(H^-)$ is a topology of locally finite type as Naimpally first proved in [19]. More precisely, Naimpally showed that the topology $\tau(H^-)$ is generated by *hit* sets $\mathbb{L}^{\mathcal{U}}$ = the collection of families of the form

$$\{U(x) : x \in Q \subset A\}, A \in CL(X), U \in \mathcal{U} \text{ and } Q \text{ is } U\text{-discrete.}$$

Note also that the upper H-B uniform topology $\tau(H^+)$ is a topology of upper proximal type, i.e. $\tau(H^+) = \sigma(\delta^+)$ (see [5]), where δ is the EF-proximity associated to $\mathcal{U}$ (see [21] or [10]).

Now, we give examples to show that the lower Hausdorff-Bourbaki or H-B uniform topology $\tau(H^-)$ is not comparable with $\sigma(\eta_2, \eta_1)^-$.

Example 2.9.

(a) Again, let $X = \mathbb{R}$ endowed with the Euclidean metric and η the metric proximity, $A = [0, 1]$, $A_n = [0, 1 - \dfrac{1}{n}]$, $n \in \mathbb{N}$. Then, $A_n \to A$ in $\tau(H^-)$, but it does not converge in the topology $\sigma(\eta, \eta_0)^-$. In fact, if $U = (1, 2)$, then $A\eta_0 U$, but $A_n \underline{\eta} U$ for all n. Hence $\sigma(\eta, \eta_0)^- \not\subset \tau(H^-)$.

(b) Let $X = \mathbb{N} = A$ with the usual metric, $A_n = \{1, 2, \ldots, n\}$. Here $\eta = \eta_0 = \eta^\star$. Moreover, the sequence (A_n) converges to A with respect the topology $\sigma(\eta, \eta_0)^- = \sigma(\eta^{\star -}) = \tau(V^-)$, but (A_n) does not converge to A with respect to the lower H-B uniform topology $\tau(H^-)$. Therefore, $\tau(H^-) \not\subset \sigma(\eta, \eta_0)^-$.

Let η_1, η_2 be LO-proximities on X with $\eta_2 \le \eta_1$, $A \in CL(X)$ and $\mathcal{E}$ a locally finite family of open sets such that $A\eta_1 U$ for all $U \in \mathcal{E}$. Then $\mathcal{E}^-_{\eta_2}$ is the set $\{B \in CL(X) : B\eta_2 U \text{ for all } U \in \mathcal{E}\}$.

Definition 2.10. Let η_1, η_2 be LO-proximities on X with $\eta_2 \le \eta_1$. Given a collection $\mathbb{L}$ of locally finite families of open sets, $\mathbb{L}_A$ denotes the subcollection of $\mathbb{L}$ such that if $\mathcal{E}$ is a family of $\mathbb{L}$ verifying $A\eta_1 U$ for all $U \in \mathcal{E}$, then $\mathcal{E} \in \mathbb{L}_A$.

Suppose $\mathbb{L}$ is a collection of locally finite families of open sets satisfying the filter condition:

($\sharp$) for each $A \in CL(X)$ whenever $\mathcal{E}, \mathcal{F} \in \mathbb{L}_A$ implies there exists a $\mathcal{G} \in \mathbb{L}_A$ such that $\mathcal{G}^-_{\eta_2} \subset \mathcal{E}^-_{\eta_2} \cap \mathcal{F}^-_{\eta_2}$.

Under the above condition, the topology $\sigma(\eta_2, \eta_1; \mathbb{L})^-$ on $CL(X)$ is the topology which has as a basic nbhds system of $A \in CL(X)$ all sets of the form

$$\{B \in CL(X) : B\eta_2 U \text{ for each } U \in \mathcal{E}\}, \text{ where } \mathcal{E} \in \mathbb{L}_A.$$

It is obvious that if $\mathbb{L}$ is a collection of locally finite families of open sets satisfying the above filter condition ($\sharp$) and $\eta^\star$ is the discrete proximity, then

$$\tau(\mathbb{L}^-) = \sigma(\eta^{\star -}; \mathbb{L}) = \sigma(\eta^\star, \eta^\star; \mathbb{L})^-.$$

We say that a collection $\mathbb{L}$ of locally finite families of open sets is *stable under locally finite families* if $\mathcal{E} \in \mathbb{L}$ and $\mathcal{F}$ is a locally finite family of open sets such that for all $V \in \mathcal{F}$ there exists $U \in \mathcal{E}$ with $V \subseteq U$, then $\mathcal{F} \in \mathbb{L}$.

Theorem 2.11. *Let X be a T_1 space, η_1, η_2 LO-proximities on X with $\eta_2 \le \eta_1$ and $\mathbb{L}$ a collection of locally finite families of open sets which satisfies the filter condition ($\sharp$) and is stable under locally finite families. If η_2 is a compatible LR-proximity on X, then $\tau(\mathbb{L}^-) \subset \sigma(\eta_2, \eta_1; \mathbb{L})^-$.*

Proof. Suppose η_2 is a LR-proximity and $A \in \mathcal{E}^-$ where $\mathcal{E} \in \mathbb{L}$. Then for each $U \in \mathcal{E}$ there exists $a_U \in U \cap A$. So, for each $U \in \mathcal{E}$ there is an open set V_U such that $a_U \in V_U \ll_{\eta_2} U$. Let $\mathcal{F} = \{V_U : a_U \in V_U \ll_{\eta_2} U \text{ and } U \in \mathcal{E}\}$. Hence, $\mathcal{F} \in \mathbb{L}_A$ and $\mathbf{W} = \{B \in CL(X) : B\eta_2 V_U \text{ for each } V_U \in \mathcal{F}\}$ is a $\sigma(\eta_2, \eta_1; \mathbb{L})^-$ nbhd. of A which is contained in $\mathcal{E}^-$. □

Corollary 2.12. *Let (X, d) be a metric space and η the associated metric proximity. Let $\mathbb{L}$ be the collection of families of open sets of the form $\{S_d(x, \frac{1}{n}) : x \in Q \subset A$, where $A \in CL(X)$, $n \in \mathbb{N}$, Q is $\frac{1}{n}$-discrete$\}$. Then we have the following:*

$$\tau(V^-) \subset \begin{matrix} \sigma(\eta, \eta_0)^- \\ \\ \tau(H^-) = \sigma(\eta^\star; \mathbb{L})^- \end{matrix} \subset \sigma(\eta, \eta_0; \mathbb{L})^-$$

Furthermore, in general, $\sigma(\eta, \eta_0)^-$ and $\tau(H^-)$ are not comparable.

Theorem 2.13. *Let (X, d) be a locally compact metric space and η the associated metric proximity. Then $\tau(V^-) \subset \sigma(\eta, \eta_0)^- = \sigma(\eta_0^-) \subset \sigma(\eta^-)$.*

Proof. We only prove $\sigma(\eta, \eta_0)^- = \sigma(\eta_0^-)$. It suffices to show $\sigma(\eta_0^-) \subset \sigma(\eta, \eta_0)^-$, since by Corollary 2.5 (b) $\sigma(\eta, \eta_0)^- \subset \sigma(\eta_0^-)$. So, suppose that the net (A_λ) of closed sets converges to a closed set A in the topology $\sigma(\eta, \eta_0)^-$. Let V be an open set and let $A \in V^-_{\eta_0}$. Then, there is an $a \in A \cap clV$. Let U be a compact nbhd. of a and set $W = U \cap V$. Note that clW is compact and that a closed set is η-near a compact set iff it is η_0-near. As a result, eventually $A_\lambda \in W^-_\eta \subset V^-_{\eta_0}$ and the claim holds. □

Remark 2.14. More generally, the equality $\sigma(\eta, \eta_0)^- = \sigma(\eta_0^-)$ in the above Theorem 2.13 holds if one of the following conditions is satisfied:

(i) the base space X is compact;
(ii) the involved proximities η, η_0 are LR, the net of closed sets (A_λ) is eventually locally finite and converges to A in the topology $\sigma(\eta, \eta_0)^-$.

Now, we compare two different lower Bombay topologies $\sigma(\gamma_2, \gamma_1)^-$, $\sigma(\eta_2, \eta_1)^-$. If η is a LO-proximity on X, then for $A \subset X$, $\eta(A) = \{E \subset X : E\eta A\}$.

