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On some variations of multifunction continuity

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Abstract. This paper considers six classes of multifunctions between topological spaces, namely almost $\ell-$continuous multifunctions, K-almost $c-$continuous multifunctions, nearly continuous multifunctions, almost nearly continuous multifunctions, super continuous multifunctions, and $\delta-$continuous multifunctions. We relate these classes of multifunctions to others, and provide characterizations of related concepts especially in terms of appropriate changes of topology.

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1. Introduction and Preliminaries

There are several notions of generalized continuity for multifunctions in the recent literature, for example almost continuity and $\ell-$continuity [17], c-continuity [10] and [17], almost c-continuity [14], nearly continuity [5], almost nearly continuity [6], super continuity [1], $\delta-$continuity [2] and [15]. In this paper we consider each of the following continuity properties of multifunctions-almost $\ell-$continuity, K-almost $c-$continuity, nearly continuity, almost nearly continuity, $\delta-$continuity, and super continuity.

Throughout this paper, the closure (resp. interior) of a subset B in a topological space (Y,τ) is denoted by clB (resp. $intB$). Then B is called regular open if $B = int(clB)$. The family of all regular open sets in (Y,τ) which is denoted by $RO(Y,\tau)$, forms a base for a topology τ_S on Y, known as the semiregularization of τ. In general $\tau_S \subseteq \tau$, and if $\tau_S = \tau$ then (Y,τ) is called a semiregular space. A set A is said to be $\delta-$closed [18] if for each point $x \notin A$ there exists a regular open set containing x which has empty intersection with A. A set B is $\delta-$open [18] if and only if its complement is $\delta-$closed. For any topological space (Y,τ) the collection of all $\delta-$open sets forms a topology on X which coincides with τ_S. For a topological space (Y,τ), the cocompact topology of τ on Y is denoted by $c(\tau)$ and defined by $c(\tau) = \{\varnothing\} \cup \{U \in \tau : Y - U$ is $\tau-$compact$\}$. The almost cocompact topology of τ on Y is denoted by $e(\tau)$ and it has as a base $e'(\tau) = \{U \in RO(Y,\tau) : Y - U$ is $\tau-$compact$\}$. These topologies are considered by Gauld [7] and [8]. If (Y,τ) is a topological space then the coLindelöf topology of τ on Y is denoted by $\ell(\tau)$ and defined by $\ell(\tau) = \{\varnothing\} \cup \{U \in \tau : Y - U$ is $\tau-$Lindelöf$\}$, considered by Gauld, Mrsevic, Reilly and Vamanamurthy [9]. Recall that a subset B of a topological space (Y,τ) is called N-closed in X if every regular open cover of B in (Y,τ) has a finite subcover. The concept of N-closed subsets was first considered by Carnahan [3]. If (Y,τ) is a topological space then the almost coN-closed topology of τ on Y is denoted by $p(\tau)$ and it has as a base $p'(\tau) = \{U \in RO(Y,\tau) : Y - U$ is N-closed relative to $\tau\}$. The almost coLindelöf topology of τ on Y is denoted by $q(\tau)$ and it has as a base $q'(\tau) = \{U \in RO(Y,\tau) : Y - U$ is $\tau-$Lindelöf$\}$. These topologies are considered by Konstadilaki-Savvopoulou and Reilly [12] and [13].

By a multifunction $F : (X,\sigma) \to (Y,\tau)$, we mean a point-to-set correspondence from (X,σ) into (Y,τ), and we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For each $B \subseteq Y$, $F^+(B) = \{x \in X : F(x) \subseteq B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \varnothing\}$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$. As usual, F is said to be a surjection if $F(X) = Y$. Moreover $F : (X,\sigma) \to (Y,\tau)$ is called upper semicontinuous, abbreviated as *u.s.c.* (resp. lower semicontinuous, abbreviated as *l.s.c.*) at a point $x \in X$ if for each open set V in Y with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \varnothing$), there exists an open set U containing x such that $F(U) \subseteq V$ (resp. $F(z) \cap V \neq \varnothing$ for every $z \in U$), or equivalently, if $F^+(V)$ (resp. $F^-(V)$) is open in (X,σ) for every open set V of (Y,τ).

