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Unified common fixed point theorems under weak reciprocal continuity or without continuity

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Abstract

The purpose of this paper is two fold. Firstly, using the notion of weak reciprocal continuity due to Pant et al. [Weak reciprocal continuity and fixed point theorems, Ann. Univ. Ferrara Sez. VII Sci. Mat. 57(1), 181–190 (2011)], we prove unified common fixed point theorems for various variants of compatible and R-weakly commuting mappings in complete metric spaces employing an implicit relation which covers a multitude of contraction conditions yielding thereby known as well as unknown results as corollaries. Secondly, we point out that more natural results can be proved under relatively tighter conditions if we replace the completeness of the space by completeness of suitable subspaces. The realized improvements in our results are also substantiated using appropriate examples.

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1. Introduction and preliminaries

Indeed, Banach Contraction Principle is a fundamental and fruitful result of metric fixed point theory. After this classical result, several authors have

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contributed to the vigorous development of metric fixed point theory in different ways (e.g. [4, 7, 10, 11, 12, 17, 30]). As patterned in Jungck [13], a result on the existence of common fixed points generally involves conditions on commutativity, continuity and contraction along with a suitable condition on the containment of range of one mapping into the range of other. Hence, one is always required to improve one or more of these conditions to prove a new fixed point theorem.

Jungck [13] obtained a common fixed point theorem for a pair of commuting mappings. In 1982, Sessa [29] formulated the notion of weak commutativity and established a common fixed point theorem for such pairs. Jungck [14] generalized the notion of weakly commuting mappings by introducing the concept of compatible mappings. Thereafter, Pant [20] defined R-weak commutativity and proved common fixed point results for R-weakly commuting mappings in metric spaces.

The study of common fixed points of non-compatible mappings is equally natural which was indeed noted in Pant [22]. In 1997, Pathak et al. [24] introduced the notions of R-weakly commuting mappings of types (A_g) and (A_f) and utilize the same to prove common fixed point theorems in metric spaces. In 1998, Pant [21] introduced the notion of reciprocal continuity and utilize the same to prove results on common fixed points which also remains a point of discontinuity of the involved mappings. Recently, Pant et al. [23] improved the notion of reciprocal continuity by introducing weak reciprocal continuity and observed that weak reciprocal continuity is applicable to compatible as well as to non-compatible mappings. In fact, they proved the following result.

Theorem 1.1. *Let f and g be weakly reciprocally continuous self-mappings of a complete metric space (X, d) such that*

(i) $f(X) \subseteq g(X)$;
(ii) $d(fx, fy) \leq ad(gx, gy) + bd(fx, gx) + cd(fy, gy)$, *for some* $a, b, c \geq 0$ *with* $a + b + c < 1$ *and for all* $x, y \in X$.

If f and g are either compatible or R-weakly commuting of type (A_g) or R-weakly commuting of type (A_f) then f and g have a unique common fixed point.

The notions and definitions utilized thus far and also to be utilized in the sequel are presented in the following multitude of definitions:

Definition 1.2. Let $f, g : X \to X$ be two self-mappings of a metric space (X, d). Then the pair (f, g) is said to be

(1) commuting if $fgx = gfx$, for all $x \in X$,
(2) weakly commuting [29] if $d(fgx, gfx) \leq d(fx, gx)$, for all $x \in X$,
(3) R-weakly commuting [20] if there exists some real number $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$ for all $x \in X$,
(4) pointwise R-weakly commuting [20] if given $x \in X$ there exists some real number $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$,

(5) R-weakly commuting of type (A_g) [24] if there exists some real number $R > 0$ such that $d(ffx, gfx) \leq Rd(fx, gx)$ for all $x \in X$,
(6) R-weakly commuting of type (A_f) [24] if there exists some real number $R > 0$ such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all $x \in X$,
(7) compatible [14] if $\lim\limits_{n\to\infty} d(fgx_n, gfx_n) = 0$ for each sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n$,
(8) g-compatible [25] if $\lim\limits_{n\to\infty} d(ffx_n, gfx_n) = 0$ for each sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n$,
(9) f-compatible [25] if $\lim\limits_{n\to\infty} d(fgx_n, ggx_n) = 0$ for each sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n$,
(10) non-compatible [22] if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n$ but $\lim\limits_{n\to\infty} d(fgx_n, gfx_n)$ is either nonzero or nonexistent,
(11) reciprocally continuous [21] if $\lim\limits_{n\to\infty} fgx_n = ft$ and $\lim\limits_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$ for some $t \in X$,
(12) weakly reciprocally continuous [23] if $\lim\limits_{n\to\infty} fgx_n = ft$ or $\lim\limits_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$ for some $t \in X$.
(13) coincidentally commuting [5] (or weakly compatible [15]) if they merely commute at their coincidence points.

In the foregoing definitions, it is assumed that there is at least one sequence in the underlying space meeting the prescribed requirements.

Definition 1.3 ([8]). Two families of self mappings $\{fi\}_{i=1}^m$ and $\{g_k\}_{k=1}^n$ are said to be pairwise commuting if

(1) $f_i f_j = f_j f_i$ for all $i, j \in \{1, 2, \ldots, m\}$,
(2) $g_k g_l = g_l g_k$ for all $k, l \in \{1, 2, \ldots, n\}$,
(3) $f_i g_k = g_k f_i$ for all $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

For more details on systematic comparisons and illustrations of earlier described notions, we refer to Murthy [19], Singh and Tomar [31], Pant et al. [23] and Kadelburg et al. [16].

In this paper, utilizing the notion of implicit relation due to Popa et al. [27], we prove common fixed point results for variants of compatible (g-compatible and f-compatible) and R-weakly commuting (R-weakly commuting of type (A_g) and R-weakly commuting of type (A_f)) mappings, employing an implicit relation which is a slightly refined form of an implicit function due to Popa [26]. Further, we prove a more general result for coincidentally commuting mappings without any requirement of weak reciprocal continuity. Some related results are also derived besides furnishing illustrative examples which exhibit the superiority of our results over some of the known ones.

2. Implicit relations

Implicit functions are often used very effectively to cover various contraction conditions in one go rather than proving a separate theorem for each contraction condition. The first ever attempt to coin an implicit relation can be traced back to Popa [26]. In 2008, Ali and Imdad [1] introduced a new class of implicit functions which covers several classes of contractions conditions. Since then, many mathematicians utilized implicit relations with various properties to prove a number of fixed point theorems (e.g. [3, 27, 28]). In order to describe the implicit function we intend to use here, let Φ be the family of lower semi-continuous functions $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(ϕ_1): ϕ is non-increasing in variables t_5 and t_6,
(ϕ_2): there exists $h \in (0,1)$ such that for $u, v \geq 0$,
(ϕ_{2a}): $\phi(u,v,v,u,u+v,0) \leq 0$ or (ϕ_{2b}): $\phi(u,v,u,v,0,u+v) \leq 0$ implies $u \leq hv$,
(ϕ_3): $\phi(u,u,0,0,u,u) > 0$, for all $u > 0$.

The following examples of functions $\phi \in \Phi$ appeared in Popa [26] and Imdad and Ali [7].

