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# Contractibility of the digital $n$-space

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Abstract

*The aim of this paper is to prove a known fact that the digital line is contractible. Hence we have that the digital space $(\mathbf{Z}^n, \kappa^n)$ is also contractible where $(\mathbf{Z}^n, \kappa^n)$ is $n$ products of the digital line $(\mathbf{Z}, \kappa)$. This is a fundamental property of homotopy theory.*

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Keywords: *Khalimsky topology; digital n-space; contractible; homotopy.*

## 1. Prelimarilies

We consider an important property of homotopy theory for the digital $n$-space.

*The digital line* $(\mathbf{Z}, \kappa)$ is the set of the integers $\mathbf{Z}$ equipped with the topology $\kappa$ having $\{\{2m-1,\ 2m,\ 2m+1\} : m \in \mathbf{Z}\}$ as a subbase.

For $x \in \mathbf{Z}$, we set

$$U(x) := \begin{cases} \{2m-1,\ 2m,\ 2m+1\} & \text{if } x = 2m, \\ \{2m+1\} & \text{if } x = 2m+1. \end{cases}$$

Then $\{U(x)\}$ is a fundamental neighborhood system at $x$. Then it it obvious that $\{2m : m \in \mathbf{Z}\}$ is closed and nowhere dense in $\mathbf{Z}$, $\{2m+1 : m \in \mathbf{Z}\}$ is open and dense in $\mathbf{Z}$. $U(x)$ is the minimal open set containing $x$ for any $x \in \mathbf{Z}$. (See [1],[2],[4],[5]).

The digital line $(\mathbf{Z}, \kappa)$ was introduced by E. Khalimsky in the late 1960's and it was made use of studying topological properties of digital images. (See [3], [6]).

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*The digital n-space* $(\mathbf{Z}^n, \kappa^n)$ is the topological product of $n$ copies of the digital line $(\mathbf{Z}, \kappa)$.

To investigate the digital $n$-space is very interesting for the application possibility. Here we focus the contractibility of one.

## 2. Contractibility of the digital line and digital $n$-space

A space $X$ is called *contractible* provided that there exists a homotopy $H : X \times I \to X$ such that $H_{X\times\{0\}}$ is the identity and $H_{X\times\{1\}}$ is a constant function.

The digital line is contractible as pointed out in Remark 4.11 of [7]. We shall show by direct computation.

**Theorem 2.1.** The digital line is contractible.

*Proof.* Defining $H : \mathbf{Z} \times I \to \mathbf{Z}$ by

$$H_{\{0\}\times I} \equiv 0$$

and for any $n \in \mathbf{Z}\backslash\{0\}$, if $n$ is an odd number,

$$H(n,t) := \begin{cases} n & \text{if } 0 \le t < 2^{-|n|}, \\ n-1(\text{if } n>0),\ n+1(\text{if } n<0) & \text{if } 2^{-|n|} \le t \le 2^{-(|n|-1)}, \\ n-2(\text{if } n>0),\ n+2(\text{if } n<0) & \text{if } 2^{-(|n|-1)} < t < 2^{-(|n|-2)}, \\ & \cdots \\ 1(\text{if } n>0),\ -1(\text{if } n<0) & \text{if } 2^{-2} < t < 2^{-1}, \\ 0 & \text{if } 2^{-1} \le t, \end{cases}$$

if $n$ is an even number,

$$H(n,t) := \begin{cases} n & \text{if } 0 \le t \le 2^{-|n|}, \\ n-1(\text{if } n>0),\ n+1(\text{if } n<0) & \text{if } 2^{-|n|} < t < 2^{-(|n|-1)}, \\ n-2(\text{if } n>0),\ n+2(\text{if } n<0) & \text{if } 2^{-(|n|-1)} \le t \le 2^{-(|n|-2)}, \\ & \cdots \\ 1(\text{if } n>0),\ -1(\text{if } n<0) & \text{if } 2^{-2} < t < 2^{-1}, \\ 0 & \text{if } 2^{-1} \le t, \end{cases}$$

then we see that $H_{\mathbf{Z}\times\{0\}} = id_{\mathbf{Z}}$ and $H_{\mathbf{Z}\times\{1\}} \equiv 0$. Since $H$ is continuous, $id_{\mathbf{Z}}$ and the constant map ($\equiv 0$) is homotopic. Therefore we have $(\mathbf{Z}, \kappa)$ is contractible.

□

Since a contractible finite product of contractible spaces is contractible, we have the following.

**Corollary 2.2.** *The digital n-space is contractible.*

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