A subset $K \subseteq F(x_0)$ is said to be a kernel [4] for F at x_0, if the multifunction $F_K : X \to Y$ defined by:

$$F_K(x) = K \quad , \text{ if } x = x_0$$
$$F_K(x) = F(x) \cap (Y - F(x_0)) \text{ , otherwise}$$

is *u.s.c.* at x_0.

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

This paper discusses how several versions of multifunction continuity behave with respect to a change of topology approach. The results presented here show that, in general, "lower" multifunction continuities behave well from this point of view. However, "upper" multifunction continuities do not behave at all well from this perspective. This behaviour of multifunctions can not be predicted from a consideration of corresponding behaviour of (single-valued) functions. It is unexpected, and a somewhat surprising situation.

2. Some properties

The following basic properties of almost $\ell-$continuity and K-almost $c-$continuity are useful in the sequel:

Definition 2.1 ([11]). A multifunction $F : X \to Y$ *is called*

(a) upper almost $\ell-$continuous (resp. upper K-almost $c-$continuous), or u.a.$\ell-$continuous (resp. u.K-a.c$-$continuous), at $x \in X$ if for each regular open subset V of Y with $F(x) \subseteq V$ and having Lindelöf (resp. compact) complement, there is an open neighbourhood U of x such that $F(U) \subseteq V$.

(b) lower almost $\ell-$continuous (resp. lower K-almost $c-$continuous), or l.a.$\ell-$continuous (resp. l.K-a.c$-$continuous), at $x \in X$ if for each regular open subset V of Y with $F(x) \cap V \neq \varnothing$ and having Lindelöf (resp. compact) complement, there is an open neighbourhood U of x such that $F(z) \cap V \neq \varnothing$ for every point $z \in U$.

(c) almost $\ell-$continuous (resp. K-almost $c-$continuous) at $x \in X$ if it is both u.a.$\ell-$continuous (resp.u.K-a.c$-$continuous) and l.a.$\ell-$continuous (resp. l.K-a.c$-$continuous) at $x \in X$.

(d) almost $\ell-$continuous (resp. u.a.$\ell-$continuous ; l.a.$\ell-$continuous, K-a.c$-$continuous, u.K-a.c$-$continuous, l.K-a.c$-$continuous) if it is almost $\ell-$continuous (resp. u.a.$\ell-$continuous; l.a.$\ell-$continuous, K-a.c$-$continuous, u.K-a.c$-$continuous, l.K-a.c$-$continuous) at each point of X.

Theorem 2.2 ([11]). *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. Then*

(a) F is upper (resp. lower) almost $\ell-$continuous if and only if $F^+(V)$ (resp. $F^-(V)$) is open for each $V \in q'(\tau)$.

(b) F is upper (resp. lower) K-almost $c-$continuous if and only if $F^+(V)$ (resp. $F^-(V)$) is open for each $V \in e'(\tau)$.

Theorem 2.3. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction and $x_0 \in X$. Then F is u.s.c. at x_0 if and only if $F(x_0)$ is a kernel for F at x_0.*

Proof. ($\Rightarrow$) Let $F : (X, \sigma) \to (Y, \tau)$ be *u.s.c.* at x_0 and $K = F(x_0)$. We will show that $F_K : (X, \sigma) \to (Y, \tau)$ is *u.s.c.* at x_0. Let $F_K(x_0) \subseteq V$ where $V \in \tau$. Since $F_K(x_0) = F(x_0)$ and F is *u.s.c.* at x_0, there exists an open set U containing x_0 such that $F(U) \subseteq V$. Therefore $F_K(U) \subseteq V$ and so F_K is *u.s.c.* at x_0 and hence $F(x_0)$ is a kernel for F at x_0.