Example 2.1. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5+t_6)\right\},$$

where $k \in (0,1)$.

Example 2.2. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - t_1(\alpha t_2 + \beta t_3 + \gamma t_4) - \eta t_5 t_6,$$

where $\alpha > 0$, $\beta, \gamma, \eta \geq 0$, $\alpha + \beta + \gamma < 1$ and $\alpha + \eta < 1$.

Example 2.3. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - \alpha t_1^2 t_2 - \beta t_1 t_3 t_4 - \gamma t_5^2 t_6 - \eta t_5 t_6^2,$$

where $\alpha > 0$, $\beta, \gamma, \eta \geq 0$, $\alpha + \beta < 1$ and $\alpha + \gamma + \eta < 1$.

Example 2.4. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - k\frac{t_3^2 t_4^2 + t_5^2 t_6^2}{1 + t_2 + t_3 + t_4},$$

where $k \in (0,1)$.

Example 2.5. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - \alpha t_2^2 - \beta\frac{t_5 t_6}{1 + t_3^2 + t_4^2},$$

where $\alpha > 0$, $\beta \geq 0$ and $\alpha + \beta < 1$.

Example 2.6. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\begin{aligned}\phi(t_1, t_2, t_3, t_4, t_5, t_6) &= t_1^2 - \alpha \max\{t_2^2, t_3^2, t_4^2\} - \beta \max\{t_3 t_5, t_4 t_6\} \\ &\quad -\gamma t_5 t_6\end{aligned}$$

where $\alpha > 0$, $\beta, \gamma \geq 0$, $\alpha + 2\beta < 1$ and $\alpha + \gamma < 1$.

Example 2.7. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{t_5}{2}, \frac{t_6}{2}\right\},$$

where $k \in (0, 1)$.

Example 2.8. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\left\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\right\},$$

where $k \in (0, 1)$.

Example 2.9. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4 + \eta t_5 + \lambda t_6),$$

where $\alpha + \beta + \gamma + \eta + \lambda < 1$.

Example 2.10. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{k}{2} \max\{t_2, t_3, t_4, t_5, t_6\},$$

where $k \in (0, 1)$.

Example 2.11. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - [\alpha t_2 + \beta t_3 + \gamma t_4 + \eta(t_5 + t_6)],$$

where $\alpha + \beta + \gamma + 2\eta < 1$.

Since verifications of requirements (ϕ_1), (ϕ_2) and (ϕ_3) for Examples 2.1–2.11 are straightforward, the details are not included. Here, one may notice that some other well known contraction conditions can also be deduced as particular cases of implicit relation of Popa [26]. In order to strengthen this viewpoint, we include some further examples and utilize them to demonstrate how this implicit relation can cover several other known contraction conditions and is also good enough to yield further natural contraction conditions as well.

Example 2.12. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = \begin{cases} t_1 - a_1 \dfrac{t_3^2 + t_4^2}{t_3 + t_4} - a_2 t_2 - a_3(t_5 + t_6), & \text{if } t_3 + t_4 \neq 0; \\ t_1, & \text{if } t_3 + t_4 = 0, \end{cases}$$

where $a_i \geq 0$ with at least one a_i non-zero and $a_1 + a_2 + 2a_3 < 1$.

Example 2.13. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = \begin{cases} t_1 - a_1 t_2 - \dfrac{a_2 t_3 t_4 + a_3 t_5 t_6}{t_3 + t_4}, & \text{if } t_3 + t_4 \neq 0; \\ t_1, & \text{if } t_3 + t_4 = 0, \end{cases}$$

where $a_1, a_2, a_3 \geq 0$ such that $1 < 2a_1 + a_2 < 2$.

Example 2.14. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\begin{aligned} \phi(t_1,t_2,t_3,t_4,t_5,t_6) &= t_1 - a_1 \big[a_2 \max\big\{ t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6) \big\} \\ &\qquad + (1 - a_2) \big[\max\big\{ t_2^2, t_3 t_4, t_5 t_6, \frac{t_3 t_6}{2}, \frac{t_4 t_5}{2} \big\} \big]^{\frac{1}{2}} \big], \end{aligned}$$

where $a_1 \in (0,1)$ and $0 \leq a_2 \leq 1$.

Example 2.15. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\begin{aligned} \phi(t_1,t_2,t_3,t_4,t_5,t_6) &= t_1^2 - a_1 \max\big\{ t_2^2, t_3^2, t_4^2 \big\} - a_2 \max\left\{ \frac{t_3 t_5}{2}, \frac{t_4 t_6}{2} \right\} \\ &\quad - a_3 t_5 t_6, \end{aligned}$$

where $a_1, a_2, a_3 \geq 0$ and $a_1 + a_2 + a_3 < 1$.

Very recently, Popa et al. [27] proved several fixed point theorems satisfying suitable implicit relations from which Husain and Sehgal [6] type contraction conditions can be deduced. A slight modification in condition (ϕ_1) is used as follows:

(ϕ_1'): ϕ is decreasing in variables $t_2, \dots, t_6$.

Hereafter, let $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ be a continuous function which satisfies the conditions ϕ_1', ϕ_2 and ϕ_3 and let Φ' be the family of such functions ϕ. In this paper, we employ such implicit relations to prove our results. But before we proceed further, let us furnish some examples to highlight the utility of the modifications instrumental herein.

Example 2.16. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \psi\left(\max\{ t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6) \} \right),$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing upper semi-continuous function with $\psi(0) = 0$ and $\psi(t) < t$ for each $t > 0$.

Example 2.17. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \psi\left(t_2, t_3, t_4, t_5, t_6 \right),$$

where $\psi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is an upper semi-continuous and non-decreasing function in each coordinate variable such that $\psi(t, t, at, bt, ct) < t$ for each $t > 0$ and $a, b, c \geq 0$ with $a + b + c \leq 3$.

Example 2.18. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \psi\left(t_2^2, t_3 t_4, t_5 t_6, t_3 t_6, t_4 t_5\right),$$

where $\psi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is an upper semi-continuous and non-decreasing function in each coordinate variable such that $\psi(t, t, at, bt, ct) < t$ for each $t > 0$ and $a, b, c \geq 0$ with $a + b + c \leq 3$.

Clearly, apart from these examples, there are many other functions which meet the requirements (ϕ_1'), (ϕ_2) and (ϕ_3).

3. Results under weak reciprocal continuity

In the following proposition, we notice that under the prescribed setting, the set of common fixed point of the involved mappings is always singleton provided such points exist.

Proposition 3.1. *Let (X, d) be a metric space and let $f, g : X \to X$ be two mappings satisfying*

$$\phi(d(fx, fy), d(gx, gy), d(fx, gx), d(fy, gy), d(gx, fy), d(gy, fx)) \leq 0, \tag{3.1}$$

for all $x, y \in X$ where ϕ enjoys the property (ϕ_3). Then f and g have at most one common fixed point in X.

