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A note on a fixed point theorem for ray oriented maps

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Abstract. In this paper, we will prove a fixed point theorem for a ray-oriented map defined on a nonempty closed bounded convex subset of a Banach space.

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Notations

Let X be a Banach space and K be a non-empty subset of X. Let $T : K \to K$ be a mapping. Let R_x be the ray passing through the segment $< x, Tx >$ and so $R_x := \{(1-\lambda)x + \lambda Tx : \lambda \in R\}$. Let $< x, y >$ be defined to be as $\{(1-\lambda)x + \lambda y : \lambda \in [0,1]\}$ and $(x, y) := \{(1-\lambda)x + \lambda y : \lambda \in (0,1)\}$.
For any $x_1, x_2 \in R_x$, we say that $x_1 \leq x_2$ whenever $\lambda_1 \leq \lambda_2$ where $x_1 = (1-\lambda_1)x + \lambda_1 Tx$ and $x_2 = (1-\lambda_2)x + \lambda_2 Tx$ for some $\lambda_1 , \lambda_2 \in R$.

1. Introduction

Let X be a normed linear space and let K be a nonempty closed bounded convex subset of X. Suppose $T : K \to K$ is a mapping satisfying the following conditions: (i). For some element x_0 of K, $R_{x_0} \bigcap K$ is invariant under T and (ii). For each element $x \in R_{x_0} \bigcap K$, $T| < x, Tx > \bigcap K$ is continuous. Then, we will prove that there exists $y_0 \in R_{x_0} \bigcap K$ such that $< y_0, Ty_0 > \subseteq R_{x_0} \bigcap K$ is invariant under T.
Moreover,the above theorem will be followed by a corollary as in the following: Suppose $T : [a,b] \to [a,b]$ is a mapping where $a, b \in \Re$. If for each $x \in [a,b]$, the map T restricted to the segment joining x and Tx is continuous. Then we will prove that there exists an invariant interval under T and so it will have a fixed point in $[a,b]$. This result extends one dimensional Brouwer's result for a larger class of mappings which need not be continuous. Also one can find some similar treatment for the convergence of fixed point in the real line by Beardon [1]. For further important fixed point results one can refer to [2].

2. Main Results

Theorem 2.1. *Let X be a normed linear space and let K be a nonempty closed bounded convex subset of X. Suppose $T : K \to K$ is a mapping satisfying the following conditions:*

(1) *For some element x_0 of K, $R_{x_0} \bigcap K$ is invariant under T and*

(2) *For each element $x \in R_{x_0} \bigcap K$, $T| < x, Tx > \bigcap K$ is continuous*

Then, T has a fixed point in $R_{x_0} \bigcap K$.

Note: When we say $T \mid < x, Tx >$ is continuous, we mean that T is right continuous at x and left continuous at Tx if $x < Tx$.

Proof. Assume that the conclusion of the theorem is false. That is, T does not have a fixed point in $R_{x_0} \bigcap K$. Therefore, for every
$b \in R_{x_0} \bigcap K$, $< b, Tb >$ is not invariant under T.
Fix $y_0 \in R_{x_0} \bigcap K$ and let $x_0 \in < y_0, Ty_0 >$ such that $Tx_0 \notin < y_0, Ty_0 >$.
Let $G_{x_0} = R_{x_0} \bigcap K$. Now we can easily prove that
$A = \{\lambda \in R : (1 - \lambda)x_0 + \lambda Tx_0 \in K\}$ is bounded.

Let $\alpha = \inf A$ and $\beta = \sup A$. Let $a = (1-\alpha)x_o + \alpha Tx_o$ and $b = (1-\beta)x_o + \beta Tx_o$. Therefore, there exists a sequence $\{\alpha_n\} \in A$ such that $\{\alpha_n\}$ converges to α. Hence $(1-\alpha_n)x_o + \alpha_n Tx_o$ converges to $(1-\alpha)x_o + \alpha Tx_o$. Therefore it is easy to see that $a \in G_{x_o}$ and $b \in G_{x_o}$. Hence $G_{x_o} = \{(1-\lambda)a + \lambda b : 0 \le \lambda \le 1\}$. Now, define a map

$$g : G_{x_0} \longrightarrow G_{x_0}$$

by

$$g(z) = \begin{cases} x_0 & \text{if } z \le x_0, \\ z & \text{if } z \in (x_0, Tx_0), \\ Tx_0 & \text{if } z \ge Tx_0, . \end{cases}$$

Since g and T are continuous,

$$goT :< x_0, Tx_0 > \longrightarrow < x_0, Tx_0 >$$

is also continuous.
Hence the map goT has a fixed point $z_0 \in < x_0, Tx_0 >$.
Case 1: $z_0 = x_0$ Then

$$x_0 = z_0 = goT(z_0) = goT(x_0) = g(Tx_0) = Tx_0.$$

Hence $x_0 = Tx_0$, contradicting our assumption.
Case 2: $z_0 \in (x_0, Tx_0)$. If $Tz_0 \le x_0$, then $z_0 = (goT)(z_0) = g(Tz_0) = x_0$, contradicting $z_0 \in (x_0, Tx_0]$.
If $Tz_0 \in < x_0, Tx_0 >$, then $z_0 = (goT)(z_0) = g(Tz_0) = Tz_0$, again contradicting our assumption, $< z_0, Tz_0 >$ is not invariant under T. Therefore,

$$Tz_0 \ge Tx_0 \tag{2.1}$$

That is ,

$$z_0 = goT(z_0) = g(Tz_0) = Tx_0 \tag{2.2}$$

Substituting (2.2) in (2.1) we get

$$T^2x_0 \geq Tx_0 \tag{2.3}$$

Now let us construct $B = \{x \in R_{x_0} \bigcap K : x < Tx < T^2x\}$.
Moreover it is bounded above and so it must have a least upper bound. Therefore let u be the least upper bound of B.
Then there exists $x_n \in B$ such that $x_n \to u$.
Suppose $Tu < u$, then there exists a positive integer N such that for all $n \geq N$, $x_n \in < u, Tu >$. Then since $T| < u, Tu >$ is continuous, $Tx_n \to Tu$. Since $x_n < Tx_n$, $u \leq Tu$, a contradiction. Therefore, $u \leq Tu$.
Since $T| < u, Tu >$ is not invariant, by 2.3 we have $T^2u \geq Tu$.
Therefore, $u < Tu < T^2u$. Hence $u \in B$.
But again, $T^3u \geq T^2u$. Therefore, $u < Tu < T^2u < T^3u$.
Hence $Tu \in B$, which is a contradiction.
Therefore there exists a $y_0 \in R_{x_0} \bigcap K$ such that $T| < y_0, Ty_0 >$ is invariant. Hence T has a fixed point in $< y_0, Ty_0 >$. □

Corollary 2.2. *Suppose $T : [a, b] \to [a, b]$ is a mapping where $a, b \in \Re$. For each element $x \in [a, b]$, $T \mid < x, Tx >$ is continuous. Then T has a fixed point in $[a, b]$.*

Remark 2.3. There exists a discontinuous mapping T satisfying the conditions of corollary 2.2. ($T : [0, 1] \to [0, 1]$ by $T(0) = 0$ and $T(x) = 1$ for $0 < x \leq 1$).

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