($\Leftarrow$) Let $K = F(x_0)$ be a kernel for F at x_0 and $F(x_0) \subseteq V$ where $V \in \tau$. Then $F_K(x_0) = F(x_0) \subseteq V$. Since F_K is *u.s.c.* at x_0, there exists an open set U containing x_0 such that $F_K(U) \subseteq V$. Therefore $F(U) \subseteq V$ since $F(x_0) \subseteq V$ and hence F is *u.s.c.* at x_0. □

Theorem 2.4. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction and $F(x_0) \in q'(\tau)$ (resp. $F(x_0) \in e'(\tau)$). Then F is upper almost $\ell-$continuous (resp. upper K-almost $c-$continuous) at $x_0 \in X$ if and only if $F(x_0)$ is a kernel for F at x_0.*

Proof. ($\Rightarrow$) Let $K = F(x_0) \in q'(\tau)$ and $F_K(x_0) = F(x_0)$. We will show that $F_K : (X, \sigma) \to (Y, \tau)$ is *u.s.c.* at x_0. Let $F_K(x_0) \subseteq V$ where $V \in \tau$.

Since $F_K(x_0) \in q'(\tau)$ and F is $u.a.\ell-$continuous multifunction, there exists an open set U containing x_0 such that $F(U) \subseteq F_K(x_0) = F(x_0) \subseteq V$. Therefore $F_K(U) \subseteq V$ and hence F_K is $u.s.c$ at $x_0 \in X$. This shows that $F(x_0)$ is a kernel for F at x_0.

($\Leftarrow$) It follows from Theorem 2.3 and the fact that every upper semi continuous multifunction is upper almost $\ell-$continuous [11].

The proof for upper K-almost $c-$continuity is similar. □

Theorem 2.5 ([16]). *A subset A of (Y, τ) is N-closed with respect to τ if and only if A is compact in (Y, τ_S).*

Definition 2.6. *A multifunction $F : (X, \sigma) \to (Y, \tau)$ is called*

(a) upper (resp. lower) $\delta-$continuous [2], or $u.\delta.c$ (resp. $l.\delta.c$), if for each $x \in X$ and for each open subset V of Y with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \varnothing$), there is an open neighbourhood U of x such that $F(int(clU)) \subseteq int(clV)$ (resp. $F(z) \cap int(clV) \neq \varnothing$ for every point $z \in int(clU)$).

(b) upper (resp. lower) super continuous [1], or $u.\sup.c$ (resp. $l.\sup.c$), if for each $x \in X$ and for each open subset V of Y with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \varnothing$), there is an open set U containing x such that $F(int(clU)) \subseteq V$ (resp. $F(z) \cap V \neq \varnothing$ for every point $z \in int(clU)$).

(c) upper (resp. lower) $c-$continuous [10], or $u.c.c$ (resp. $l.c.c$), if for each $x \in X$ and for each open subset V of Y with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \varnothing$) and having compact complement, there is an open set U containing x such that $F(U) \subseteq V$ (resp. $F(z) \cap V \neq \varnothing$ for every point $z \in U$).

(d) upper (resp. lower) $\ell-$continuous [17], or $u.\ell.c$ (resp. $l.\ell.c$), if $F^+(V)$ (resp. $F^-(V)$) is open for each open subset $V \subseteq Y$ having Lindelöf complement.

(e) upper (resp. lower) nearly continuous [5] if for each $x \in X$ and for each open subset V of Y with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \varnothing$) and having N-closed complement, there is an open set U containing x such that $F(U) \subseteq V$ (resp. $F(z) \cap V \neq \varnothing$ for every point $z \in U$).

(f) upper (resp. lower) almost nearly continuous [6] if for each $x \in X$ and for each open subset V of Y with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \varnothing$) and having N-closed complement, there is an open set U containing x such that $F(U) \subseteq int(clV)$ (resp. $F(z) \cap int(clV) \neq \varnothing$ for every point $z \in U$).