Proof. Let, on the contrary, the mappings f and g have two common fixed points w and w' such that $w \neq w'$. On using (3.1), we have

$$\phi(d(fw, fw'), d(gw, gw'), d(fw, gw), d(fw', gw'), d(gw, fw'), d(gw', fw)) \leq 0,$$

or

$$\phi(d(w, w'), d(w, w'), d(w, w), d(w', w'), d(w, w'), d(w', w)) \leq 0,$$

so that

$$\phi(d(w, w'), d(w, w'), 0, 0, d(w, w'), d(w', w)) \leq 0,$$

which contradicts (ϕ_3), yielding thereby $w = w'$. □

Let f and g be two mappings from a metric space (X, d) into itself satisfying the following condition:

$$f(X) \subseteq g(X). \tag{3.2}$$

For an arbitrary point $x_0 \in X$ there exists a point $x_1 \in X$ such that $fx_0 = gx_1$. Continuing in this way, one can inductively define a sequence $\{y_n\}$ in X such that

$$y_n = fx_n = gx_{n+1}, \quad \text{for all } n = 0, 1, 2, \ldots. \tag{3.3}$$

Lemma 3.2. *Let f and g be two mappings from a metric space (X, d) into itself which satisfy conditions (3.1) (for some $\phi \in \Phi'$) and (3.2), and let $\{y_n\}$ be defined by (3.3). Then $\{y_n\}$ is a Cauchy sequence.*

Proof. On using (3.1) with $x = x_n$ and $y = x_{n+1}$, we have

$$\begin{aligned}\phi\left(\begin{array}{c} d(fx_n, fx_{n+1}), d(gx_n, gx_{n+1}), d(fx_n, gx_n), \\ d(fx_{n+1}, gx_{n+1}), d(gx_n, fx_{n+1}), d(gx_{n+1}, fx_n) \end{array}\right) &\le 0, \\ \text{or } \phi\left(\begin{array}{c} d(y_n, y_{n+1}), d(y_{n-1}, y_n), d(y_n, y_{n-1}), \\ d(y_{n+1}, y_n), d(y_{n-1}, y_{n+1}), d(y_n, y_n) \end{array}\right) &\le 0, \\ \text{or } \phi\left(\begin{array}{c} d(y_n, y_{n+1}), d(y_{n-1}, y_n), d(y_{n-1}, y_n), \\ d(y_n, y_{n+1}), d(y_{n-1}, y_n) + d(y_n, y_{n+1}), 0 \end{array}\right) &\le 0,\end{aligned}$$

implying thereby $d(y_n, y_{n+1}) \le hd(y_{n-1}, y_n)$ (due to (ϕ_{2a})) where $h \in (0, 1)$. In general, for all $n = 0, 1, 2, \ldots$, we have $d(y_n, y_{n+1}) \le hd(y_{n-1}, y_n)$. Again using (3.1) with $x = x_{n+2}$ and $y = x_{n+1}$, we get

$$\begin{aligned}\phi\left(\begin{array}{c} d(fx_{n+2}, fx_{n+1}), d(gx_{n+2}, gx_{n+1}), d(fx_{n+2}, gx_{n+2}), \\ d(fx_{n+1}, gx_{n+1}), d(gx_{n+2}, fx_{n+1}), d(gx_{n+1}, fx_{n+2}) \end{array}\right) &\le 0, \\ \text{or } \phi\left(\begin{array}{c} d(y_{n+2}, y_{n+1}), d(y_{n+1}, y_n), d(y_{n+2}, y_{n+1}), \\ d(y_{n+1}, y_n), d(y_{n+1}, y_{n+1}), d(y_n, y_{n+2}) \end{array}\right) &\le 0, \\ \text{or } \phi\left(\begin{array}{c} d(y_{n+1}, y_{n+2}), d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), \\ d(y_n, y_{n+1}), 0, d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) \end{array}\right) &\le 0.\end{aligned}$$

In view of (ϕ_{2b}), we have $d(y_{n+1}, y_{n+2}) \le hd(y_n, y_{n+1})$ where $h \in (0, 1)$. For all $n = 0, 1, 2, \ldots$ we get $d(y_{n+1}, y_{n+2}) \le hd(y_n, y_{n+1})$. Moreover, for every integer $p > 0$, we obtain

$$\begin{aligned}d(y_n, y_{n+p}) &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p}) \\ &\le (1 + h + \cdots + h^{p-1}) d(y_n, y_{n+1}) \\ &\le \left(\frac{1}{1-h}\right) h^n d(y_0, y_1),\end{aligned}$$

i.e., $d(y_n, y_{n+p}) \to 0$ as $n \to \infty$. Therefore $\{y_n\}$ is a Cauchy sequence in X. □

Utilizing the recently introduced notion of weak reciprocal continuity due to Pant et al. [23], we prove a common fixed point theorem in a complete metric space for variants of compatible (g-compatible and f-compatible) and R-weakly commuting (R-weakly commuting of type (A_g) and R-weakly commuting of type (A_f)) mappings via an implicit function studied by Popa [27] which is a slightly refined version of some known implicit relations. Some related results are also derived besides furnishing illustrative examples which establish the superiority of our results over some of the known results especially the ones contained in Pant et al. [23].

Theorem 3.3. *Let f and g be weakly reciprocally continuous self-mappings of a complete metric space (X, d) satisfying conditions (3.1) (for some $\phi \in \Phi'$) and (3.2). If the mappings f and g are either compatible or g-compatible or f-compatible or R-weakly commuting or R-weakly commuting of type (A_g) or R-weakly commuting of type (A_f), then f and g have a unique common fixed point in X.*

Proof. Let $\{y_n\}$ be a sequence defined by (3.3). By Lemma 3.2, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point z in X such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z.$$

Case I: Suppose that the mappings f and g are compatible. The weak reciprocal continuity of f and g implies that $fgx_n \to fz$ or $gfx_n \to gz$ as $n \to \infty$.

(i) Let $gfx_n \to gz$ as $n \to \infty$. The compatibility of f and g yields $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, hence $fgx_n \to gz$ as $n \to \infty$. By (3.2), this yields $fgx_{n+1} = ffx_n \to gz$ as $n \to \infty$. On using (3.1) with $x = z$ and $y = fx_n$, we get

$$\phi(d(fz, ffx_n), d(gz, gfx_n), d(fz, gz), d(ffx_n, gfx_n), d(gz, ffx_n), d(gfx_n, fz)) \leq 0.$$

Taking the limit as $n \to \infty$, we have

$$\phi(d(fz, gz), d(gz, gz), d(fz, gz), d(gz, gz), d(gz, gz), d(gz, fz)) \leq 0,$$

or, equivalently,

$$\phi(d(fz, gz), 0, d(fz, gz), 0, 0, d(gz, fz)) \leq 0,$$

implying thereby $d(fz, gz) = 0$ (due to (ϕ_{2b})). Hence $fz = gz$ which shows that z is a coincidence point of the mappings f and g. Also compatibility of f and g implies commutativity at a coincidence point. Hence $ffz = fgz = gfz = ggz$. Now we assert that fz is a common fixed point of f and g. To accomplish this, using (3.1) with $x = z$ and $y = fz$, we have

$$\phi(d(fz, ffz), d(gz, gfz), d(fz, gz), d(ffz, gfz), d(gz, ffz), d(gfz, fz)) \leq 0.$$

and so

$$\phi(d(fz, ffz), d(fz, ffz), 0, 0, d(fz, ffz), d(ffz, fz)) \leq 0,$$

yielding thereby $d(fz, ffz) = 0$ (due to (ϕ_3)). Therefore, $fz = ffz = gfz$ which shows that fz is a common fixed point of the mappings f and g.