Theorem 2.15 (Main Theorem)**.** *Let (X, τ) be a T_1 space with LO-proximities γ_1, γ_2, η_1, η_2 with $\gamma_2 \leq \gamma_1$ and $\eta_2 \leq \eta_1$. If γ_2 and η_2 are compatible, then the following are equivalent:*

(a) $\sigma(\gamma_2, \gamma_1)^- \subset \sigma(\eta_2, \eta_1)^-$;
(b) *whenever $A \in CL(X)$ and $U \in \tau$ with $A\gamma_1 U$, there exists a $V \in \tau$ such that: (i) $A\eta_1 V$, and (ii) $\eta_2(V) \subset \gamma_2(U)$.*

Proof. $(a) \Rightarrow (b)$. Let $A \in CL(X)$ and $U \in \tau$ with $A\gamma_1 U$. Then $U^-_{\gamma_2}$ is a $\sigma(\gamma_2, \gamma_1)^-$-nbhd. of A. By assumption there exists a $\sigma(\eta_2, \eta_1)^-$-nbhd. $\mathcal{L}$ of A such that $A \in \mathcal{L} \subset U^-_{\gamma_2}$. $\mathcal{L}$ has the form:

$$\bigcap_{k=1}^{n} \{(V_k)^-_{\eta_2} \text{ with } V_k \in \tau \text{ and } A\eta_1 V_k \text{ for each } k \in \{1, \ldots, n\}\}.$$

We claim that there exists $k_0 \in \{1, \ldots, n\}$ such that $\eta_2(V_{k_0}) \subset \gamma_2(U)$. Assume not. Then, for each $k \in \{1, \ldots, n\}$ there is a closed set F_k with $F_k \in \eta_2(V_k) \setminus \gamma_2(U)$. Set $F = \bigcup_{k=1}^n F_k$. Then $F \in \mathcal{L}$, but $F \notin U^-_{\gamma_2}$; a contradiction.

$(b) \Rightarrow (a)$. Let $A \in CL(X)$ and $U^-_{\gamma_2}$ ($U \in \tau$ and $A\gamma_1 U$) be a subbasic $\sigma(\gamma_2, \gamma_1)^-$-nbhd. of A. By assumption, there is an open subset V with $A\eta_1 V$ and $\eta_2(V) \subset \gamma_2(U)$. It follows that $V^-_{\eta_2}$ is a $\sigma(\eta_2, \eta_1)^-$-nbhd. of A with $V^-_{\eta_2} \subset U^-_{\gamma_2}$. □

Corollary 2.16. *Let (X, τ) be a T_1 space with a compatible LO-proximity η. If $\eta^\star$ is the discrete proximity, then the following are equivalent:*

(a) $\tau(V^-) = \sigma(\eta^{\star -}) \subset \sigma(\eta, \eta^\star)^-$;
(b) *whenever $A \in CL(X)$ and $U \in \tau$ with $A \cap U \neq \varnothing$, there exists a $V \in \tau$ such that $A \cap V \neq \varnothing$ and $\eta(V) \subset \eta^\star(U)$.*

3. First and second countability of lower and upper Bombay topologies.

Lemma 3.1. *Let (X,τ) be a T_1 space with a compatible LO-proximity η and W, V open subsets of X. The following are equivalent:*

(a) $clW \subset clV$;
(b) $\eta(W) \subset \eta(V)$.

Proof. $(a) \Rightarrow (b)$. Let $F \in \eta(W)$. Then $F\eta W$. Since $clW \subset clV$, we have $F\eta V$ and hence $F \in \eta(V)$.

$(b) \Rightarrow (a)$. Assume not. Then, there exists an $x \in clW \setminus clV$. Set $F = \{x\}$. We have $F \in \eta(W)$, but $F \notin \eta(V)$; a contradiction. □

Definition 3.2 (see [8])**.** Let (X,τ) be a T_1 space, η a compatible LO-proximity on X and $A \in CL(X)$.

A family $\mathbf{N}_A$ of open subsets of X is an *external proximal local base of A with respect to η* (or, briefly a *η-external proximal local base of A*) if for any U open subset of X with $A\eta U$, there exists $V \in \mathbf{N}_A$ satisfying $A\eta V$ and $clV \subset clU$.

The *external proximal character of A with respect to η* (or, briefly the *η-external proximal character of A*) is defined as the smallest (infinite) cardinal number of the form $|\mathbf{N}_A|$, where $\mathbf{N}_A$ is a η-external proximal local base of A, and it is denoted by $E\chi(A,\eta)$.

The *external proximal character of $CL(X)$ with respect to η* (or, briefly the *η-external proximal character*) is defined as the supremum of all cardinal numbers $E\chi(A,\eta)$, where $A \in CL(X)$ and is denoted by $E\chi(CL(X),\eta)$.

Note that if $\eta = \eta^\star$, then a family $\mathbf{N}_A$ of open subsets of X is an *external local base of A* if for any U, an open subset of X with $A\eta^\star U$ (i.e. $A \cap U \neq \varnothing$), there exists a $V \in \mathbf{N}_A$ satisying $A\eta^\star V$ (i.e. $A \cap V \neq \varnothing$) and $V \subset U$ (cf. [2]).

In a similar way, we can define the *external character $E\chi(A)$* of A and the *external character $E\chi(CL(X))$* of $CL(X)$ (cf. [2]).

Remark 3.3. Obviously, if the external character $E\chi(CL(X))$ is countable, then the base space X is first countable and each closed subset of X is separable. On the other hand, if we consider the proximal case, X might not be separable even if the η-external character $E\chi(CL(X),\eta)$ is countable. However, we have the following:

If X is a T_3 space, η is a compatible LR-proximity and the η-external proximal character $E\chi(CL(X),\eta)$ is countable, then X is separable.

We now consider the first countability of the lower Bombay topology $\sigma(\eta_2,\eta_1)^-$.

If $\eta_2 = \eta_1 = \eta^\star$, then $\sigma(\eta_2,\eta_1)^-$ is the lower Vietoris topology $\tau(V^-)$. Its first countability has been studied since 1971 ([2], see also [6] and [15]) and holds if and only if X is first countable and each closed subset of X is separable.

Hence, we investigate the case $\sigma(\eta_2,\eta_1)^- \neq \tau(V^-)$, i.e. $\eta_2 \neq \eta^\star$.

First, we study the case $\eta_2 \leq \eta_1 \neq \eta^\star$.

Theorem 3.4. *Let (X,τ) be a T_1 space with compatible LO-proximities η_1, η_2, $\eta_2 \leq \eta_1$ and $\eta_1 \neq \eta^\star$. The following are equivalent:*

(a) $(CL(X), \sigma(\eta_2,\eta_1)^-)$ *is first countable;*
(b) *the η_1-external proximal character $E\chi(CL(X),\eta_1)$ is countable;*
(c) $(CL(X), \sigma(\eta_1^-))$ *is first countable.*

Proof. $(a) \Rightarrow (b)$. It suffices to show that for each $A \in CL(X)$ there exists a countable family $\mathbf{N}_A$ of open subsets of X which is a η_1-external proximal local base of A. So, let $A \in CL(X)$ and $\mathbf{Z}$ be a countable $\sigma(\eta_2,\eta_1)^-$-nbhd. system of A. Then, $\mathbf{Z} = \{\mathcal{L}_n : n \in \mathbb{N}\}$, where each $\mathcal{L}_n$ has the form

$$\bigcap_{k\in I_n} \{(V_k)^-_{\eta_2} \text{ with } V_k \in \tau, A\eta_1 V_k \text{ for every } k \in I_n \text{ and } I_n \text{ finite subset of } \mathbb{N}\}.$$

Set $\mathbf{N}_A = \{V_k : k \in I_n, n \in \mathbb{N}\}$. $\mathbf{N}_A$ is a countable family and if $V_k \in \mathbf{N}_A$, then $A\eta_1 V_k$ (by construction). We claim that for each open set U with $A\eta_1 U$, there is a $V_k \in \mathbf{N}_A$ with $clV_k \subset clU$. So, let U be open with $A\eta_1 U$ and consider $U^-_{\eta_2}$ (which is a $\sigma(\eta_2,\eta_1)^-$-nbhd. of A). By assumption, there is some $\mathcal{L}_n \in \mathbf{Z}$ with $\mathcal{L}_n \subset U^-_{\eta_2}$. Since $\mathcal{L}_n$ has the form $\bigcap_{k\in I_n}\{(V_k)^-_{\eta_2}$ with $V_k \in \tau$, $A\eta_1 V_k$ for every $k \in I_n$ and I_n finite subset of $\mathbb{N}\}$, we have that $\bigcap_{k\in I_n}(V_k)^-_{\eta_2} \subset U^-_{\eta_2}$. We claim that there exists some $k_0 \in I_n$ such that $clV_{k_0} \subset clU$. Assume not and for each $k \in I_n$ let $x_k \in clV_k \setminus clU$. Set $E = \bigcup_{k\in I_n}\{x_k\}$. Then $E \in CL(X)$ as well as $E \in \mathcal{L}_n$, but $E \notin U^-_{\eta_2}$; a contradiction.

$(b) \Rightarrow (a)$. Let $A \in CL(X)$ and $\mathbf{N}_A = \{V_n : n \in \mathbb{N}\}$ be a countable η_1-external proximal local base of A. Obviously, $\mathbf{Z} = \{(V_n)^-_{\eta_2} : V_n \in \mathbf{N}_A\}$ is a countable $\sigma(\eta_2,\eta_1)^-$-subbasic nbhds system of A.

$(b) \Leftrightarrow (c)$. Theorem 2.11 in [8]. □

The case $\eta_2 \leq \eta_1 = \eta^\star$ can be handled similarly. So, we have

Theorem 3.5. *Let (X,τ) be a T_1 space with a compatible LO-proximity η, and the discrete proximity $\eta^\star$ on X. The following are equivalent:*

(a) $(CL(X), \sigma(\eta,\eta^\star)^-)$ *is first countable;*
(b) *the external character $E\chi(CL(X))$ is countable;*
(c) $(CL(X), \tau(V^-))$ *is first countable.*

Remark 3.6. From the above discussion the following unexpected, but natural result holds.