The following characterizations are important for our discussion.

Theorem 2.7. *A multifunction $F : (X, \sigma) \to (Y, \tau)$ is*

(a) upper (resp. lower) $\delta-$continuous [2] if and only if $F^+(V)$ (resp. $F^-(V)$) is $\delta-$open for each regular open subset $V \subseteq Y$.

(b) upper (resp. lower) super continuous [1] if and only if $F^+(V)$ (resp. $F^-(V)$) is $\delta-$open for each open subset $V \subseteq Y$.

(c) *upper (resp. lower) $c-$continuous [10] and [17] if and only if* $F^+(V)$ (*resp.* $F^-(V)$) *is open for each open subset* $V \subseteq Y$ *having compact complement*.
(d) *upper (resp. lower) nearly continuous [5] if and only if* $F^+(V)$ (*resp.* $F^-(V)$) *is an open set for any open subset* $V \subseteq Y$ *having N-closed complement.*
(e) *upper (resp. lower) almost nearly continuous [6] if and only if* $F^+(V)$ (*resp.* $F^-(V)$) *is an open set for any regular open subset* $V \subseteq Y$ *having N-closed complement.*

The next six results indicate how change of topology is related to these concepts of near continuity of multifunctions.

Theorem 2.8. *Let* $F : (X, \sigma) \to (Y, \tau)$ *be a multifunction. Then* $F : (X, \sigma) \to (Y, \tau_S)$ *is upper (resp. lower) $c-$continuous if and only if* $F : (X, \sigma) \to (Y, \tau)$ *is upper (resp. lower) nearly continuous.*

Proof. ($\Rightarrow$) Let V be an open subset having N-closed complement. Then $Y - V$ is compact in (Y, τ_S) by Theorem 2.5. Since $F : (X, \sigma) \to (Y, \tau_S)$ is upper $c-$continuous, $F^+(V) \in \sigma$. Hence $F : (X, \sigma) \to (Y, \tau)$ is upper nearly continuous. The proof for the case F lower nearly continuous is analogous.

($\Leftarrow$) Let $V \in \tau_S$ and having compact complement. Then $Y - V$ is N-closed with respect to τ by Theorem 2.5. Since $F : (X, \sigma) \to (Y, \tau)$ is upper nearly continuous, $F^+(V) \in \sigma$. Hence $F : (X, \sigma) \to (Y, \tau)$ is upper $c-$continuous. The proof for the case F lower $c-$continuous is analogous. □

Corollary 2.9. *Let* $F : (X, \sigma) \to (Y, \tau)$ *be a multifunction and* Y *be semi regular. Then* $F : (X, \sigma) \to (Y, \tau)$ *is upper (resp. lower) $c-$continuous if and only if* $F : (X, \sigma) \to (Y, \tau)$ *is upper (resp. lower) nearly continuous.*

Theorem 2.10. *Let* $F : (X, \sigma) \to (Y, \tau)$ *be a multifunction. Then* $F : (X, \sigma) \to (Y, \tau_S)$ *is upper (resp. lower) K-almost $c-$continuous if and only if* $F : (X, \sigma) \to (Y, \tau)$ *is upper (resp. lower) almost nearly continuous.*

Proof. ($\Rightarrow$) Let V be a regular open subset having N-closed complement. Then $Y - V$ is compact in (Y, τ_S) by Theorem 2.5. Since $F : (X, \sigma) \to (Y, \tau_S)$ is upper K-almost $c-$continuous, $F^+(V) \in \sigma$. Hence $F : (X, \sigma) \to (Y, \tau)$ is upper almost nearly continuous. The proof for the case F lower almost nearly continuous is analogous.