(ii) Assume next that $fgx_n \to fz$ as $n \to \infty$. By virtue of (3.2), we have $fz = gt$ for some $t \in X$ hence $fgx_n \to gt$ as $n \to \infty$. The compatibility of the mappings f and g implies $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, hence $gfx_n \to gt$ as $n \to \infty$. Again by (3.2), $fgx_{n+1} = ffx_n \to gt$ as $n \to \infty$. On using (3.1) with $x = t$ and $y = fx_n$, we get

$$\phi(d(ft, ffx_n), d(gt, gfx_n), d(ft, gt), d(ffx_n, gfx_n), d(gt, ffx_n), d(gfx_n, ft)) \leq 0.$$

Taking the limit as $n \to \infty$, we have

$$\phi(d(ft, gt), 0, d(ft, gt), 0, 0, d(gt, ft)) \leq 0,$$

which implies $d(ft, gt) = 0$ (due to (ϕ_{2b})). Hence $ft = gt$ which shows that t is a coincidence point of the mappings f and g. Also, compatibility of f and g implies commutativity at a coincidence point. Hence $fft = fgt = gft = ggt$.

We show that ft is a common fixed point of f and g. To assert this, using (3.1) with $x = t$ and $y = ft$, we have

$$\phi(d(ft, fft), d(gt, gft), d(ft, gt), d(fft, gft), d(gt, fft), d(gft, ft)) \leq 0,$$

or, equivalently,

$$\phi(d(ft, fft), d(ft, fft), 0, 0, d(ft, fft), d(fft, ft)) \leq 0,$$

implying thereby $d(ft, fft) = 0$ (due to (ϕ_3)). Therefore, $ft = fft = gft$ which shows that ft is a common fixed point of the mappings f and g.

Case II: Let the mappings f and g be g-compatible. The weak reciprocal continuity of f and g implies that $fgx_n \to fz$ or $gfx_n \to gz$ as $n \to \infty$.

(i) Suppose that $gfx_n \to gz$ as $n \to \infty$. Then g-compatibility of f and g yields $\lim\limits_{n\to\infty} d(ffx_n, gfx_n) = 0$, hence $ffx_n \to gz$ as $n \to \infty$. On using (3.1) with $x = z$ and $y = fx_n$,

$$\phi(d(fz, ffx_n), d(gz, gfx_n), d(fz, gz), d(ffx_n, gfx_n), d(gz, ffx_n), d(gfx_n, fz)) \leq 0.$$

Taking the limit as $n \to \infty$, we get $fz = gz$ (due to (ϕ_{2b})) which shows that z is a coincidence point of the mappings f and g. Since g-compatibility of f and g implies commutativity at a coincidence point, we have $ffz = fgz = gfz = ggz$. Now, we assert that fz is a common fixed point of f and g. On using (3.1) with $x = z$ and $y = fz$, we have

$$\phi(d(fz, ffz), d(gz, gfz), d(fz, gz), d(ffz, gfz), d(gz, ffz), d(gfz, fz)) \leq 0,$$

implying thereby $fz = ffz = gfz$ (due to (ϕ_3)). Hence fz is a common fixed point of the mappings f and g.

(ii) Assume that $fgx_n \to fz$ as $n \to \infty$. By (3.2), we have $fz = gt$ for some $t \in X$. Thus $fgx_n \to gt$ as $n \to \infty$. Also (3.2) implies $fgx_{n+1} = ffx_n \to gt$ as $n \to \infty$. The g-compatibility of the mappings f and g implies $\lim\limits_{n\to\infty} d(ffx_n, gfx_n) = 0$, hence $gfx_n \to gt$ as $n \to \infty$. On using (3.1) with $x = t$, $y = fx_n$ and taking the limit as $n \to \infty$, we get $ft = gt$ (due to (ϕ_{2b})). Hence t is a coincidence point of the mappings f and g. Also, g-compatibility of f and g implies commutativity at a coincidence point. Hence $fft = fgt = gft = ggt$. We assert that ft is a common fixed point of f and g. On using (3.1) with $x = t$, $y = ft$ we obtain $ft = fft = gft$ (due to (ϕ_3)). Therefore, ft is a common fixed point of the mappings f and g.

Case III: Suppose that the pair (f, g) is f-compatible. The weak reciprocal continuity of f and g implies that $fgx_n \to fz$ or $gfx_n \to gz$ as $n \to \infty$.

(i) Let us consider $gfx_n \to gz$ as $n \to \infty$. Then f-compatibility of f and g implies $\lim\limits_{n\to\infty} d(fgx_n, ggx_n) = 0$. By (3.2), we get $gfx_n = ggx_{n+1} \to gz$ as $n \to \infty$ and so $fgx_{n+1} = ffx_n \to gz$ as $n \to \infty$. Putting $x = z$ and $y = fx_n$ in (3.1) and taking the limit as $n \to \infty$, we get $fz = gz$ (in view of (ϕ_{2b})). Hence z is a coincidence point of the mappings f and g. Since f-compatibility of f and g implies commutativity at a coincidence point, we have $ffz = fgz = gfz = ggz$. Now we show that fz is a common fixed point of f

and g. Putting $x = z$ and $y = fz$ in (3.1), we have $fz = ffz = gfz$ (due to (ϕ_3)). Thus fz is a common fixed point of the mappings f and g.

(ii) Assume that $fgx_n \to fz$ as $n \to \infty$. By (3.2), we have $fz = gt$ for some $t \in X$. Thus $fgx_n \to gt$ as $n \to \infty$. Then f-compatibility of the mappings f and g implies $\lim_{n\to\infty} d(fgx_n, ggx_n) = 0$, hence $ggx_n \to gt$ as $n \to \infty$. By (3.2), we get $fgx_{n+1} = ffx_n \to gt$ as $n \to \infty$ and so $gfx_n = ggx_{n+1} \to gt$ as $n \to \infty$. On using (3.1) with $x = t$, $y = fx_n$ and taking the limit as $n \to \infty$, we obtain $ft = gt$ (in view of (ϕ_{2b})). Thus t is a coincidence point of the mappings f and g. Also, f-compatibility of f and g implies commutativity at a coincidence point. Hence $fft = fgt = gft = ggt$. By putting $x = t$, $y = ft$ in (3.1), we get $ft = fft = gft$ (due to (ϕ_3)). Hence ft is a common fixed point of the mappings f and g.

Case IV: Let the pair (f, g) be R-weakly commuting. The weak reciprocal continuity of f and g implies that $fgx_n \to fz$ or $gfx_n \to gz$ as $n \to \infty$.