Let (X,τ) be a T_1 space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$. The following are equivalent:

(a) $(CL(X), \sigma(\eta_2,\eta_1)^-)$ is first countable;
(b) $(CL(X), \sigma(\eta_1^-))$ is first countable.

Now, we study the first countability of the upper Bombay Δ topology $\sigma(\gamma_1,\gamma_2;\Delta)^+$.

First, we need the following definition.

Definition 3.7. Let (X,τ) be a T_1 space with a compatible LO-proximity γ, $A \in CL(X)$ and $\Delta \subset CL(X)$ a cobase.

A family $\mathbf{L}_A$ of open nbhds. of A is a *local proximal* Δ *base with respect to* γ (or, briefly a γ*-local proximal* Δ *base of* A) if for any open subset U of X with $U^c \in \Delta$ and $A \ll_\gamma U$, there exists $V \in \mathbf{L}_A$ with $V^c \in \Delta$ and $A \ll_\gamma V \subset U$.

The γ*-proximal* Δ *character* of A is defined as the smallest (infinite) cardinal number of the form $|\mathbf{L}_A|$, where $\mathbf{L}_A$ is a γ-local proximal Δ base of A, and it is denoted by $\chi(A,\gamma,\Delta)$.

The γ*-proximal* Δ *character* of $CL(X)$ is defined as the supremum of all cardinal numbers $\chi(A,\gamma,\Delta)$, where $A \in CL(X)$, and is denoted by $\chi(CL(X),\gamma,\Delta)$.

Theorem 3.8. *Let (X,τ) be a T_1 space with compatible LO-proximities γ_1, γ_2, $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cobase. The following are equivalent:*

(a) $(CL(X),\sigma(\gamma_1,\gamma_2;\Delta)^+)$ *is first countable;*
(b) *the γ_1-proximal Δ character $\chi(CL(X),\gamma_1,\Delta)$ of $CL(X)$ is countable;*
(c) $(CL(X),\sigma(\gamma_1^+;\Delta))$ *is first countable.*

Proof. $(a) \Rightarrow (b)$. Let $A \in CL(X)$ and $\mathbf{U}_A$ be a countable $\sigma(\gamma_1,\gamma_2;\Delta)^+$-nbhd. system of A. Then, $\mathbf{U}_A = \{\mathcal{O}_n : n \in \mathbb{N}\}$, where $\mathcal{O}_n = V^{++}_{\gamma_2}$ with $V^c \in \Delta$ and $A \ll_{\gamma_1} V$. Let $\mathbf{L}_A = \{V : A \ll_{\gamma_1} V$ and $V^{++}_{\gamma_2} = \mathcal{O}_n$ for some $\mathcal{O}_n \in \mathbf{U}_A\}$. By construction $\mathbf{L}_A$ is countable. We claim that $\mathbf{L}_A$ is a γ_1-proximal local Δ base of A. So, let U be an open subset of X with $A \ll_{\gamma_1} U$ and $U^c \in \Delta$. Then, $U^{++}_{\gamma_2}$ is a $\sigma(\gamma_1,\gamma_2;\Delta)^+$-nbhd. of A. By assumption, there is $\mathcal{O}_n \in \mathbf{U}_A$ with $A \in \mathcal{O}_n \subset U^{++}_{\gamma_2}$. Note that $\mathcal{O}_n = V^{++}_{\gamma_2}$, where $V^c \in \Delta$ and $A \ll_{\gamma_1} V$. We claim $V \subset U$. Assume not. Let $x \in V \setminus U$ and set $F = \{x\}$. Then, $F \in \mathcal{O}_n$, but $F \notin U^{++}_{\gamma_2}$; a contradiction.

$(b) \Rightarrow (a)$. Let $A \in CL(X)$ and $\mathbf{L}_A$ be a countable γ_1-proximal local Δ base of A. Set $\mathbf{U}_A = \{V^{++}_{\gamma_2} : V \in \mathbf{L}_A\}$. By construction $\mathbf{U}_A$ is a countable family of open $\sigma(\gamma_1,\gamma_2;\Delta)^+$-nbhd. of A. We claim that $\mathbf{U}_A$ is a $\sigma(\gamma_1,\gamma_2;\Delta)^+$-nbhd. system of A. So, let $\mathcal{B}$ be a $\sigma(\gamma_1,\gamma_2;\Delta)^+$-nbhd. of A. Then, $\mathcal{B}$ has the form $\{U^{++}_{\gamma_2} : U^c \in \Delta$ and $A \ll_{\gamma_1} U\}$. Let V be an open subset with $V \in \mathbf{L}_A$ and $A \ll_{\gamma_1} V \subset U$ and consider $V^{++}_{\gamma_2}$. Then $V^{++}_{\gamma_2} \in \mathbf{U}_A$ and since $V \subset U$ we have $V^{++}_{\gamma_2} \subset U^{++}_{\gamma_2}$.

$(b) \Leftrightarrow (c)$. It is straightforward. □

Remark 3.9. Let (X,τ) be a T_1 space with compatible LO-proximities γ_1, γ_2, $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cobase. The following are equivalent:

(a) $(CL(X),\sigma(\gamma_1,\gamma_2;\Delta)^+)$ is first countable;
(b) $(CL(X),\sigma(\gamma_1^+;\Delta))$ is first countable.

Definition 3.10 (cf. [8])**.** Let (X,τ) be a T_1 space and η a compatible LO-proximity on X.

A family $\mathbf{N}$ of open subsets of X is an *external proximal base with respect to* η (or, briefly a η*-external proximal base*) if for any $A \in CL(X)$ and any $U \in \tau$ with $A\eta U$, there exists $V \in \mathbf{N}$ satisfying $A\eta V$ and $clV \subset clU$.

The *external proximal weight of* $CL(X)$ *with respect to* η (or, briefly the η-*external proximal weight of* $CL(X)$) is the smallest (infinite) cardinality of its η-external proximal bases and it is denoted by $EW(CL(X), \eta)$.

Note that if $\eta = \eta^\star$, then a family $\mathbf{N}$ of open subsets of X is an *external base* if for any $A \in CL(X)$ and any $U \in \tau$ with $A\eta^\star U$ (i.e. $A \cap U \neq \varnothing$), there exists a $V \in \mathbf{N}$ satisfying $A\eta^\star V$ (i.e. $A \cap V \neq \varnothing$) and $V \subset U$ (see [2]).

The *external character* $E\chi(A)$ of A and the *external weight* $EW(CL(X))$ of $CL(X)$ can be defined similarly (see [2]).

Now, we study the second countability of the lower Bombay topology $\sigma(\eta_2, \eta_1)^-$.

If $\eta_2 = \eta_1 = \eta^\star$, then $\sigma(\eta_2, \eta_1)^-$ is the lower Vietoris topology $\tau(V^-)$. Its second countability has been studied by [6] and [15] and holds if and only if X is second countable.

Hence we investigate the case $\sigma(\eta_2, \eta_1)^- \neq \tau(V^-)$, i.e. $\eta_2 \neq \eta^\star$.

First, the case $\eta_2 \leq \eta_1 \neq \eta^\star$.

Theorem 3.11. *Let* (X, τ) *be a* T_1 *space with compatible LO-proximities* η_1, η_2, $\eta_2 \leq \eta_1$ *and* $\eta_1 \neq \eta^\star$. *The following are equivalent:*

(a) $(CL(X), \sigma(\eta_2, \eta_1)^-)$ *is second countable;*
(b) *the* η_1*-external proximal weight* $EW(CL(X), \eta_1)$ *of* $CL(X)$ *is countable;*
(c) $(CL(X), \sigma(\eta_1^-))$ *is second countable.*

Now, we discuss the case $\eta_2 \leq \eta_1 = \eta^\star$.

Theorem 3.12. *Let* (X, τ) *be a* T_1 *space with a compatible LO-proximity* η *and* $\eta^\star$ *the discrete proximity on* X. *The following are equivalent:*

(a) $(CL(X), \sigma(\eta, \eta^\star)^-)$ *is second countable;*
(b) *the external proximal weight* $EW(CL(X))$ *of* $CL(X)$ *is countable;*
(c) $(CL(X), \tau(V^-))$ *is second countable.*

Remark 3.13. Let (X, τ) be a T_1 space and η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$. The following are equivalent:

(a) $(CL(X), \sigma(\eta_2, \eta_1)^-)$ is second countable;
(b) $(CL(X), \sigma(\eta_1^-))$ is second countable.

Definition 3.14. Let (X, τ) be a T_1 space with a compatible LO-proximity γ and $\Delta \subset CL(X)$ a cobase.

A family $\mathbf{B}$ of open subsets of X is a γ-*proximal base with respect to* Δ if whenever $A \ll_\gamma U$ with $U^c \in \Delta$, there exists $V \in \mathbf{B}$ with $V^c \in \Delta$ and $A \ll_\gamma V \subset U$.