($\Leftarrow$) Let $V \in q'(\tau_S)$. Then $Y - V$ is N-closed with respect to τ by Theorem 2.5. Since $F : (X, \sigma) \to (Y, \tau)$ is upper almost nearly continuous, $F^+(V) \in \sigma$. Hence $F : (X, \sigma) \to (Y, \tau)$ is upper K-almost $c-$continuous. The proof for the case F lower K-almost $c-$continuous is analogous. □

Corollary 2.11. *Let* $F : (X, \sigma) \to (Y, \tau)$ *be a multifunction and* Y *be semi regular. Then* $F : (X, \sigma) \to (Y, \tau)$ *is upper (resp. lower) K-almost $c-$continuous if and only if* $F : (X, \sigma) \to (Y, \tau)$ *is upper (resp. lower) almost nearly continuous.*

We have the following corollary since upper (resp. lower) almost $\ell-$continuous multifunctions are upper (resp. lower) K-almost $c-$continuous [11].

Corollary 2.12. *If $F : (X,\sigma) \to (Y,\tau_S)$ is upper (resp. lower) almost $\ell-$continuous, then $F : (X,\sigma) \to (Y,\tau)$ is upper (resp. lower) almost nearly continuous.*

Theorem 2.13. *Let $F : (X,\sigma) \to (Y,\tau)$ be a multifunction. Then $F : (X,\sigma) \to (Y,\tau)$ is lower almost nearly continuous if and only if $F : (X,\sigma) \to (Y,p(\tau))$ is lower semi continuous.*

Proof. ($\Rightarrow$) Let $V \in p(\tau)$. We can write $V = \cup_{\alpha\in\Lambda} V_\alpha$ where V_α is a regular open set having N-closed complement for each $\alpha \in \Lambda$. We have $F^-(\cup_{\alpha\in\Lambda} V_\alpha) = \cup_{\alpha\in\Lambda} F^-(V_\alpha)$. From Theorem 2.7(e) we have that $F^-(V_\alpha)$ is an open set for each $\alpha \in \Lambda$. So $F^-(V)$ is an open set. Hence $F : (X,\sigma) \to (Y,p(\tau))$ is *l.s.c.*

($\Leftarrow$) Obvious. □

A result analogous to Theorem 2.13 for upper almost nearly continuous does not hold as the following example shows.

Example 2.14 ([11]). Let $X = Y = \{1,2,3,4\}$ and $\sigma = \{\{1\},\{1,3,4\},X,\varnothing\}$ be the topology on X and $\tau = \{\{1\},\{2\},\{1,2\},Y,\varnothing\}$ be the topology on Y. Let F be defined as $F(1) = \{4\}, F(2) = \{1,2\}, F(3) = \{3\}, F(4) = \{4\}$. The family $p'(\tau) = \{\{1\},\{2\},Y,\varnothing\}$ is a base consisting of regular open sets having N-closed complement in Y for $p(\tau) = \tau$. Then for any regular open set V having N-closed complement we have $F^+(V) \in \sigma$. Therefore $F : (X,\sigma) \to (Y,\tau)$ is upper almost nearly continuous. The topology $p(\tau)$ contains the set $\{1,2\}$ but $F^+(\{1,2\}) = \{2\} \notin \sigma$. Hence $F : (X,\sigma) \to (Y,p(\tau))$ is not *u.s.c.*

We have the following corollary by Theorem 2.13 and since lower semi continuous multifunctions are lower almost $\ell-$continuous multifunctions [11].

Corollary 2.15. *If the multifunction $F : (X,\sigma) \to (Y,\tau)$ is lower almost nearly continuous, then $F : (X,\sigma) \to (Y,p(\tau))$ is $l.a.\ell-$continuous.*

The proof of each of the next three results is straightforward from Theorem 2.7 and Definition 2.6(d). Note that Propositions 2.16 and 2.23 provide change of topology results for upper super continuity, and that this is an unusual circumstance. These are exceptional results, as the counter-examples 2.14 and 2.20 show.