(i) First we assume that $gfx_n \to gz$ as $n \to \infty$. Then R-weak commutativity of the pair (f, g) implies $d(fgx_n, gfx_n) \leq Rd(fx_n, gx_n)$. Taking the limit as $n \to \infty$, we have $fgx_n \to gz$ as $n \to \infty$. By (3.2), we get $fgx_{n+1} = ffx_n \to gz$ as $n \to \infty$. On using (3.1) with $x = z$, $y = fx_n$ and taking the limit as $n \to \infty$, we get $fz = gz$ (in view of (ϕ_{2b})), hence z is a coincidence point of the mappings f and g. It is obvious that R-weakly commuting mappings commute at their coincidence points, i.e., $d(fgz, gfz) \leq Rd(fz, gz)$, implying $fgz = gfz$ and so $ffz = fgz = gfz = ggz$. Now we assert that fz is a common fixed point of f and g. On using (3.1) $x = z$, $y = fz$, we have $fz = ffz = gfz$ (due to (ϕ_3)) and fz is a common fixed point of the mappings f and g.

(ii) Suppose that $fgx_n \to fz$ as $n \to \infty$. By (3.2), we have $fz = gt$ for some $t \in X$. Thus $fgx_n \to gt$ as $n \to \infty$. Then R-weak commutativity of the pair (f, g), i.e., $d(fgx_n, gfx_n) \leq Rd(fx_n, gx_n)$ implies $gfx_n \to gt$ as $n \to \infty$. By (3.2), we get $fgx_{n+1} = ffx_n \to gt$ as $n \to \infty$. On using (3.1) with $x = t$, $y = fx_n$ and taking the limit as $n \to \infty$, we get $ft = gt$ (in view of (ϕ_{2b})) which shows that t is a coincidence point of the mappings f and g. Also, R-weak commutativity of the pair (f, g), $d(fgt, gft) \leq Rd(ft, gt)$ implies $fft = fgt = gft = ggt$. By putting $x = t$, $y = ft$ in (3.1), we can easily obtain $ft = fft = gft$ (due to (ϕ_3)). Hence ft is a common fixed point of the mappings f and g.

Case V: Assume that the pair (f, g) is R-weakly commuting of type (A_g). The weak reciprocal continuity of f and g implies that $fgx_n \to fz$ or $gfx_n \to gz$ as $n \to \infty$.

(i) Let us assume that $gfx_n \to gz$ as $n \to \infty$. Then R-weak commutativity of type (A_g) of the pair (f, g) implies $d(ffx_n, gfx_n) \leq Rd(fx_n, gx_n)$. On letting $n \to \infty$, we have $ffx_n \to gz$ as $n \to \infty$. On using (3.1) with $x = z$, $y = fx_n$ and taking the limit as $n \to \infty$, we have $fz = gz$ (in view of (ϕ_{2b})) which shows that z is a coincidence point of the mappings f and g. R-weak commutativity of type (A_g), $d(ffz, gfz) \leq Rd(fz, gz)$ yields $ffz = gfz$ and so $ffz = fgz = gfz = ggz$. Now we show that fz is a common fixed point of

f and g. On using (3.1) $x = z$, $y = fz$, we have $fz = ffz = gfz$ (due to (ϕ_3)) and fz is a common fixed point of the mappings f and g.

(ii) Suppose that $fgx_n \to fz$ as $n \to \infty$. In view of (3.2), $fz = gt$ for some $t \in X$. Thus $fgx_n \to gt$ as $n \to \infty$. By (3.2), $fgx_{n+1} = ffx_n \to gt$ as $n \to \infty$. Then R-weak commutativity of type (A_g) of the pair (f, g), $d(ffx_n, gfx_n) \le Rd(fx_n, gx_n)$ implies $gfx_n \to gt$ as $n \to \infty$. On using (3.1) with $x = t$, $y = fx_n$ and taking the limit as $n \to \infty$, we get $ft = gt$ (in view of (ϕ_{2b})), hence t is a coincidence point of the mappings f and g. Also, R-weak commutativity of type (A_g) of the pair (f, g), $d(fft, gft) \le Rd(ft, gt)$ implies $fft = fgt = gft = ggt$. Again using (3.1) with $x = t$, $y = ft$, we get $ft = fft = gft$ (due to (ϕ_3)). Thus ft is a common fixed point of the mappings f and g.

Case VI: Finally, let us consider that the pair (f, g) be R-weakly commuting of type (A_f). The weak reciprocal continuity of f and g implies that $fgx_n \to fz$ or $gfx_n \to gz$ as $n \to \infty$.

(i) Suppose that $gfx_n \to gz$ as $n \to \infty$. By (3.2), we get $gfx_n = ggx_{n+1} \to gz$ as $n \to \infty$. Then R-weak commutativity of type (A_f) of the pair (f, g) implies $d(fgx_n, ggx_n) \le Rd(fx_n, gx_n)$. This yields $fgx_n \to gz$ as $n \to \infty$. In view of (3.2), $fgx_{n+1} = ffx_n \to gz$ as $n \to \infty$. Putting $x = z$ and $y = fx_n$ in (3.1) with limit as $n \to \infty$, we get $fz = gz$ (in view of (ϕ_{2b})). Thus z is a coincidence point of the mappings f and g. Since R-weak commutativity of type (A_f), $d(fgz, ggz) \le Rd(fz, gz)$ yields $fgz = ggz$ and so $ffz = fgz = gfz = ggz$. Now we assert that fz is a common fixed point of f and g. On using (3.1) $x = z$, $y = fz$, we have $fz = ffz = gfz$ (due to (ϕ_3)). Hence fz is a common fixed point of the mappings f and g.

(ii) Assume that $fgx_n \to fz$ as $n \to \infty$. In view of (3.2), $fz = gt$ for some $t \in X$. Thus $fgx_n \to gt$ as $n \to \infty$. Then R-weak commutativity of type (A_f) of the pair (f, g), $d(fgx_n, ggx_n) \le Rd(fx_n, gx_n)$ implies $ggx_n \to gt$ as $n \to \infty$. By (3.2), we get $gfx_n = ggx_{n+1} \to gt$ and $fgx_{n+1} = ffx_n \to gt$ as $n \to \infty$. On using (3.1) with $x = t$, $y = fx_n$ and taking the limit as $n \to \infty$, we get $ft = gt$ (in view of (ϕ_{2b})) which shows that t is a coincidence point of the mappings f and g. Also, R-weak commutativity of type (A_f) of the pair (f, g), $d(fgt, ggt) \le Rd(ft, gt)$ implies $fft = fgt = gft = ggt$. Again using (3.1) with $x = t$, $y = ft$, we get $ft = fft = gft$ (due to (ϕ_3)). Thus ft is a common fixed point of the mappings f and g.

In view of Proposition 3.1, z is the unique common fixed point of the mappings f and g. □

Obviously, Theorem 1.1 follows as a special case of the obtained result (using $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - bt_3 - ct_4$, with $a, b, c \ge 0$, $a + b + c < 1$). We state some other corollaries.