The γ-*proximal weight of* $CL(X)$ *with respect to* Δ (or, briefly the γ-*proximal weight with respect to* Δ) is the smallest (infinite) cardinality of its γ-proximal bases with respect to Δ and it is denoted by $W(CL(X), \gamma, \Delta)$.

Theorem 3.15. *Let (X, τ) be a T_1 space with compatible LO-proximities γ_1, γ_2, $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cobase. The following are equivalent:*

(a) $(CL(X), \sigma(\gamma_1, \gamma_2; \Delta)^+)$ *is second countable;*
(b) *the γ_1-proximal weight $W(CL(X), \gamma_1, \Delta)$ of $CL(X)$ with respect to Δ is countable;*
(c) $(CL(X), \sigma(\gamma_1^+; \Delta))$ *is second countable.*

Remark 3.16. Let (X, τ) be a T_1 space with compatible LO-proximities γ_1, γ_2, $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cobase. The following are equivalent:

(a) $(CL(X), \sigma(\gamma_1, \gamma_2; \Delta)^+)$ is second countable;
(b) $(CL(X), \sigma(\gamma_1^+; \Delta))$ is second countable.

4. Symmetric Bombay topology and some of its properties.

Let (X, τ) be a T_1 space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cobase. The lower Bombay topology $\sigma(\eta_2, \eta_1)^-$ combined with the upper one $\sigma(\gamma_1, \gamma_2; \Delta)^+$ yields a new hypertopology, namely

the Δ-*symmetric Bombay topology* with respect to η_2, η_1, γ_1, γ_2 denoted by $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta) = \sigma(\eta_2, \eta_1)^- \vee \sigma(\gamma_1, \gamma_2; \Delta)^+$.

If $\eta_2 = \eta_1 = \eta^\star$, then we have the standard γ_1-γ_2-Δ-*Bombay topology* $\sigma(\gamma_1, \gamma_2; \Delta) = \pi(\eta^\star, \eta^\star, \gamma_1, \gamma_2; \Delta) = \tau(V^-) \vee \sigma(\gamma_1, \gamma_2; \Delta)^+$, investigated in [7].

If $\eta_2 = \eta_1 = \eta$ and $\gamma_1 = \gamma_2 = \gamma$, then we have the η-γ-Δ-*symmetric proximal topology* $\pi(\eta, \gamma; \Delta) = \sigma(\eta, \eta)^- \vee \sigma(\gamma, \gamma; \Delta)^+ = \sigma(\eta^-) \vee \sigma(\gamma^+; \Delta)$, studied in [8].

We now consider some basic properties of $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$.

In general, the space X is not embedded in $(CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ (cf. Example 2.1) and so $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$ is not an admissible topology.

Lemma 4.1. *Let (X, τ) be a T_1 space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$, $\Delta \subset CL(X)$ a cobase and $A \in CL(X)$. If $\gamma_1 \leq \eta_1$, then a base for the nbhd. system of A with respect to the $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$ topology consists of all sets of the form:*

$V_{\gamma_2}^{++} \cap \bigcap_{j \in J} (S_j)_{\eta_2}^-$, *with $A \ll_{\gamma_1} V$, $V^c \in \Delta$, $A \eta_1 S_j$, $S_j \in \tau$ for each $j \in J$, J finite and $\bigcup_{j \in J} S_j \subset V$.*

Proof. Let $A \in V_{\gamma_2}^{++} \cap \bigcap_{j \in J} (S_j)_{\eta_2}^-$, where $A \ll_{\gamma_1} V$, $V^c \in \Delta$, $A \eta_1 S_j$, $S_j \in \tau$ for each $j \in J$ and J finite. We may replace each S_j with $S_j \cap V$. In fact, from $\gamma_1 \leq \eta_1$ we have $A \underline{\eta}_1 V^c$ and thus $A \eta_1 S_j$ iff $A \eta_1 (S_j \cap V)$. □

Remark 4.2. The condition $\gamma_1 \leq \eta_1$ in the above Lemma 4.1 is indeed a natural one. In fact, in the presentation $V_{\gamma_2}^{++} \cap \bigcap_{j \in J} (S_j)_{\eta_2}^-$ we may assume that $\bigcup_{j \in J} S_j \subset V$ as in the classic Vietoris topology. When $\gamma_1 \leq \eta_1$, they associated symmetric Bombay topology $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$ is called **standard**, otherwise *abstract*. We will see that the most significant result hold for **standard** symmetric Bombay topologies. Often, we will omit the term standard.

We point out that all the symmetric Bombay topologies investigated in this section are standard.

Remark 4.3. Let (X,τ) be a T_1 space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$, $\Delta \subset CL(X)$ a cobase, $\gamma_1 \leq \eta_1$, i.e. $\pi(\eta_2,\eta_1,\gamma_1,\gamma_2;\Delta)$ is standard. If D is a dense subset of X, then the family of all finite subsets of D is dense in $(CL(X),\pi(\eta_2,\eta_1,\gamma_1,\gamma_2;\Delta))$.

Theorem 4.4. *Let (X,τ) be a T_1 space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$, $\Delta \subset CL(X)$ a cobase and $\gamma_1 \leq \eta_1$. The following are equivalent:*

(a) $(CL(X),\pi(\eta_2,\eta_1,\gamma_1,\gamma_2;\Delta))$ *is first countable;*
(b) $(CL(X),\sigma(\eta_2,\eta_1)^-)$ *and* $(CL(X),\sigma(\gamma_1,\gamma_2;\Delta)^+)$ *are both first countable.*

Proof. $(b) \Rightarrow (a)$ is clear, since $\pi(\eta_2,\eta_1,\gamma_1,\gamma_2;\Delta) = \sigma(\eta_2,\eta_1)^- \vee \sigma(\gamma_1,\gamma_2;\Delta)^+$.

$(a) \Rightarrow (b)$. Let $A \in CL(X)$. Assume

$\mathbf{Z} = \{\mathcal{L} = V^{++}_{\gamma_2} \cap \bigcap_{j\in J}(S_j)^-_{\eta_2}$, with $A \ll_{\gamma_1} V$, $V^c \in \Delta$, S_j open, $A\eta_1 S_j$ and J finite $\}$

is a countable local base of A with respect to the topology $\pi(\eta_2,\eta_1,\gamma_1,\gamma_2;\Delta)$. We claim that the family $\mathbf{Z}^+ = \{V^{++}_{\gamma_2} : V^{++}_{\gamma_2}$ occurs in some $\mathcal{L} \in \mathbf{Z}\} \bigcup \{CL(X)\}$ forms a local base of A with respect to the topology $\sigma(\gamma_1,\gamma_2;\Delta)^+$. Indeed, if there is no open subset U with $A \ll_{\gamma_1} U$, $U^c \in \Delta$, then $CL(X)$ is the only open set in $\sigma(\gamma_1,\gamma_2;\Delta)^+$ containing A. If there is U with $A \ll_{\gamma_1} U$, $U^c \in \Delta$, then $U^{++}_{\gamma_2}$ is a $\pi(\eta_2,\eta_1,\gamma_1,\gamma_2;\Delta)$-nbhd. of A. Therefore, there exists $\mathcal{L} \in \mathbf{Z}$ with $A \in \mathcal{L} \subset U^{++}_{\gamma_2}$. Note that $\mathcal{L}$ cannot be of the form $\bigcap_{j\in J}(S_j)^-_{\eta_2}$, otherwise by setting $F = A \cup U^c$, we have $F \in \mathcal{L}$, but $F \notin U^{++}_{\gamma_2}$; a contradiction. Thus, $\mathcal{L}$ has the form $V^{++}_{\gamma_2} \cap \bigcap_{j\in J}(S_j)^-_{\eta_2}$. We claim that $V^{++}_{\gamma_2} \subset U^{++}_{\gamma_2}$. Assume not and let $E \in \{V^{++}_{\gamma_2}$, with $A \ll_{\gamma_1} V$, $V^c \in \Delta\} \setminus U^{++}_{\gamma_2}$. Set $F = E \cup A$, we have $F \in \mathcal{L} \setminus U^{++}_{\gamma_2}$; a contradiction.

Now, we show that there is a countable local base of A with respect to the topology $\sigma(\eta_2,\eta_1)^-$. Without any loss of generality, we may assume that in the expression of every element from $\mathbf{Z}$ the family of index set J is non-empty. In fact $\{V^{++}_{\gamma_2} : A \ll_{\gamma_1} V, V^c \in \Delta\} = \{V^{++}_{\gamma_2} : A \ll_{\gamma_1} V, V^c \in \Delta\} \cap V^-_{\eta_2}$. Moreover, by Lemma 4.1, if $\mathcal{L} \in \mathbf{Z}$ then $\mathcal{L} = V^{++}_{\gamma_2} \cap \bigcap_{j\in J}(S_j)^-_{\eta_2}$, where $A \ll_{\gamma_1} V$, $V^c \in \Delta$, $A\eta_1 S_j$, $S_j \in \tau$ for each $j \in J$, $\bigcup_{j\in J} S_j \subset V$ and J finite.