Proposition 2.16. *Let $F : (X,\sigma) \to (Y,\tau)$ be a multifunction. Then $F : (X,\sigma) \to (Y,\ell(\tau))$ is $u.\sup.c.$(resp. $l.\sup.c.$) if and only if $F : (X,\sigma_S) \to (Y,\tau)$ is $u.\ell.c.$(resp. $l.\ell.c.$)*

Proposition 2.17. *Let $F : (X,\sigma) \to (Y,\tau)$ be a multifunction. Then $F : (X,\sigma) \to (Y,\tau)$ is $l.\delta.c.$ if and only if $F : (X,\sigma) \to (Y,\tau_S)$ is $l.\sup.c.$*

Proposition 2.18. *Let $F : (X,\sigma) \to (Y,\tau)$ be a multifunction. Then $F : (X,\sigma_S) \to (Y,\tau)$ is $l.a.\ell-$continuous if and only if $F : (X,\sigma) \to (Y,q(\tau))$ is $l.\delta.c.$*

Theorem 2.19. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. Then the following statements are equivalent:*

(a) $F : (X, \sigma_S) \to (Y, \tau)$ is $l.a.\ell-$continuous.
(b) $F : (X, \sigma_S) \to (Y, q(\tau))$ is $l.\sup.c.$
(c) $F : (X, \sigma_S) \to (Y, q(\tau))$ is $l.\delta.c.$
(d) $F : (X, \sigma_S) \to (Y, q(\tau))$ is $l.s.c.$
(e) $F : (X, \sigma_S) \to (Y, q(\tau))$ is $l.\ell.c.$

Proof. The proofs of (b)$\Rightarrow$ (c), (c)$\Rightarrow$ (d), (d)$\Rightarrow$ (e), (e)$\Rightarrow$ (a) are immediate. We prove only (a)$\Rightarrow$ (b).

Let $V \in q(\tau)$. Then we can write $V = \cup_{\alpha\in\Lambda} V_\alpha$ where V_α is a regular open set having Lindelöf complement, for each $\alpha \in \Lambda$. We have $F^-(\cup_{\alpha\in\Lambda} V_\alpha) = \cup_{\alpha\in\Lambda} F^-(V_\alpha)$. Therefore $F^-(V_\alpha) \in \sigma_S$ by hypothesis. So $F^-(V) \in \sigma_S$ and therefore is $\delta-$open. Hence $F : (X, \sigma_S) \to (Y, q(\tau))$ is $l.\sup.c.$ □

A result analogous to Theorem 2.19 for $u.a.\ell-$continuous, $u.\sup.c.$, $u.\delta.c.$, $u.s.c.$, $u.\ell.c.$ does not hold as the following example shows.

Example 2.20. Let us redefine the topology σ in Example 2.14 as follows $\sigma = \{\{1, 2\}, \{3, 4\}, \varnothing, X\}$. No other changes are made to Example 2.14. It is obvious that $\sigma_S = \sigma$ and $q(\tau) = \tau$. Then F is $u.a.\ell-$continuous and $u.\delta.c$ but not $u.\sup.c.$, $u.s.c.$ and $u.\ell.c$ even if X is regular and Lindelöf and Y is semi regular and Lindelöf.

Proposition 2.21 ([12]). *If (Y, τ) is a Lindelöf and semi regular space, then $q(\tau) = \tau$.*

We have the following corollary by Theorem 2.19 and Proposition 2.21.

Corollary 2.22. *Let X be a semi regular topological space and let Y be a semi regular and Lindelöf topological space and $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. Then the following statements are equivalent:*

(a) $F : (X, \sigma) \to (Y, \tau)$ is $l.a.\ell-$continuous.
(b) $F : (X, \sigma) \to (Y, \tau)$ is $l.\sup.c.$
(c) $F : (X, \sigma) \to (Y, \tau)$ is $l.\delta.c.$
(d) $F : (X, \sigma) \to (Y, \tau)$ is $l.s.c.$
(e) $F : (X, \sigma) \to (Y, \tau)$ is $l.\ell.c.$

A result analogous to Corollary 2.22 for $u.a.\ell-$continuous, $u.\sup.c.$, $u.\delta.c.$, $u.s.c.$, $u.\ell.c.$ does not hold as Example 2.20 shows.