Corollary 3.4. *The conclusions of Proposition 3.1, Lemma 3.2 and Theorem 3.3 remain true if condition (3.1) is replaced by one of the following contraction conditions (each inequality is supposed to hold for all $x, y \in X$).*

$$d(fx,fy) \leq k \max\left\{d(gx,gy), d(fx,gx), d(fy,gy), \frac{1}{2}[d(gx,fy)+d(gy,fx)]\right\}, \tag{3.4}$$

where $k \in (0,1)$.

$$\begin{aligned} d^2(fx,fy) &\leq d(fx,fy)[\alpha d(gx,gy)+\beta d(fx,gx)+\gamma d(fy,gy)] \\ &\quad +\eta d(gx,fy)d(gy,fx), \end{aligned} \tag{3.5}$$

where $\alpha > 0$, $\beta, \gamma, \eta \geq 0$, $\alpha+\beta+\gamma < 1$ and $\alpha+\eta < 1$.

$$\begin{aligned} d^3(fx,fy) \leq{}& \alpha d^2(fx,fy)d(gx,gy)+\beta d(fx,fy)d(fx,gx)d(fy,gy) \\ &+\gamma d(gx,fy)^2 d(gy,fx)+\eta d(gx,fy)d(gy,fx)^2, \end{aligned} \tag{3.6}$$

where $\alpha > 0$, $\beta, \gamma, \eta \geq 0$, $\alpha+\beta < 1$ and $\alpha+\gamma+\eta < 1$.

$$d^3(fx,fy) \leq k\frac{d^2(fx,gx)d^2(fy,gy)+d^2(gx,fy)d^2(gy,fx)}{1+d(gx,gy)+d(fx,gx)+d(fy,gy)}, \tag{3.7}$$

where $k \in (0,1)$.

$$d^2(fx,fy) \leq \alpha d^2(gx,gy)+\beta\frac{d(gx,fy)d(gy,fx)}{1+d^2(fx,gx)+d^2(fy,gy)}, \tag{3.8}$$

where $\alpha > 0$, $\beta \geq 0$ and $\alpha+\beta < 1$.

$$\begin{aligned} d^2(fx,fy) \leq{}& \alpha \max\left\{d^2(gx,gy), d^2(fx,gx), d^2(fy,gy)\right\} \\ &+\beta \max\{d(fx,gx)d(gx,fy), d(fy,gy)d(gy,fx)\} \\ &+\gamma d(gx,fy)d(gy,fx), \end{aligned} \tag{3.9}$$

where $\alpha > 0$, $\beta, \gamma \geq 0$, $\alpha+2\beta < 1$ and $\alpha+\gamma < 1$.

$$d(fx,fy) \leq k \max\left\{d(gx,gy), d(fx,gx), d(fy,gy), \frac{1}{2}d(gx,fy), \frac{1}{2}d(gy,fx)\right\}, \tag{3.10}$$

where $k \in (0,1)$.

$$d(fx,fy) \leq k \max\left\{d(gx,gy), \frac{1}{2}[d(fx,gx)+d(fy,gy)], \frac{1}{2}[d(gx,fy)+d(gy,fx)]\right\}, \tag{3.11}$$

where $k \in (0,1)$.

$$d(fx,fy) \leq \alpha d(gx,gy)+\beta d(fx,gx)+\gamma d(fy,gy)+\eta d(gx,fy)+\lambda d(gy,fx), \tag{3.12}$$

where $\alpha+\beta+\gamma+\eta+\lambda < 1$.

$$d(fx,fy) \leq \frac{k}{2} \max\left\{d(gx,gy), d(fx,gx), d(fy,gy), d(gx,fy), d(gy,fx)\right\}, \tag{3.13}$$

where $k \in (0,1)$.
(3.14)
$$d(fx,fy) \leq \alpha d(gx,gy) + \beta d(fx,gx) + \gamma d(fy,gy) + \eta[d(gx,fy) + d(gy,fx)],$$
where $\alpha + \beta + \gamma + 2\eta < 1$.
(3.15)
$$d(fx,fy) \leq \begin{cases} a_1 \frac{d^2(fx,gx)+d^2(fy,gy)}{d(fx,gx)+d(fy,gy)} + a_2 d(gx,gy) & \\ +a_3[d(gx,fy)+d(gy,fx)], & \text{if } d(fx,gx)+d(fy,gy) \neq 0; \\ 0, & \text{if } d(fx,gx)+d(fy,gy) = 0, \end{cases}$$
where $a_i \geq 0$ *with at least one* a_i *non-zero and* $a_1 + a_2 + 2a_3 < 1$.
(3.16)
$$d(fx,fy) \leq \begin{cases} a_1 d(gx,gy) & \\ +\frac{a_2 d(fx,gx)d(fy,gy)+a_3 d(gx,fy)d(gy,fx)}{d(fx,gx)+d(fy,gy)}, & \text{if } d(fx,gx)+d(fy,gy) \neq 0; \\ 0, & \text{if } d(fx,gx)+d(fy,gy) = 0, \end{cases}$$

(3.17)
$$\begin{aligned} d(fx,fy) \leq a_1[a_2 &\max\{d(gx,gy), d(fx,gx), d(fy,gy), (d(gx,fy)+d(gy,fx))/2\} \\ &+ (1-a_2)(\max\{d^2(gx,gy), d(fx,gx)d(fy,gy), d(gx,fy)d(gy,fx), \\ &(d(fx,gx)d(gy,fx))/2, (d(fy,gy)d(gx,fy))/2\})^{\frac{1}{2}}], \end{aligned}$$
where $a_1 \in (0,1)$ *and* $0 \leq a_2 \leq 1$.
(3.18)
$$\begin{aligned} d^2(fx,fy) \leq a_1 &\max\{d^2(gx,gy), d^2(fx,gx), d^2(fy,gy)\} \\ &+ a_2 \max\left\{\frac{1}{2}[d(fx,gx)d(gx,fy)], \frac{1}{2}[d(fy,gy)d(gy,fx)]\right\} \\ &+ a_3 d(gx,fy)d(gy,fx), \end{aligned}$$
where $a_1, a_2, a_3 \geq 0$ *and* $a_1 + a_2 + a_3 < 1$.
(3.19)
$$d(fx,fy) \leq \psi\left(\max\left\{d(gx,gy), d(fx,gx), d(fy,gy), \frac{1}{2}[d(gx,fy)+d(gy,fx)]\right\}\right),$$
where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is an increasing upper semi-continuous function with* $\psi(0) = 0$ *and* $\psi(t) < t$ *for each* $t > 0$.
(3.20)
$$d(fx,fy) \leq \psi\left(\max\{d(gx,gy), d(fx,gx), d(fy,gy), d(gx,fy), d(gy,fx)\}\right),$$
where $\psi : \mathbb{R}_+^5 \to \mathbb{R}_+$ *is an upper semi-continuous and non-decreasing function in each coordinate variable such that* $\psi(t,t,at,bt,ct) < t$ *for each* $t > 0$ *and* $a,b,c \geq 0$ *with* $a + b + c \leq 3$.
(3.21)
$$d^2(fx,fy) \leq \psi\left(\max\left\{\begin{array}{c} d^2(gx,gy), d(fx,gx)d(fy,gy), d(gx,fy)d(gy,fx), \\ d(fx,gx)d(gy,fx), d(fy,gy)d(gx,fy) \end{array}\right\}\right),$$

where $\psi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is an upper semi-continuous and non-decreasing function in each coordinate variable such that $\psi(t, t, at, bt, ct) < t$ for each $t > 0$ and $a, b, c \geq 0$ with $a + b + c \leq 3$.

