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Unitary representability of free abelian topological groups

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Abstract. For every Tikhonov space X the free abelian topological group $A(X)$ and the free locally convex vector space $L(X)$ admit a topologically faithful unitary representation. For compact spaces X this is due to Jorge Galindo.

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1. Introduction

With every Tikhonov space X one associates the free topological group $F(X)$, the free abelian topological group $A(X)$, and the free locally convex vector space $L(X)$. They are characterized by respective universal properties. For example, $L(X)$ is defined by the following: X is an (algebraic) basis of $L(X)$, and for every continuous mapping $f : X \to E$, where E is a Hausdorff locally convex space, the linear extension $\bar{f} : L(X) \to E$ of f is continuous. There are two versions of $L(X)$, real and complex. There also are versions of all these free objects for spaces with a distinguished point. We consider non-pointed spaces.

The *unitary group* $U(H)$ of a Hilbert space H will be equipped with the strong operator topology, which is the topology inherited from the Tikhonov product H^H, or, equivalently, the topology of pointwise convergence. We use the notation $U_s(H)$ to indicate this topology. A *unitary representation* of a topological group G is a continuous homomorphism $f : G \to U_s(H)$. Such a representation is *faithful* if f is injective, and *topologically faithful* if f is a homeomorphic embedding. A topological group is *unitarily representable* if it is isomorphic to a topological subgroup of $U_s(H)$ for a Hilbert space H (which may be non-separable), or, equivalently, if it admits a topologically faithful unitary representation.

All locally compact groups are unitarily representable. For groups beyond the class of locally compact groups this is no longer true: there exist abelian topological groups (even monothetic groups, that is, topologically generated by one element) for which every unitary representation is trivial (sends the whole group to the identity), see [1, Theorem 5.1 and Remark 5.2]. Thus one may wonder what happens in the case of free topological groups: are they unitarily representable?

In the non-abelian case, this question is open even if X is compact metric, see [9, Questions 35, 36]. The aim of the present note is to answer the question in the positive for $L(X)$ and $A(X)$.

Theorem 1.1. *For every Tikhonov space X the free locally convex space $L(X)$ and the free abelian topological group $A(X)$ admit a topologically faithful unitary representation.*

It suffices to consider the case of $L(X)$, since $A(X)$ is isomorphic to the subgroup of $L(X)$ generated by X [12], [15], see also [5][1]. For compact X (or, more generally, for k_ω-spaces X) Theorem 1.1 is due to Jorge Galindo [4]. However, it was claimed in an early version of [4] that there exists a metrizable space X for which the group $A(X)$ is not isomorphic to a subgroup of a unitary group. This claim was wrong, as Theorem 1.1 shows.

It is known that (the additive group of) the space $L^1(\mu)$ is unitarily representable for every measure space (Ω, μ). (For the reader's convenience, we remind the proof in Section 3, see Fact 3.5.) In particular, if μ is the counting measure on a set A, we see that the Banach space $l^1(A)$ of summable sequences is unitarily representable. Since the product of any family of unitarily representable groups is unitarily representable (consider the Hilbert sum of the spaces of corresponding representations), we see that Theorem 1.1 is a consequence of the following:

Theorem 1.2. *For every Tikhonov space X the free locally convex space $L(X)$ is isomorphic to a subspace of a power of the Banach space $l^1(A)$ for some A.*

For A we can take any infinite set such that the cardinality of every discrete family of non-empty open sets in X does not exceed $\mathrm{Card}\,(A)$.

Theorem 1.1 implies the following result from [5]: every Polish abelian group is the quotient of a closed abelian subgroup of the unitary group of a separable Hilbert space (the non-abelian version of the reduction is explained in Section 4, the argument for the abelian case is the same). It is an open question (A. Kechris) whether a similar assertion holds for non-abelian Polish groups, see [8, Section 5.2, Question 16], [9, Question 34].

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[1]The result was stated in [12], but the proof was incomplete. I gave a proof in [15] — not knowing that I was rediscovering the Wasserstein metric (which is also known under many other names: Monge – Kantorovich, Kantorovich – Rubinstein, transportation, Earth Mover's) and its basic properties, such as the Integer Value Property. The proof is reproduced in [5], where a historic account is given.