The next two results are analogues of Propositions 2.16 and 2.18 respectively. Their proofs are straightforward from Theorem 2.7.

Proposition 2.23. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. Then $F : (X, \sigma) \to (Y, c(\tau))$ is $u.\sup.c.$(resp. $l.\sup.c.$) if and only if $F : (X, \sigma_S) \to (Y, \tau)$ is $u.c.c.$(resp. $l.c.c.$).*

Proposition 2.24. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. Then $F : (X, \sigma_S) \to (Y, \tau)$ is $l.K$-$a.c-$continuous if and only if $F : (X, \sigma) \to (Y, e(\tau))$ is $l.\delta.c.$*

Similarly to Theorem 2.19, we can obtain the following characterizations.

Theorem 2.25. *Let $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. Then the following statements are equivalent:*

(a) $F : (X, \sigma_S) \to (Y, \tau)$ *is $l.K$-$a.c-$continuous.*
(b) $F : (X, \sigma_S) \to (Y, e(\tau))$ *is $l.\sup.c.$*
(c) $F : (X, \sigma_S) \to (Y, e(\tau))$ *is $l.\delta.c.$*
(d) $F : (X, \sigma_S) \to (Y, e(\tau))$ *is $l.s.c.$*
(e) $F : (X, \sigma_S) \to (Y, e(\tau))$ *is $l.c.c.$*

A result analogous to Theorem 2.25 for $u.K$-$a.c-$continuous, $u.\sup.c.$, $u.\delta.c.$, $u.s.c.$, $u.c.c.$ does not hold as Example 2.20 shows.

Corresponding to Proposition 2.21 we have the following result, which seems to be new.

Proposition 2.26. *If (Y, τ) is a compact and semi regular space, then $e(\tau) = \tau$.*

Corollary 2.27. *Let X be a semi regular topological space and let Y be a semi regular and compact topological space and $F : (X, \sigma) \to (Y, \tau)$ be a multifunction. Then the following statements are equivalent:*

(a) $F : (X, \sigma) \to (Y, \tau)$ *is $l.K$-$a.c-$continuous.*
(b) $F : (X, \sigma) \to (Y, \tau)$ *is $l.\sup.c.$*
(c) $F : (X, \sigma) \to (Y, \tau)$*is $l.\delta.c.$*
(d) $F : (X, \sigma) \to (Y, \tau)$*is $l.s.c.$*
(e) $F : (X, \sigma) \to (Y, \tau)$*is $l.c.c.$*
(f) $F : (X, \sigma) \to (Y, \tau)$ *is lower nearly continuous.*
(g) $F : (X, \sigma) \to (Y, \tau)$*is lower almost nearly continuous.*

Proof. This is an immediate consequence of Theorem 2.25, Proposition 2.26, and Corollaries 2.9 and 2.11. □

A result analogous to Corollary 2.27 for $u.K$-$a.c-$continuous, $u.\sup.c.$, $u.\delta.c.$, $u.s.c.$, $u.c.c.$, upper almost nearly continuous, upper nearly continuous does not hold as Example 2.20 shows.

Several properties of upper and lower almost $\ell-$continuous multifunctions have been given in [11]. We now provide some more. We can also obtain similar sets of results [which are parallel to the following results] for $u.K$-$a.c-$continuous (resp. $l.K$-$a.c-$continuous) multifunctions. We shall leave them unstated.

The following result is immediate. The proof is left to the reader.