Set $\mathbf{Z}^- = \{(S_j)^-_{\eta_2} : (S_j)^-_{\eta_2}$ occurs in some $\mathcal{L} \in \mathbf{Z}\}$. We claim that the family $\mathbf{Z}^-$ is a local subbase of A with respect to the topology $\sigma(\eta_2,\eta_1)^-$. Take S open with $A\eta_1 S$. Then, $S^-_{\eta_2}$ is a $\pi(\eta_2,\eta_1,\gamma_1,\gamma_2;\Delta)$-nbhd. of A. Hence, there exists $\mathcal{L} \in \mathbf{Z}$ with $A \in \mathcal{L} = V^{++}_{\gamma_2} \cap \bigcap_{j\in J}(S_j)^-_{\eta_2} \subset S^-_{\eta_2}$. We claim that there exists a $j_0 \in J$ such that $(S_{j_0})^-_{\eta_2} \subset S^-_{\eta_2}$. It suffices to show that there exists a $j_0 \in J$ such that $S_{j_0} \subset S^{\eta_2}$ (where $S^{\eta_2} = \{x \in X : x\eta_2 S\}$). Assume not and for each $j \in J$ let $x_j \in S_j \setminus S^{\eta_2}$. The set $F = \bigcup_{j\in J}\{x_j\} \in \mathcal{L} \setminus S^-_{\eta_2}$; a contradiction. □

By Theorems 3.4, 3.8 and 4.3 we have

Corollary 4.5. *Let (X, τ) be a T_1 space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$, $\Delta \subset CL(X)$ a cobase and $\gamma_1 \leq \eta_1$. The following are equivalent:*

(a) *$(CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is first countable;*
(b) *the η_1-external proximal character $E\chi(CL(X), \eta_1)$ of $CL(X)$ and the γ_1-proximal Δ character $\chi(CL(X), \gamma_1, \Delta)$ of $CL(X)$ are both countable;*
(c) *the η_1-γ_1-Δ-symmetric proximal topology $\pi(\eta_1, \gamma_1; \Delta)$ on $CL(X)$ is first countable.*

Proof. Note that $(b) \Leftrightarrow (c)$ follows from Theorem 4.9 in [8]. □

Theorem 4.6. *Let (X, τ) be a T_1 space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$, $\Delta \subset CL(X)$ a cobase and $\gamma_1 \leq \eta_1$. The following are equivalent:*

(a) *$(CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is second countable;*
(b) *$(CL(X), \sigma(\eta_2, \eta_1)^-)$ and $(CL(X), \sigma(\gamma_1, \gamma_2; \Delta)^+)$ are both second countable.*

The next Corollary follows from Theorems 3.11, 3.15 and Corollary 4.5.

Corollary 4.7. *Let (X, τ) be a T_1 space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$, $\Delta \subset CL(X)$ a cobase and $\gamma_1 \leq \eta_1$. The following are equivalent:*

(a) *$(CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is second countable;*
(b) *the η_1-external proximal weight $EW(CL(X), \eta_1)$ of $CL(X)$ and the γ_1-proximal weight $W(CL(X), \gamma_1, \Delta)$ of $CL(X)$ with respect to Δ are both countable;*
(c) *the η_1-γ_1-Δ-symmetric proximal topology $\pi(\eta_1, \gamma_1; \Delta)$ on $CL(X)$ is second countable.*

Proof. Note that $(b) \Leftrightarrow (c)$ follows from Theorem 4.12 in [8]. □

Theorem 4.8. *Let (X, τ) be a Tychonoff space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$, $\Delta \subset CL(X)$ a cobase, $\gamma_1 \leq \eta_1$. If η_1 is a compatible LR-proximity, γ_1 a compatible EF-proximity, then the following are equivalent:*

(a) *$(CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is metrizable;*
(b) *$(CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is second countable and uniformizable;*
(c) *$(CL(X), \pi(\eta_1, \gamma_1; \Delta))$ is metrizable.*

Proof. $(b) \Rightarrow (a)$. It follows from the Urysohn's Metrization Theorem.

$(a) \Rightarrow (b)$. Since $(CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is first countable, X is separable (use Theorem 4.4 and Remark 3.3). Thus, $(CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is second countable (see Remark 4.3).

$(b) \Leftrightarrow (c)$. Use Corollary 4.7. □

Now, we compare two symmetric standard Bombay topologies.

Theorem 4.9. *Let (X, τ) be a T_1 space; η_1, η_2, α_1, α_2 LO-proximities on X with $\eta_2 \leq \eta_1$, $\alpha_2 \leq \alpha_1$; γ_1, γ_2, δ_1, δ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$, $\delta_1 \leq \delta_2$ and Δ and $\Lambda \subset CL(X)$ cobases. If η_2, α_2 are compatible and $\gamma_1 \leq \eta_1$ as well as $\delta_1 \leq \alpha_1$, then the following are equivalent:*

(a) *$\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta) \subset \pi(\alpha_2, \alpha_1, \delta_1, \delta_2; \Lambda)$;*

(b) *(1) for each $F \in CL(X)$ and $U \in \tau$ with $F\eta_1 U$ there are $W \in \tau$ and $L \in \Lambda$ such that $(1i)$ $F \in [\alpha_1(W) \setminus \delta_1(L)]$, and $(1ii)$ $[\alpha_2(W) \setminus \delta_2(L)] \subset \eta_2(U)$;*
(2) for each $B \in \Delta$ and $W \in \tau$, $W \neq X$ with $B \ll_{\gamma_1} W$ there exists $M \in \Lambda$ such that $(2i)$ $M \ll_{\delta_1} W$, and $(2ii)$ $\gamma_2(B) \subset \delta_2(M)$.

Proof. $(a) \Rightarrow (b)$. We start by showing (1). So, let $F \in CL(X)$ and $U \in \tau$ with $F\eta_1 U$. Then $U^-_{\eta_2}$ is a $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$-nbhd. of F. By assumption there is a $\pi(\alpha_2, \alpha_1, \delta_1, \delta_2; \Lambda)$-nbhd. $\mathbf{W}$ of F such that $\mathbf{W} \subset U^-_{\eta_2}$.

$\mathbf{W} = W^{++}_{\delta_2} \cap \bigcap_{i=1}^n (W_i)^-_{\alpha_2}$, $F\alpha_1 W_i$, $W_i \in \tau$, $i \in \{1, \ldots, n\}$, $\bigcup_{i=1}^n W_i \subset W$, $W^c \in \Lambda$ and $F \ll_{\delta_1} W$. Set $L = W^c$. By construction $F\underline{\delta}_1 L$ as well as $F\alpha_1 W_i$, i.e. $F \in [\alpha_1(W_i) \setminus \delta_1(L)]$, for $i \in \{1, \ldots, n\}$. We claim that there exists $i_0 \in \{1, \ldots, n\}$ such that $[\alpha_2(W_{i_0}) \setminus \delta_2(L)] \subset \eta_2(U)$. Assume not. Then, for each $i \in \{1, \ldots, n\}$ there exists $T_i \in CL(X)$ with $T_i \in [\alpha_2(W_i) \setminus \delta_2(L)]$ and $T_i \notin \eta_2(U)$, i.e. $T_i\alpha_2 W_i$, $T_i \ll_{\delta_2} W = L^c$ and $T_i\underline{\eta}_2 U$. Set $T = \bigcup_{i=1}^n T_i$. $T \in CL(X)$, $T \in \mathbf{W} = W^{++}_{\delta_2} \cap \bigcap_{i=1}^n (W_i)^-_{\alpha_2}$ and $T \notin U^-_{\eta_2}$ which contradicts $\mathbf{W} \subset U^-_{\eta_2}$.

Now, we show (2). So, let $B \in \Delta$ and $W \in \tau$, $W \neq X$ with $B \ll_{\gamma_1} W$. Set $A = W^c$. Then, $A \in CL(X)$ and $A \in (B^c)^{++}_{\gamma_2} \in \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$. Thus, there exists a $\pi(\alpha_2, \alpha_1, \delta_1, \delta_2; \Lambda)$-nbhd. $\mathbf{O}$ of A such that $\mathbf{O} \subset (B^c)^{++}_{\gamma_2}$.

$\mathbf{O} = O^{++}_{\delta_2} \cap \bigcap_{i=1}^n (O_i)^-_{\alpha_2}$, $A\alpha_1 O_i$, $O_i \in \tau$ for each $i \in \{1, \ldots, n\}$, $\bigcup_{i=1}^n O_i \subset O$, $O^c \in \Lambda$ and $A \ll_{\delta_1} O$. Set $M = O^c$. By construction $A\underline{\delta}_1 M$. Therefore, $M \ll_{\delta_1} W = A^c$. We claim that $\gamma_2(B) \subset \delta_2(M)$. Assume not and let $E \in [\gamma_2(B) \setminus \delta_2(M)]$ with $E \in CL(X)$. Set $F = A \cup E$. Then $F \in CL(X)$, $F \in \mathbf{O} = O^{++}_{\delta_2} \cap \bigcap_{i=1}^n (O_i)^-_{\alpha_2}$ and $F \notin (B^c)^{++}_{\gamma_2}$; a contradiction.