The *fine uniformity* μ_X on a Tikhonov space X is the finest uniformity compatible with the topology. It is generated by the family of all continuous pseudometrics on X. It is also the uniformity induced on X by the group uniformity of $A(X)$ or $L(X)$. A *fine uniform space* is a space of the form (X, μ_X). We note the following corollary of Theorem 1.2 which may be of some independent interest.

Corollary 1.3. *For every Tikhonov space X the fine uniform space (X, μ_X) is isomorphic to a uniform subspace of a power of a Hilbert space.*

This need not be true for uniform spaces which are not fine: many separable Banach spaces (for example, c_0 or l_p for $p > 2$) do not admit a uniform embedding in a Hilbert space [3, Chapter 8]. Since the countable power of an infinite-dimensional Hilbert space H uniformly embeds in H (use the fact that H uniformly embeds in the unit sphere of itself [3, Corollary 8.11], and uniformly embed the countable power of the unit sphere into the Hilbert sum of countably many copies of H), it easily follows that c_0 or l_p for $p > 2$ are not uniformly isomorphic to a subspace of a power of a Hilbert space.

To deduce Corollary 1.3 from Theorem 1.1, note that the left uniformity on the unitary group $U_s(H)$ is induced by the product uniformity of H^H, hence the same is true for the left uniformity on every unitarily representable group.

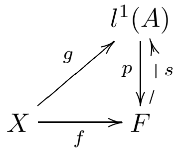
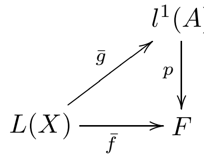
For any uniform space X one defines the free abelian group $A(X)$ and free locally convex space $L(X)$ in an obvious way. The objects $A(X)$ and $L(X)$ for Tikhonov spaces X considered in this paper are special cases of the same objects for uniform spaces, corresponding to fine uniform spaces.

Question 1.4 (Megrelishvili)**.** *For what uniform spaces X are the groups $A(X)$ and $L(X)$ unitarily representable? Is it sufficient that X be a uniform subspace of a product of Hilbert spaces?*

We prove Theorem 1.2 in Section 2. The proof depends on two facts: (1) every Banach space is a quotient of a Banach space of the form $l^1(A)$; (2) every onto continuous linear map between Banach spaces admits a (possibly non-linear) continuous right inverse. We remind the proof of these facts in Section 3.

2. Proof of Theorem 1.2

Let X be a Tikhonov space. Let $\mathcal{T}_0$ be the topology of the free locally convex space $L(X)$. Let $\mathcal{T}_1$ be the topology on $L(X)$ generated by the linear extensions of all possible continuous maps of X to spaces of the form $l^1(A)$. Theorem 1.2 means that $\mathcal{T}_1 = \mathcal{T}_0$. In order to prove this, it suffices to verify that $(L(X), \mathcal{T}_1)$ has the following universal property: for every continuous map $f : X \to F$, where F is a Hausdorff LCS, the linear extension of f, say $\bar{f} : L(X) \to F$, is $\mathcal{T}_1$-continuous. Since every Hausdorff LCS embeds in a product of Banach spaces, we may assume that F is a Banach space. Represent F as a quotient of $l^1(A)$ (Fact 3.1), let $p : l^1(A) \to F$ be linear and onto.

There exists a lift $g : X \to l^1(A)$ of f, i.e. such a map g for which $f = pg$. Indeed, find a non-linear right inverse $s : F \to l^1(A)$ of p (Fact 3.2), and set $g = sf$. Then $f = psf = pg$. By the definition of $\mathcal{T}_1$, the linear map $\bar{g} : L(X) \to l^1(A)$ is $\mathcal{T}_1$-continuous. Hence $\bar{f} = p\bar{g}$ is $\mathcal{T}_1$-continuous as well.

3. Basic facts: reminders

Fact 3.1. *Every Banach space X is a Banach quotient of a Banach space of the form $l^1(A)$.*

By a Banach quotient we mean a Banach space of the form E/F with the norm $\|x + F\| = \inf\{\|y\| : y \in x + F\}$.