Proof. The proof of each case (3.4)–(3.21) easily follows from Theorem 3.3 in view of Examples 2.1–2.18. □

The following example demonstrates the genuineness of Theorem 1.1 over Theorem 3.3.

Example 3.5. Let $X = [1, 5.5]$ (with the standard metric d) and consider the mappings $f, g : X \to X$ given by

$$fx = \begin{cases} 1, & \text{for } x = 1, \\ 3, & \text{for } 1 < x \leq 2.5, \\ 1, & \text{for } 2.5 < x \leq 5.5; \end{cases} \qquad gx = \begin{cases} 1, & \text{for } x = 1, \\ 4, & \text{for } 1 < x \leq 2.5, \\ \frac{2x+1}{6}, & \text{for } 2.5 < x < 5.5, \\ 3, & \text{for } x = 5.5. \end{cases}$$

Firstly, we show that Theorem 1.1 does not work in the context of this example. To accomplish this, we show that the contraction condition (ii) of Theorem 1.1 is not satisfied. To accomplish this, take $1 < x \leq 2.5$ and $2.5 < y \leq 5.5$ so that $d(fx, fy) = 2$, and

$$ad(gx, gy) + bd(fx, gx) + cd(fy, gy) = a\left(4 - \frac{2y+1}{6}\right) + b + c\left(\frac{2y+1}{6} - 1\right).$$

The right-hand side of the last equality is equal to $2a + b + c$ for $y = 5.5$ and tends to $3a + b$ for $y \to 2.5$. Hence, in order that conditon (ii) of Theorem 1.1 holds, one must simultaneously require $a, b, c \geq 0$, $a + b + c < 1$, $2a + b + c \geq 2$ and $3a + b \geq 2$ which is indeed impossible.

Next, we show that all the conditions of Theorem 3.3 are fulfilled with the function ϕ taken from Example 2.9. First of all, the mappings f and g are R-weakly commuting of type (A_g) (the condition $d(ffx, gfx) \leq Rd(fx, gx)$ can be easily checked). Also, $f(X) = \{1, 3\} \subseteq [1, 2) \cup \{3, 4\} = g(X)$. In order to show that f and g are weakly reciprocally continuous, suppose that $\{x_n\}$ is a sequence in X such that $fx_n \to t$ and $gx_n \to t$ yielding thereby $t = 1$. Then, we are left with two possibilities: either $x_n = 1$ for almost each n, or $x_n = 2.5 + \alpha_n$ where $\alpha_n \to +0$ as $n \to \infty$. In the first case, obviously, $fgx_n \to 1 = f1$ and $gfx_n \to 1 = g1$ as $n \to \infty$. In the second case $gx_n = 1 + \frac{1}{3}\alpha_n$ and $fgx_n = 3 \neq f1$, but $fx_n = 1$ and $gfx_n = 1 = g1$. Hence, f and g are weakly reciprocally continuous which is not reciprocally continuous as well as non-compatible.

Finally, we check the contractive condition (3.12) of Corollary 3.4 with $\eta = \frac{2}{3}$, $\lambda = \frac{2}{9}$ and $\alpha = \beta = \gamma = 0$ (note that $\alpha + \beta + \gamma + \eta + \lambda < 1$) so that

$$d(fx, fy) \leq \frac{2}{3}\, d(gx, fy) + \frac{2}{9}\, d(fx, gy). \tag{3.22}$$

Without loss of generality, we can assume that $x \leq y$. Consider the following cases: 1° $x = y = 1$; 2° $x = 1$, $1 < y \leq 2.5$; 3° $x = 1$, $y > 2.5$; 4° $1 <$

$x, y \le 2.5$; 5° $1 < x \le 2.5$, $y > 2.5$; 6° $x, y > 2.5$. The cases 1°, 3°, 4° and 6° are trivial as $d(fx, fy) = 0$. In the Case 2° we have $d(fx, fy) = 2$ and $\frac{2}{3}d(gx, fy) + \frac{2}{9}d(fx, gy) = \frac{2}{3} \cdot 2 + \frac{2}{9} \cdot 3 = 2$, while in Case 5° we have $d(fx, fy) = 2$ and $\frac{2}{3}d(gx, fy) + \frac{2}{9}d(fx, gy) \ge \frac{2}{3} \cdot 3 + \frac{2}{9} \cdot 0 = 2$. Therefore, condition (3.22) is satisfied.

Thus all the conditions of Theorem 3.3 are satisfied and the mappings f and g have a unique common fixed point at $z = 1$.

Remark 3.6. Notice that the preceding example confirms the importance of weak reciprocal continuity instead of reciprocal continuity when the given pair of mappings is not even compatible. In the following proposition, we notice that a non-compatible reciprocally continuous pair of self-mappings cannot have a common fixed point under certain contractive assumptions (see also [23, Remark 1]).

Proposition 3.7. *Let (X, d) be a metric space and let $f, g : X \to X$ satisfy the following assumptions:*

(1) *f and g are noncompatible.*
(2) *f and g are reciprocally continuous.*
(3) *f and g satisfy the condition (3.1) for some $\phi \in \Phi'$.*

Then f and g cannot have a common fixed point.

Proof. Suppose, to the contrary, that there is a point $z \in X$ such that

$$fz = gz = z. \tag{3.23}$$

By the assumption (1), there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is non-zero or non-existent. The assumption (2) implies that $fgx_n \to ft$ and $gfx_n \to gt$ as $n \to \infty$, which means that $ft \ne gt$.

Now, put $x = z$ and $y = x_n$ in the condition (3.1) to obtain

$$\phi\left(\begin{array}{c} d(fz, fx_n), d(gz, gx_n), d(fz, gz), \\ d(fx_n, gx_n), d(gz, fx_n), d(fz, gx_n) \end{array}\right) \le 0,$$
$$\text{or } \phi\left(\begin{array}{c} d(z, fx_n), d(z, gx_n), d(z, z), \\ d(fx_n, gx_n), d(z, fx_n), d(z, gx_n) \end{array}\right) \le 0.$$

On taking the limit as $n \to \infty$, we get that

$$\phi\big(d(z,t), d(z,t), 0, 0, d(z,t), d(z,t)\big) \le 0,$$

which implies that $d(z, t) = 0$ (by (ϕ_3)) so that $z = t$, which is a contradiction to (3.23) as $ft \ne gt$. □

4. Results without continuity

A critical examination of the recent literature reveals the fact that more natural results can be proved if we replace the completeness of the entire space by completeness of suitable subspaces (e.g. Imdad et al. [12]) under relatively tighter conditions. Concretely speaking, a common fixed point theorem under entire space completeness condition requires five conditions while the similar results can be proved under four conditions if the completeness of suitable subspaces are assumed. On the lines of the results proved in Imdad et al. [12], we state and prove our next theorem which is an improvement over Theorem 3.3 in the following three respects:

(1) to relax the weak reciprocal continuity of the involved mappings,
(2) to reduce the commutativity requirements of the mappings to coincidence points,
(3) to replace the completeness of the whole space by the completeness of any one of the underlying subspaces.