$(b) \Rightarrow (a)$. Let $F \in CL(X)$, $\mathbf{U} = U^{++}_{\gamma_2} \cap \bigcap_{i=1}^n (U_i)^-_{\eta_2}$ be a $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$-nbhd. of F. Then, $F\eta_1 U_i$, $U_i \in \tau$, $i \in \{1, \ldots, n\}$, $\bigcup_{i=1}^n U_i \subset U$, $F \ll_{\gamma_1} U$ and $B = U^c \in \Delta$. By (1) for each $i \in \{1, \ldots, n\}$ there exist $W_i \in \tau$ and $L_i \in \Lambda$ such that $F \in [\alpha_1(W_i) \setminus \delta_1(L_i)]$ and $[\alpha_2(W_i) \setminus \delta_2(L_i)] \subset \eta_2(U_i)$. By (2), there exists $M \in \Lambda$ such that $M \ll_{\delta_1} F^c$ (i.e. $M\underline{\delta}_1 F$) and $\gamma_2(B) \subset \delta_2(M)$.

Set $N = \bigcup_{i=1}^n L_i \cup M \in \Lambda$, $O = N^c$ and for each $i \in \{1, \ldots, n\}$ $O_i = W_i \setminus N$. Note that $F\underline{\delta}_1 N$ and by construction $O_i \subset O$ for each $i \in \{1, \ldots, n\}$. Thus $\bigcup_{i=1}^n O_i \subset O$. We claim that $F\alpha_1 O_i$ for each $i \in \{1, \ldots, n\}$. Assume not. Since, $F\underline{\alpha}_1 N$ and $F\underline{\alpha}_1 W_{i_0} \cap N^c$, then $F\underline{\alpha}_1 N \cup (W_{i_0} \cap N^c)$, and hence $F \ll_{\alpha_1} N^c \cap W^c_{i_0} \subseteq W^c_{i_0}$ and hence $F\underline{\alpha}_1 W_{i_0}$. But $F\alpha_1 W_{i_0}$, a contradiction.

It follows that $\mathbf{O} = O^{++}_{\delta_2} \cap \bigcap_{i=1}^n (O_i)^-_{\alpha_2}$ is a $\pi(\alpha_2, \alpha_1, \delta_1, \delta_2; \Lambda)$-nbhd. of F. We claim that $\mathbf{O} \subset \mathbf{U}$. Assume not. Then there exists $E \in \mathbf{O}$, but $E \notin \mathbf{U}$. Hence either $(\Diamond^\star)$ $E\underline{\eta}_2 U_i$ for some i or $(\Diamond^\star\Diamond^\star)$ $E\gamma_2 U^c$.

If $(\diamondsuit^\star)$ occurs, then since $E\alpha_2 O_i$, $O_i \subset W_i$, $E \ll_{\delta_2} O = N^c$ and $L_i \subset N$ we have $E \in [\alpha_2(W_i) \setminus \delta_2(L_i)] \setminus \eta_2(U_i)$, and hence $E \in [\alpha_2(W_i) \setminus \delta_2(L_i)] \not\subset \eta_2(U_i)$ which contradicts $(1ii)$.

If $(\diamondsuit^\star\diamondsuit^\star)$ occurs, then since $E\gamma_2 B = U^c$, $E \ll_{\delta_2} O = N^c$ and $M \subset N$ we have $E \in \gamma_2(B) \setminus \delta_2(M)$, i.e. $\gamma_2(B) \not\subset \delta_2(M)$; which contradicts $(2ii)$. □

5. Uniformizable symmetric abstract Bombay topologies.

This section is devoted to find conditions which guarantee the uniformizability of a Δ-symmetric abstract Bombay topology $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$.

First we need the following definitions.

Definition 5.1 (cf. [8]). Let (X, τ) be a T_1 space, δ a compatible LO-proximity on X and $\Delta \subset CL(X)$.

(a) Δ is *δ-Urysohn* iff whenever $D \in \Delta$ and $A \in CL(X)$ are δ-far, there exists an $E \in \Delta$ with $D \ll_\delta E \ll_\delta A^c$ (see also [9], [3]).

(b) Δ is *Urysohn* iff (a) above is true w.r.t. the LO-proximity δ_0, i.e. whenever $D \in \Delta$ and $A \in CL(X)$ are disjoint, there exists $E \in \Delta$ with $D \subset intE \subset E \subset A^c$.

Lemma 5.2 (cf. Theorem 1.6 in [9]). *Let (X, τ) be a Tychonoff space, γ a compatible LO-proximity on X and $\Delta \subset CL(X)$ a cobase. If Δ is γ-Urysohn, then the relation δ defined on the power set of X by*

$$(*) \qquad A\underline{\delta}B \text{ iff } clA\underline{\gamma}clB \text{ and either } clA \in \Delta \text{ or } clB \in \Delta$$

is a compatible EF-proximity on X. Moreover, $\delta \leq \gamma$ and Δ is γ-Urysohn iff Δ is δ-Urysohn.

We recall that if (X, τ) is a Tychonoff space with a compatible EF-proximity δ, then a uniformity $\mathcal{U}$ on X is compatible w.r.t. δ if the proximity relation $\delta(\mathcal{U})$ defined by $A\delta(\mathcal{U})B$ iff $A \cap U(B) \neq \varnothing$ for each $U \in \mathcal{U}$ equals δ (see [21] or [10]). We point out that δ admits a unique compatible totally bounded uniformity $\mathcal{U}_w(\delta)$ ([21], [10]).

We will omit reference to δ if this is clear from the context.

Let $\mathcal{U}$ be a compatible uniformity on X and Δ a cobase. For each $D \in \Delta$ and $U \in \mathcal{U}$ set
$[D, U] = \{(A_1, A_2) \in CL(X) \times CL(X) : A_1 \cap D \subset U(A_2) \text{ and } A_2 \cap D \subset U(A_1)\}$.

The family $\{[D, U] : D \in \Delta \text{ and } U \in \mathcal{U}\}$ is a base for a filter $\mathcal{U}_\Delta$ on $CL(X)$ called the *Δ-Attouch-Wets filter*. $\mathcal{U}_\Delta$ induces the topology $\tau(\mathcal{U}_\Delta)$ called the *Δ-Attouch-Wets topology* (cf. [1] and [9]).

We recall that a cobase Δ is a *cover* on X iff it is closed hereditary (cf. [9]).

The following Theorem is given in [9].

Theorem 5.3 (cf. Theorem 2.1 in [9]). *Let (X,τ) be a Tychonoff space with a compatible EF-proximity δ, $\mathcal{U}_w$ the unique totally bounded uniformity which induces δ and $\Delta \subseteq CL(X)$ a cover of X. Then the following are equivalent:*

(a) *Δ is δ-Urysohn;*

(b) *1) the Δ-Attouch-Wets filter $\mathcal{U}_{w\Delta}$ is a Hausdorff uniformity, and*
2) the proximal Δ-topology $\sigma(\delta,\Delta)$ equals $\tau(\mathcal{U}_{w\Delta})$.

Lemma 5.4 (cf. Theorem 2.2 in [9]). *Let (X,τ) be a Tychonoff space, γ_1, γ_2, compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cover of X. If Δ is γ_1-Urysohn, δ the compatible EF-proximity on X defined by*

$$(*) \qquad A\underline{\delta}B \text{ iff } clA\underline{\gamma}_1 clB \text{ and either } clA \in \Delta \text{ or } clB \in \Delta$$

and $\mathcal{U}_w$ the unique totally bounded uniformity on X compatible w.r.t. δ, then the Bombay topology $\sigma(\gamma_1,\gamma_2;\Delta)$, the proximal Δ-topology $\sigma(\delta;\Delta)$ and the topology $\tau(\mathcal{U}_{w\Delta})$ induced by the Δ-Attouch-Wets uniformity $\mathcal{U}_{w\Delta}$ all coincide. Thus the Bombay topology $\sigma(\gamma_1,\gamma_2;\Delta)$ is Tychonoff.

Proof. We omit the proof that is similar to that of Theorem 2.2 in [9]. □

By Theorem 2.4 and Lemma 2.2 (b) we know that if η_2 is a compatible LR-proximity on X, then $\tau(V^-) \subset \sigma(\eta_2,\eta_1)^- \subset \sigma(\eta_2^-) \cap \sigma(\eta_1^-)$. So, in order to get $\sigma(\eta_2,\eta_1)^-$ we have to augment a typical entourage $[D,U] \in \mathcal{U}_{w\Delta}$ by adding sets of the type

$$P_{\{V_k\}} = \{(A,B) \in CL(X) \times CL(X) : A\eta_1 V_k \text{ and } B\eta_1 V_k\} \text{ and}$$
$$Q_{\{V_k\}} = \{(A,B) \in CL(X) \times CL(X) : A\eta_2 V_k \text{ and } B\eta_2 V_k\}$$

for a finite family of open sets $\{V_k\}$.