Theorem 2.28. *Let $(X,\sigma),(Y,\tau),(Z,\vartheta)$ be topological spaces and let $F : X \to Y$ and $G : Y \to Z$ be multifunctions. If $F : X \to Y$ is upper (resp. lower) semi continuous and $G : Y \to Z$ is upper (resp. lower) almost $\ell-$continuous, then $F \circ G : X \to Z$ is upper(resp. lower) almost $\ell-$continuous multifunction.*

Theorem 2.29. *Let $F : (X,\sigma) \to (Y,\tau)$ be a multifunction and let $A \subseteq X$ be a nonempty subset. If F is u.a.$\ell-$continuous (resp. l.a.$\ell-$continuous) then the restriction multifunction $F|_A : A \to Y$ is u.a.$\ell-$continuous (resp. l.a.$\ell-$continuous).*

Proof. We prove only the assertion for $F|_A$ $u.a.\ell-$continuous, the proof for $F|_A$ $l.a.\ell-$continuous being analogous. Let $x \in A$ and V be a regular open subset of (Y,τ) having Lindelöf complement such that $(F|_A)(x) \subseteq V$. Since F is $u.a.\ell-$continuous and $(F|_A)(x) = F(x)$, there exists $U \in \sigma$ containing x such that $F(U) \subseteq V$. Then $x \in A \cap U$ and $A \cap U$ is open in A. Moreover $(F|_A)(A \cap U) \subseteq V$. This shows that $F|_A$ is $u.a.\ell-$continuous. □

Theorem 2.30. *Let $F : (X,\sigma) \to (Y,\tau)$ be a multifunction. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X. If the restriction multifunction $F\alpha = F|_{V_\alpha}$ is u.a.$\ell-$continuous (resp. l.a.$\ell-$continuous) multifunction for each $\alpha \in \Lambda$, then F is u.a.$\ell-$continuous (resp. l.a.$\ell-$continuous).*

Proof. Let $V \in q'(\tau)$. Since F_α is $u.a.\ell-$continuous $F_\alpha^+(V) \subseteq int_{V_\alpha}(F^+(V))$ and since V_α is open, we have $F^+(V) \cap V_\alpha \subseteq int_{V_\alpha}(F^+(V) \cap V_\alpha)$ and $F^+(V) \cap V_\alpha \subseteq int(F^+(V)) \cap V_\alpha$. Since $\{V_\alpha : \alpha \in \Lambda\}$ is an open cover of X, $F^+(V) = int(F^+(V))$. Hence F is $u.a.\ell-$continuous.

The proof for $l.a.\ell-$continuous is similar. □

Corollary 2.31. *Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of X. A multifuntion $F : X \to Y$ is u.a.$\ell-$continuous (resp. l.a.$\ell-$continuous) if and only if the restriction $F|_{V_\alpha} : V_\alpha \to Y$ is u.a.$\ell-$continuous (resp. l.a.$\ell-$continuous) for each $\alpha \in \Lambda$.*

Proof. This is an immediate consequence of Theorems 2.29 and 2.30. □

Theorem 2.32. *Let $F : (X,\sigma) \to (Y,\tau)$ be multifunction and let $X \times Y$ be a Lindelöf space. If the graph multifunction of F is lower (resp. upper) almost $\ell-$continuous multifunction, then F is lower (resp. upper) almost $\ell-$continuous multifuction.*

Proof. Let $x \in X$ and let $V \in q'(\tau)$ with $F(x) \cap V \neq \varnothing$. Then $G_F(x) \cap (X \times V) \neq \varnothing$ and $X \times V$ is a regular open set having Lindelöf complement. Since the graph multifunction G_F is lower almost $\ell-$continuous, there exists an open neighbourhood U of x such that $G_F(z) \cap (X \times V) \neq \varnothing$ for every point $z \in U$. Thus $F(z) \cap V \neq \varnothing$. Hence F is lower almost $\ell-$continuous. The proof of the upper almost $\ell-$continuity of F is similar. □

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