Proof. Take for A a dense subset of the unit ball of X, and consider the natural map $p : l^1(A) \to X$. □

Fact 3.2 (the Bartle–Graves theorem [13, C.1.2])**.** *Every linear onto map $p : E \to F$ between Banach spaces (or locally convex Fréchet spaces) has a (possibly non-linear) continuous right inverse $s : F \to E$, i.e. such a map that $ps = 1_F$.*

Proof. This is a consequence of Michael's Selection Theorem for convex-valued maps [7, 13]. The theorem reads as follows. Suppose X is paracompact, E is a locally convex Fréchet space, and for every $x \in X$ a closed convex non-empty set $\Phi(x) \subset E$ is given. Suppose further that Φ is *lower semicontinuous*: for every U open in E, the set $\Phi^{-1}(U) = \{x \in X : \Phi(x) \text{ meets } U\}$ is open in X. Then Φ has a continuous selection: there exists a continuous map $s : X \to E$ such that $s(x) \in \Phi(x)$ for every $x \in X$.

To prove Fact 3.2, apply Michael's selection theorem to $X = F$ and Φ defined by $\Phi(x) = p^{-1}(x)$. The lower semicontinuity of Φ follows from the Open Mapping theorem, according to which p is open.

For various proofs of Michael's selection theorem, see [13]. In particular, note a nice reduction of the convex-valued selection theorem to the zero-dimensional selection theorem [13, A, §3], based on the notion of a Milyutin map. □

For a complex matrix $A = (a_{ij})$ we denote by A^* the matrix $(\bar{a}_{ji})$. A is *Hermitian* if $A = A^*$. A Hermitian matrix A is *positive* if all the eigenvalues of A are ≥ 0 or, equivalently, if $A = B^2$ for some Hermitian B. A complex function p on a group G is *positive-definite*, or *of positive type*, if $p(g^{-1}) = \overline{p(g)}$ for every $g \in G$, and for every $g_1, \dots, g_n \in G$ the Hermitian $n \times n$-matrix $(p(g_i^{-1}g_j))$ is positive. If $f : G \to U(H)$ is a unitary representation of G, then for every vector $v \in H$ the function $p = p_v$ on G defined by $p(g) = (gv, v)$ is positive-definite. (We denote by (x, y) the scalar product of $x, y \in H$.) Conversely, let p be a positive-definite function on G. Then $p = p_v$ for some unitary representation $f : G \to U(H)$ and some $v \in H$. Indeed, consider the group algebra $\mathbb{C}[G]$, equip it with the scalar product defined by $(g, h) = p(h^{-1}g)$, quotient out the kernel, and take the completion for H. The regular representation of G on $\mathbb{C}[G]$ gives rise to a unitary representation on H with the required property. (This is the so-called *GNS-construction*, see e.g. [2, Theorem C.4.10] for more details.) If G is a topological group and the function p is continuous, the resulting unitary representation is continuous as well. These arguments yield the following (see e.g. [16, Proposition 2.1]):

Fact 3.3. *A topological group G is unitarily representable (in other words, is isomorphic to a subgroup of $U_s(H)$ for some Hilbert space H) if and only if for every neighbourhood U of the neutral element e of G there exist a continuous positive-definite function $p : G \to \mathbb{C}$ and $a > 0$ such that $p(e) = 1$ and $|1 - p(g)| > a$ for every $g \in G \setminus U$.*

For a measure space (Ω, μ) we denote by $L^1(\mu)$ the complex Banach space of (equivalence classes of) complex integrable functions, and by $L^1_{\mathbf{R}}(\mu)$ the real Banach space of (equivalence classes of) real integrable functions.

Fact 3.4 (Schoenberg [10, 11])**.** *If (Ω, μ) is a measure space and $X = L^1_{\mathbf{R}}(\mu)$, the function $x \mapsto \exp(-\|x\|)$ on X is positive-definite. In other words, for any $f_1, \dots, f_n \in L^1_{\mathbf{R}}(\mu)$ the symmetric real matrix $(\exp(-\|f_i - f_j\|))$ is positive.*

Proof. We invoke Bochner's theorem: positive-definite continuous functions on $\mathbf{R}^n$ (or any locally compact abelian group) are exactly the Fourier transforms of positive measures. For $f \in L^1(\mathbf{R}^n)$ we define the Fourier transform $\hat{f}$ by

$$\hat{f}(y) = \int_{\mathbf{R}^n} f(x) \exp\left(-2\pi i(x, y)\right) dx.$$

Here $(x, y) = \sum_{k=1}^n x_k y_k$ for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

The positive functions p and q on $\mathbf{R}$ defined by $p(x) = \exp(-|x|)$ and $q(y) = 2/(1 + 4\pi^2 y^2)$ are the Fourier transforms of each other. Hence each of them is positive-definite. Similarly, the positive functions p_n and q_n on $\mathbf{R}^n$ defined by $p_n(x_1, \dots, x_n) = \exp(-\sum_{k=1}^n |x_k|) = \prod_{k=1}^n p(x_k)$ and $q_n(y_1, \dots, y_n) = \prod_{k=1}^n q(y_k)$ are positive-definite, being the Fourier transforms of each other. If $m_1, \dots, m_n$ are strictly positive masses, the function $x \mapsto \exp(-\sum_{k=1}^n m_k |x_k|)$ on $\mathbf{R}^n$ is positive-definite, since it is the composition of p_n and a linear automorphism of $\mathbf{R}^n$. This is exactly Fact 3.4 for finite measure spaces.