Then, we have

Theorem 5.5. *Let (X,τ) be a Tychonoff space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cover of X. If η_2 is a compatible LR-proximity, Δ γ_1-Urysohn, δ the compatible EF-proximity defined by*

$$(*) \qquad A\underline{\delta}B \text{ iff } clA\underline{\gamma}_1 clB \text{ and either } clA \in \Delta \text{ or } clB \in \Delta$$

and $\mathcal{U}_w$ the unique totally bounded uniformity on X compatible w.r.t. δ, then the family

$$\mathbb{S} = \mathcal{U}_{w\Delta} \cup \{[D,U] \cap P_{\{V_k\}} : [D,U] \in \mathcal{U}_{w\Delta}, \{V_k\} \text{ finite family of open sets }\} \cup \{[D,U] \cap Q_{\{V_k\}} : [D,U] \in \mathcal{U}_{w\Delta}, \{V_k\} \text{ finite family of open sets}\}$$

defines a compatible uniformity on $(CL(X), \pi(\eta_2,\eta_1,\gamma_1,\gamma_2;\Delta))$.

Proof. It is easy to show that the above family $\mathbb{S}$ is a base for a uniformity on $CL(X)$. Nevertherless, it is indeed tricky to prove the compatibility, i.e. that $\tau(\mathbb{S})$ equals $\pi(\eta_2,\eta_1,\gamma_1,\gamma_2;\Delta)$ on $CL(X)$. So, let (A_λ) be a net converging to A w.r.t. $\tau(\mathbb{S})$. If $A \in V_{\eta_1}^-$, where $V \in \tau$, then $V_{\eta_1}^- \in \tau(\mathbb{S})$ and eventually $A_\lambda \in V_{\eta_1}^-$. But $V_{\eta_1}^- \subset V_{\eta_2}^-$ (because $\eta_2 \leq \eta_1$). Thus, eventually $A_\lambda \in V_{\eta_2}^-$. If $A \in V_{\gamma_1}^{++}$, where $V^c \in \Delta$, then $V^c \ll_\delta A^c$, where $D = V^c$ and δ is the compatible EF-proximity defined in $(*)$. By Lemma 5.2 there is $S \in \Delta$ such that $D \ll_\delta S \ll_\delta A^c$. Hence, there is $W \in \mathcal{U}_w$ such that $W(A) \cap S = \varnothing$. Eventually $A_\lambda \in [S, W](A) \subset V_{\gamma_1}^{++}$. So, eventually $A_\lambda \in V_{\gamma_2}^{++}$ (because $\gamma_1 \leq \gamma_2$). Thus, $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta) \subset \tau(\mathbb{S})$.

On the other hand, let (A_λ) be a net converging to A w.r.t. $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$, $D \in \Delta$ and $U \in \mathcal{U}_w$. Let $W \in \mathcal{U}_w$, W symmetric, be such that $W \circ W \subset U$. By Lemma 5.4 two cases arise:

i) $A \in (D^c)_\delta^{++}$. Then eventually $A_\lambda \in (D^c)_\delta^{++}$ and obviously, $A_\lambda \cap D = \varnothing \subset W(A)$ and $A \cap D = \varnothing \subset W(A_\lambda)$.

ii) $A \notin (D^c)_\delta^{++}$. Then $W(A) \cap D \neq \varnothing$.

Since W is totally bounded, there are $x_k \in A$, $k \in \{1, \ldots, n\}$, such that $A \subset \bigcup_{k=1}^n W(x_k) \subset W^2(A)$. Note that since η_2 is a compatible LR-proximity we have $\tau(V^-) \subset \sigma(\eta_2, \eta_1)^-$ (cf. Theorem 2.4). Now, $A \cap W(x_k) \neq \varnothing$ for $k \in \{1, \ldots, n\}$. Hence, for each k there is a V_k with $x_k \in V_k$ and $clV_k \underline{\eta}_2 [W(x_k)]^c$. But, eventually $A_\lambda \eta_2 V_k$ and since $clV_k \underline{\eta}_2 [W(x_k)]^c$ we have that eventually $A_\lambda \cap W(x_k) \neq \varnothing$. Therefore, eventually $x_k \in W(A_\lambda)$. Hence, eventually $A \cap D \subset \bigcup_{k=1}^n W(x_k) \subset W^2(A_\lambda) \subset U(A_\lambda)$. Furthermore, note that $[D \cap (W(A))^c] \in \Delta$ and $A \in [D^c \cup W(A)]_\delta^{++}$. So, eventually $A_\lambda \in [D^c \cup W(A)]_\delta^{++}$. Thus, eventually $A_\lambda \cap D = [A_\lambda \cap D \cap W(A)] \subset U(A)$, i.e. eventually $A_\lambda \in [D, U](A)$. Therefore, $\tau(\mathbb{S}) \subset \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$.

Combining the earlier part we get $\tau(\mathbb{S}) = \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$. □

6. Appendix (Admissibility).

It is a well known fact that if (X, τ) is a T_1 space, then the lower Vietoris topology $\tau(V^-)$ is an admissible topology, i.e. the map $i\colon (X, \tau) \to (CL(X), \tau(V^-))$, defined by $i(x) = \{x\}$, is an embedding.

On the other hand (as observed in Example 2.1), if the involved proximities η_1, η_2 are different from the discrete proximity $\eta^\star$, then the map $i\colon (X, \tau) \to (CL(X), \sigma(\eta_2, \eta_1)^-)$ is, in general, not even continuous.

So, we study the behaviour of $i\colon (X, \tau) \to (CL(X), \sigma(\eta_2, \eta_1)^-)$, when $\eta_2 \leq \eta_1$ and $\eta_1 \neq \eta^\star$. First, we state the following Lemma.

Lemma 6.1. *Let (X, τ) be a T_1 space, $U \in \tau$ with $clU \neq X$ and $V = (clU)^c$. If $z \in clU \cap clV$, then there exists a net (z_λ) τ-converging to z such that for all λ either*

i) $z_\lambda \in U$ *and* $z_\lambda \neq z$, *or*

ii) $z_\lambda \in V$ *and* $z_\lambda \neq z$.

Proof. Let $\mathcal{N}(z)$ be the filter of open neighbouhoods of z. For each $I \in \mathcal{N}(z)$, select $w_I \in I \cap V$ and $y_I \in I \cap U$. Then, the net (w_I) τ-converges to z and $(w_I) \subset V$ as well as the net (y_I) τ-converges to z and $(y_I) \subset U$.

We claim that for all $I \in \mathcal{N}(z)$ either $w_I \neq z$ or $y_I \neq z$.

Assume not. Then there exist I and $J \in \mathcal{N}(z)$ such that $y_I = z$ and $w_J = z$. As a result, $z \in U \cap V \subset clU \cap V = \varnothing$; a contradiction. □

Recall that a Hausdorff space X is *extremally disconnected* if for every open set $U \subset X$, clU is open in X (see [10] Page 368).

Proposition 6.2. *Let (X,τ) be a Hausdorff space and η_1, η_2 two compatible LO-proximities on X with $\eta_2 \leq \eta_1$ and $\eta_1 \neq \eta^\star$. Then the following are equivalent:*

(a) *X is extremally disconnected;*
(b) *the map $i\colon (X,\tau) \to (CL(X), \sigma(\eta_2,\eta_1)^-)$, defined by $i(x) = \{x\}$, is continuous.*

Proof. $(a) \Rightarrow (b)$. Let $x \in X$ and (x_λ) a net τ-converging to x. Let $V \subset X$ with V open and $\{x\}\eta_1 V$. Since $\{x\}\eta_1 V$ and $\eta_2 \leq \eta_1$, then $\{x\}\eta_2 V$ and so $x \in clV$. By assumption clV is an open subset of X and the net (x_λ) τ-converges to x. Thus, eventually $x_\lambda \in clV$.

$(b) \Rightarrow (a)$. By contradiction, suppose (a) fails. Then, there exists an open set $U \subset X$ such that clU is not open in X. Then, $clU \neq X$. Set $V = (clU)^c$. V is non-empty and open in X. We claim that $clU \cap clV \neq \varnothing$. Assume not, i.e. $clU \cap clV = \varnothing$. Then, $clU \subset (clV)^c \subset V^c = clU$. Thus, $clU = (clV)^c$, i.e. clU is open; a contradiction.

Let $z \in clU \cap clV$. By Lemma 6.1, there exists a net (z_λ) τ-converging to z such that for all λ either (1) $z_\lambda \in U$ and $z_\lambda \neq z$ or (2) $z_\lambda \in V$ and $z_\lambda \neq z$. In both cases, there exists an open subset W such that $z \in clW$ and $z_\lambda \notin clW$ for all λ. In fact if (1) holds, then set $W = V$, otherwise set $W = U$. Thus, the net (z_λ) τ-converges to z, but there exists an open subset W such that $\{z\}\eta_1 W$ (because $z \in clW$ and η_1 is a compatible LO-proximity on X) as well as $\{z_\lambda\}\underline{\eta}_2 W$ (again because $z_\lambda \notin clW$ and η_2 is a compatible LO-proximity on X) for all λ. Hence, the map $i\colon (X,\tau) \to (CL(X), \sigma(\eta_2,\eta_1)^-)$ fails to be continuous. □

Remark 6.3. If $\eta_1 = \eta^\star$, then the map $i\colon (X,\tau) \to (CL(X), \sigma(\eta_2,\eta_1)^-)$ is always continuous.