Theorem 4.1. *Let f and g be two self-mappings of a metric space (X, d) satisfying conditions (3.1) (for some $\phi \in \Phi'$) and (3.2). If one of $f(X)$ and $g(X)$ is a complete subspace of X, then f and g have a coincidence point.*

Moreover, f and g have a unique common fixed point provided the pair (f, g) is coincidentally commuting.

Proof. Let $\{y_n\}$ be a sequence defined by (3.3). By Lemma 3.2, $\{y_n\}$ is a Cauchy sequence in X. Now assume that $g(X)$ is a complete subspace of X, then the subsequence $fx_{2n+1} = gx_{2n}$ which is contained in $g(X)$ must has a limit $z \in g(X)$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore $\{y_n\}$ also converges implying thereby the convergence of the subsequence $\{y_{2n}\}$, i.e.

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = z.$$

Let $u \in g^{-1}(z)$ such that $gu = z$. By using inequality (3.1) with $x = x_{2n}$, $y = u$, we have

$$\phi\left(\begin{array}{c} d(fx_{2n}, fu), d(gx_{2n}, gu), d(fx_{2n}, gx_{2n}), \\ d(fu, gu), d(gx_{2n}, fu), d(gu, fx_{2n}) \end{array}\right) \leq 0,$$

which on making $n \to \infty$, gives rise

$$\begin{aligned} \phi\left(d(z, fu), d(z, z), d(z, z), d(fu, z), d(z, fu), d(z, z)\right) &\leq 0, \\ \phi\left(d(z, fu), 0, 0, d(fu, z), d(z, fu) + 0, 0\right) &\leq 0, \end{aligned}$$

implying thereby $d(z, fu) = 0$ (due to (ϕ_{2a})). Hence $z = fu = Su$ which shows that u is a coincidence point of the mappings f and g.

If one assumes that $f(X)$ is a complete subspace of X, then $z \in f(X) \subseteq g(X)$. Similarly, we can easily show that f and g have a coincidence point.

Since f and g are weakly compatible and $fu = gu = z$, then $fgu = gfu$ which implies $fz = gz$. By using inequality (3.1) with $x = z$, $y = u$, we have

$$\phi\left(d(fz, fu), d(gz, gu), d(fz, gz), d(fu, gu), d(gz, fu), d(gu, fz)\right) \leq 0,$$

or, equivalently,

$$\phi\left(d(fz, z), d(gz, z), 0, 0, d(z, fu), d(z, fz)\right) \leq 0,$$

a contradiction to (ϕ_3) if $d(z, fz) > 0$. Thus $z = fz = gz$ which shows that z is a common fixed point of the mappings f and g. In view of Proposition 3.1, z is the unique common fixed point of the mappings f and g. $\square$

Remark 4.2. The conclusion of Theorem 4.1 remains valid if we replace condition (3.1) by anyone of the contraction conditions (3.4)-(3.21) shown in Corollary 3.4 (each inequality is supposed to hold for all $x, y \in X$).

Example 4.3. Let $X = [0, 5]$ (with the standard metric d) and consider the mappings $f, g : X \to X$ given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } 0 < x \leq 5. \end{cases} \quad f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 5, & \text{if } 0 < x < 5; \\ 1, & \text{if } x = 5. \end{cases}$$

Notice that the mappings f and g are discontinuous at '0' which is also their common fixed point. Also the pair (f, g) is coincidentally commuting with $f(X) = \{0, 1\} \subseteq \{0, 1, 5\} = g(X)$. Define $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi\left(\max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}\right), \tag{4.1}$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing upper semi-continuous function with $\psi(0) = 0$ and $\psi(t) < t$ for each $t > 0$.

In view of (4.1), condition (3.1) implies (for all $x, y \in X$)

$$d(fx, fy) \leq \psi\left(\max\left\{d(gx, gy), d(fx, gx), d(fy, gy), \frac{1}{2}[d(gx, fy) + d(gy, fx)]\right\}\right). \tag{4.2}$$

By a routine calculation one can verify that implicit contraction condition (4.2) is satisfied if we choose $\psi(s) = \sqrt{s}$. Thus, all the conditions of Theorem 4.1 are satisfied. Notice that 0 is the unique common fixed point of the involved mappings f and g.

As an application of Theorem 4.1, we have the following interesting result involving two finite families of self mappings.

Corollary 4.4. *Let $\{f_i\}_{i=1}^p$ and $\{g_j\}_{j=1}^q$ be two finite families of self-mappings of a metric space (X, d) with $f = f_1 f_2 \dots f_p$ and $g = g_1 g_2 \dots g_q$ satisfying conditions (3.1) (for some $\phi \in \Phi'$) and (3.2). Then (f, g) has a point of coincidence.*

Moreover, $\{f_i\}_{i=1}^p$ and $\{g_j\}_{j=1}^q$ have a unique common fixed point if the families $(\{f_i\}, \{g_j\})$ commutes pairwise wherein $i \in \{1, 2, \dots, p\}$ and $j \in \{1, 2, \dots, q\}$.

Proof. The proof of this theorem can be completed on the lines of Theorem 2.2 of Imdad et al. [12]. $\square$

By setting $f_1 = f_2 = \dots = f_p = f$ and $g_1 = g_2 = \dots = g_q = g$ in Corollary 4.4, one deduces the following corollary which is a slight but partial generalization of Theorem 4.1 as the commutativity requirements are slightly stronger as compared to Theorem 4.1.

Corollary 4.5. *Let f and g be two self-mappings of a metric space (X, d) satisfying*

$$f^p(X) \subseteq g^q(X), \tag{4.3}$$

$$\phi(d(f^p x, f^p y), d(g^q x, g^q y), d(f^p x, g^q x), d(f^p y, g^q y), d(g^q x, f^p y), d(g^q y, f^p x)) \leq 0, \tag{4.4}$$

for all $x, y \in X$, some $\phi \in \Phi'$ and p, q are fixed positive integers. If one of $f^p(X)$ and $g^q(X)$ is a complete subspace of X, then the pair (f^p, g^q) has a coincidence point.

Moreover, the mappings f and g have a unique common fixed point provided that the pair (f, g) commutes.

Conclusion

Theorem 3.3 extends a result of Pant et al. [23, Theorem 1] to various variants of compatible and R-weakly commuting mappings which is substantiated well by an appropriate example. Proposition 3.7 is designed to exhibit the importance of weak reciprocal continuity over reciprocal continuity. Theorem 4.1 scores a superiority over all those results wherein the continuity of the mappings and completeness of the whole spaces are assumed for the existence of coincidence point. However, Corollary 4.4 demonstrates the validity of Theorem 4.1 to two finite families of self-mappings. The implicit function employed in our results cover much wider class of contraction conditions than ones covered by the implicit function of Popa [26].

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