The general case easily follows: given finitely many functions $f_1, \dots, f_n \in L^1_{\mathbf{R}}(\mu)$, we can approximate them by finite-valued functions. In this way we see that the symmetric matrix $A = (\exp(-\|f_i - f_j\|))$ is in the closure of the set of matrices A' of the same form arising from finite measure spaces. The result of the preceding paragraph means that each A' is positive. Hence A is positive. □

As a topological group, $L^1(\mu)$ is isomorphic to the square of $L^1_{\mathbf{R}}(\mu)$. Combining Facts 3.3 and 3.4, we obtain:

Fact 3.5. *The additive group of the space $L^1(\mu)$ is unitarily representable for every measure space (Ω, μ).*

See [3, 6] for more on unitarily and reflexively representable Banach spaces.

4. Open questions

Let us say that a metric space M is of *L^1-type* if it is isometric to a subspace of the Banach space $L^1_{\mathbf{R}}(\mu)$ for some measure space (Ω, μ). A non-abelian version of Theorems 1.1 and 1.2 might be the following:

Conjecture 4.1. *For any Tikhonov space X the free topological group $F(X)$ is isomorphic to a subgroup of the group of isometries* $\mathrm{Iso}\,(M)$ *for some metric space M of L^1-type.*

It follows from [16, Theorem 3.1] (apply it to the positive-definite function p on $\mathbf{R}$ used in the proof of Fact 3.4 and defined by $p(x) = \exp(-|x|))$ and Fact 3.4 that for every $M \subset L^1_{\mathbf{R}}(\mu)$ the group $\mathrm{Iso}\,(M)$ is unitarily representable. Thus, if conjecture 4.1 is true, every $F(X)$ is unitarily representable. This would imply a positive answer to the question of Kechris mentioned in Section 1: is every Polish group a quotient of a closed subgroup of the unitary group of a separable Hilbert space? Indeed:

Proposition 4.2. *Let P be the space of irrationals. If the group $F(P)$ is unitarily representable, then every Polish group is a quotient of a closed subgroup of the unitary group of a separable Hilbert space.*

Proof. A topological group is *uniformly Lindelöf* (or, in another terminology, *ω-bounded*) if for every neighbourhood U of the unity the group can be covered by countably many left (equivalently, right) translates of U. If G is a uniformly Lindelöf group of isometries of a metric space M, then for every $x \in M$ the orbit Gx is separable (see e.g. the section "Guran's theorems" in [14]). If G is a uniformly Lindelöf subgroup of the unitary group $U_s(H)$, where H is a (non-separable) Hilbert space, it easily follows that H is covered by separable closed G-invariant linear subspaces and therefore G embeds in a product of unitary groups of separable Hilbert spaces.

If G is a Polish group, there exists a quotient onto map $F(P) \to G$ (because there exists a continuous open onto map $P \to G$, see Lemma 4.3 below). The group $F(P)$, like any separable topological group, is uniformly Lindelöf. Assume that $F(P)$ is unitarily representable. Then, as we saw in the first paragraph of the proof, $F(P)$ is isomorphic to a topological subgroup of a power of $U_s(H)$, where H is a separable Hilbert space. An easy factorization argument (see Lemma 4.4) shows that there is a group N lying in a countable power of $U_s(H)$ (and hence isomorphic to a subgroup of $U_s(H)$) such that G is a quotient of N. The quotient homomorphism $N \to G$ can be extended over the closure of N, so we may assume that N is closed in $U_s(H)$. □

The following lemmas were used in the proof above:

Lemma 4.3. *For every non-empty Polish space X there exists an open onto map $P \to X$, where P, as above, is the space of irrationals.*

Proof. Consider open covers $\{\mathcal{U}_n\}$ of X such that:

- $\operatorname{diam} U < 2^{-n}$ for every $U \in \mathcal{U}_n$;
- each $\mathcal{U}_n$ is indexed by $A_n = \mathbb{N}^n$;
- if $t \in A_n$, then $U_t = \bigcup\{U_s : s \in A_{n+1} \text{ and } t = s|n\}$
- if $s \in A_{n+1}$ and $t = s|n$, then $\overline{U_s} \subset U_t$.