Definition 6.4 (cf. Definition 6.3 in [8])**.** A T_1 space (X,τ) is *nearly regular* iff whenever $x \in U$ with $U \in \tau$ there exists $V \in \tau$ with $x \in clV \subset U$.

Proposition 6.5. *Let (X,τ) be T_1 space and η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$. If η_2 is a compatible LO-proximity, then the following are equivalent:*

(a) *(X,τ) is nearly regular;*
(b) *the map $i\colon (X,\tau) \to (CL(X), \sigma(\eta_2,\eta_1)^-)$ is open.*

Proof. Left to the reader. □

Note that if (X,τ) is a T_1 space, γ_1, γ_2 are compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cobase, then the map $i\colon (X,\tau) \to (CL(X), \sigma(\gamma_1,\gamma_2;\Delta)^+)$ is always continuous. So, we have:

Proposition 6.6. *Let (X, τ) be a T_1 space, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cobase. The following are equivalent:*

(a) *the map $i\colon (X, \tau) \to (CL(X), \sigma(\gamma_1, \gamma_2; \Delta)^+)$, defined by $i(x) = \{x\}$, is an embedding;*
(b) *the map $i\colon (X, \tau) \to (CL(X), \sigma(\gamma_1, \gamma_2; \Delta)^+)$, defined by $i(x) = \{x\}$, is an open map;*
(c) *whenever $U \in \tau$ and $x \in U$, there exists a $B \in \Delta$ such that $x \in B^c \subset U$.*

Finally, we have the following results dealing with admissibility of the symmetric **standard** Bombay Δ topology $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$. Obviously, we investigate just the significant case $\eta_2 \neq \eta^\star$ (the standard Bombay Δ topology $\sigma(\gamma_1, \gamma_2; \Delta)$ is always admissible). We have to distinguish the two subcases (1) $\eta_1 \neq \eta^\star$, (2) $\eta_1 = \eta^\star$.

Proposition 6.7. *Let (X, τ) be a regular Hausdorff space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cobase. Suppose that $\eta_1 \neq \eta^\star$ and $\gamma_1 \leq \eta_1$. Then the map $i\colon (X, \tau) \to (CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is an embedding if and only if the following three conditions are fulfilled:*

(a) *$i\colon (X, \tau) \to (CL(X), \sigma(\eta_2, \eta_1)^-)$ is continuous;*
(b) *$i\colon (X, \tau) \to (CL(X), \sigma(\eta_2, \eta_1)^-)$ is open;*
(c) *$i\colon (X, \tau) \to (CL(X), \sigma(\gamma_1, \gamma_2; \Delta)^+)$ is open.*

Proof. Only necessity requires a proof. Namely just (b) and (c). Now, we show that $i\colon (X, \tau) \to (CL(X), \sigma(\eta_2, \eta_1)^-)$ is open.

Assume not. Then there exist $x \in X$ and $U \in \mathcal{N}(x)$, where $\mathcal{N}(x)$ is the filter of open nhoods at x such that $i(U) \notin \sigma(\eta_2, \eta_1)^- \cap i(X)$.

So, there is $y \in U$ such that for each $W^-_{\eta_2}$ with $W \subset X$ open and $y \eta_1 W$, we have $W^-_{\eta_2} \cap i(X) \not\subset i(U)$. But $i\colon (X, \tau) \to (CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is an embedding. Thus, for each $U \in \mathcal{N}(x)$, $i(U) \in \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta) \cap i(X)$. Hence there exists $\mathbf{O} \in \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$ such that $\{y\} \in \mathbf{O} \cap i(X) \subset i(U)$. Note that $\mathbf{O}$ has the form $O^{++}_{\gamma_2} \cap \bigcap_{j=1}^n (O_j)^-_{\eta_2}$ with $y \eta_1 O_j$, $O_j \in \tau$, $O^c \in \Delta$ and $y \ll_{\gamma_1} O$.

Furthermore, since $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$ is standard we may assume that $\cup_{j=1}^n O_j \subset O$. Now, since $i\colon (X, \tau) \to (CL(X), \sigma(\eta_2, \eta_1)^-)$ is continuous, there exists a $V_j \in \mathcal{N}(y)$ such that $i(V_j) \subset (O_j)^-_{\eta_2}$ for each $j \in \{1, \ldots, n\}$. Because X is regular, there exists L_j, $j \in \{1, \ldots, n\}$, $L_j \in \mathcal{N}(y)$ such that $L_j \subset clL_j \subset V_j$. Since X is regular select $W \in \mathcal{N}(y)$ with $W = \cap_{j=1}^n L_j$ and $clW \subset O$. It follows $i(W) \subset i(clW) = W^-_{\eta_2} \cap i(X) \subset i(U)$. Thus $i(U) \in \sigma(\eta_2, \eta_1)^- \cap i(X)$, a contradiction.

Now we prove that $i\colon (X, \tau) \to (CL(X), \sigma(\gamma_1, \gamma_2; \Delta)^+)$ is open. Assume not. So, there exists $x' \in X$ and $V \in \mathcal{N}(x')$, such that $i(V) \notin \sigma(\gamma_1, \gamma_2; \Delta)^+ \cap i(X)$.

So there is $y' \in V$ such that for each $W^{++}_{\gamma_2}$ with $W^c \in \Delta$ and $y' \ll_{\gamma_1} W$, we have $W^{++}_{\gamma_2} \cap i(X) \not\subset i(V)$. But $i\colon (X, \tau) \to (CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is an embedding. Thus, for each $V \in \mathcal{N}(x')$, $i(V) \in \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta) \cap i(X)$.

Hence there exists $\mathbf{T} \in \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$ such that $\{y'\} \in \mathbf{T} \cap i(X) \subset i(V)$. Note that $\mathbf{T}$ has the form $T_{\gamma_2}^{++} \cap \bigcap_{j=1}^{n} (T_j)_{\eta_2}^{-}$ with $y' \eta_1 T_j$, $T_j \in \tau$, $T^c \in \Delta$ and $y' \ll_{\gamma_1} T$. Moreover, since $\pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$ is standard, we may assume that $T_j \subset T$ for each $j \in \{1, \ldots, n\}$. Again, since $i\colon (X, \tau) \to (CL(X), \sigma(\eta_2, \eta_1)^-)$ is continuous, there exists a $S_j \in \mathcal{N}(y')$ such that $i(S_j) \subset (T_j)_{\eta_2}^{-}$ for each $j \in \{1, \ldots, n\}$. Select $M_j \in \mathcal{N}(y')$, $j \in \{1, \ldots, n\}$ such that $M_j \subset clM_j \subset S_j$.

Set $W = T \cap \bigcap_{j=1}^{n} M_j$. It follows that $W \in \mathcal{N}(y')$. Again, since $i\colon (X, \tau) \to (CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is an embedding, there exists $\mathbf{O} \in \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta)$ such that $\{y'\} \in \mathbf{O} \cap i(X) \subset i(W)$. Note that $\mathbf{O}$ has the form $O_{\gamma_2}^{++} \cap \bigcap_{k=1}^{m} (O_k)_{\eta_2}^{-}$ with $y' \eta_1 O_k$, $O_k \in \tau$, $O^c \in \Delta$ and $y' \ll_{\gamma_1} O$. We may assume $\cup_{k=1}^{n} O_k \subset O$.

Moreover, from $\mathbf{O} \cap i(X) \subset i(W)$ we have $O \subset W$ (otherwise, select $z \in O \setminus W$ and $z_k \in O_k$; the set $F = \{z\} \cup \bigcup_{k=1}^{n} z_k \in \mathbf{O} \setminus \mathbf{T}$, a contradiction). As a result, $i(O) = i(X) \cap O_{\gamma_2}^{++} \subset \mathbf{T} \subset i(V)$. Thus $i(V) \in \sigma(\gamma_1, \gamma_2; \Delta)^{+} \cap i(X)$; a contradiction. □

Theorem 6.8. *Let (X, τ) be a regular Hausdorff space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cobase. If $\eta_1 \neq \eta^\star$ and $\gamma_1 \leq \eta_1$, then the map $i\colon (X, \tau) \to (CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is an embedding if and only if the following conditions are fulfilled:*

(a) *X is extremally disconnected;*
(b) *whenever $U \in \tau$ and $x \in U$, there exists a $B \in \Delta$ such that $x \in B^c \subset U$.*

Theorem 6.9. *Let (X, τ) be a regular Hausdorff space, η_1, η_2 LO-proximities on X with $\eta_2 \leq \eta_1$, γ_1, γ_2 compatible LO-proximities on X with $\gamma_1 \leq \gamma_2$ and $\Delta \subset CL(X)$ a cobase. If $\eta_1 = \eta^\star$ and η_2 is a compatible LO-proximity, then the following are equivalent:*

(a) *the map $i\colon (X, \tau) \to (CL(X), \pi(\eta_2, \eta_1, \gamma_1, \gamma_2; \Delta))$ is an embedding;*
(b) *whenever $U \in \tau$ and $x \in U$, there exists a $B \in \Delta$ such that $x \in B^c \subset U$.*

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