For every infinite sequence $s \in \mathbb{N}^{\mathbb{N}}$ let x_s be the only point in the intersection $\bigcap U_{s|n} = \bigcap \overline{U_{s|n}}$. Then the map $s \mapsto x_s$ from $\mathbb{N}^{\mathbb{N}}$ (which is homeomorphic to P) to X is open and onto. □

Lemma 4.4. *Let $\{G_\alpha : \alpha \in A\}$ be a family of topological groups, K a subgroup of $\prod G_\alpha$, H a metrizable topological group, $f : K \to H$ a continuous homomorphism. Then there exists a countable subset $B \subset A$ such that $f = g \circ p_B$ for some continuous homomorphism $g : K_B \to H$, where $p_B : K \to \prod\{G_\alpha : \alpha \in B\}$ is the projection and $K_B = p_B(K)$. If f is open and onto, then so is g.*

Proof. Let $\mathcal{B}$ be a countable base at unity of H. For each $U \in \mathcal{B}$ pick a finite set $F = F_U \subset A$ such that $p_F^{-1}(V) \subset f^{-1}(U)$ for some neighbourhood V of unity of $\prod\{G_\alpha : \alpha \in F\}$. Put $B = \bigcup\{F_U : U \in \mathcal{B}\}$. □

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References

[1] W. Banaszczyk, *On the existence of exotic Banach – Lie groups,* Math. Ann. **264** (1983), 485–493.

[2] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan property (T),* available at http://poncelet.sciences.univ-metz.fr/~bekka/.

[3] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis, Vol. 1*, AMS Colloquium Publications, vol. 48 (AMS, Providence, RI, 2000).

[4] J. Galindo, *On unitary representability of topological groups,* arXiv:math.GN/0607193.

[5] S. Gao and V. Pestov, *On a universality property of some abelian Polish groups*, Fund. Math. **179** (2003), 1–15; arXiv:math.GN/0205291.

[6] M. Megrelishvili, *Reflexively but not unitarily representable topological groups,* Topology Proceedings **25** (2000), 615–625.

[7] E. Michael, *Continuous selections I,* Ann. Math. **63** (1956), 361–382.

[8] V. Pestov, *Dynamics of infinite-dimensional groups and Ramsey-type phenomena,* (Publicações dos Colóquios de Matemática, IMPA, Rio de Janeiro, 2005).

[9] V. Pestov, *Forty-plus annotated questions about large topological groups*, arXiv:math.GN/0512564.

[10] I. J. Shoenberg, *On certain metric spaces arising from Euclidean spaces by a change of metric and their imbedding in Hilbert space*, Ann. Math. **38** (1937), 787–793.

[11] I. J. Schoenberg, *Metric spaces and positive definite functions*, Trans. Amer. Math. Soc. **44** (1938), 522–536.

[12] M. G. Tkachenko, *On completeness of free abelian topological groups,* Soviet Math. Dokl. **27** (1983), 341–345.
[13] D. Repovš and P. V. Semenov, *Continuous selections of multivalued mappings,* Kluwer Academic Publishers, Dordrecht–Boston–London, 1998.
[14] V. V. Uspenskii, *Why compact groups are dyadic*, in book: *General Topology and its relations to modern analysis and algebra VI: Proc. of the 6th Prague topological Symposium 1986*, Frolik Z. (ed.) (Berlin: Heldermann Verlag, 1988), 601–610.
[15] V. V. Uspenskiĭ, *Free topological groups of metrizable spaces*, Izvestiya Akad. Nauk SSSR, Ser. Matem. **54** (1990), 1295–1319; English transl.: Math. USSR-Izvestiya **37** (1991), 657–680.
[16] V. V. Uspenskij, *On unitary representations of groups of isometries,* in book: *Contribuciones Matemáticas. Homenaje al profesor Enrique Outerelo Domínguez,* E. Martin-Peinador (ed.), Universidad Complutense de Madrid, 2004, 385–389; arXiv:math.RT/0406253.

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