Applied General Topology
© Universidad Politécnica de Valencia
Volume 9, No. 2, 2008
pp. 169-175

Spread of balleans

M. Filali and I. V. Protasov

Abstract. A ballean is a set endowed with some family of balls in such a way that a ballean can be considered as an asymptotic counterpart of a uniform topological space. We introduce and study a new cardinal invariant called the spread of a ballean. In particular, we show that, for every ordinal ballean $\mathcal{B}$, spread of $\mathcal{B}$ coincides with density of $\mathcal{B}$.

2000 AMS Classification: 54A25, 54E25, 05A18.

Keywords: ballean, pseudodiscrete subset, density, cellularity, spread.

1. Introduction

A *ball structure* is a triplet $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called the *ball of radius* α *around* x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X, \alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radiuses.*

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^{\star}(x, \alpha) = \{y \in X : x \in B(y, \alpha)\},$$

$$B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure $\mathcal{B}$ is called a *ballean* (or a *coarse structure*) if

• for any $\alpha, \beta \in P$, there exist α', $\beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^{\star}(x, \alpha'), \quad B^{\star}(x, \beta) \subseteq B(x, \beta');$$

• for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma);$$

• for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$.

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \longrightarrow X_2$ is called a $\prec$- *mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that $f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$ for every $x \in X$. If f is a bijection such that f

and f^{-1} are $\prec$- mappings, we say that f is an *asymorphism.* If $X_1 = X_2$ and the identity mapping $id: X_1 \longrightarrow X_2$ is an asymorphism, we identify $\mathcal{B}_1$ and $\mathcal{B}_2$, and write $\mathcal{B}_1 = \mathcal{B}_2$. For each ballean $\mathcal{B} = (X, P, B)$, replacing every ball $B(x, \alpha)$ with $B(x, \alpha) \bigcap B^*(x, \alpha)$, we obtain the same ballean, so every ballean can be determined in such a way that $B(x, \alpha) = B^*(x, \alpha)$ for all $x \in X$, $\alpha \in P$.

For motivations for the study of balleans as the asymptotic counterparts of the uniform topological spaces see [1, 5, 6].

2. Spread and density

Let $\mathcal{B} = (X, P, B)$ be a ballean. A subset $V \subseteq X$ is called *bounded* if there exist $x \in X$ and $\alpha \in P$ such that $V \subseteq B(x, \alpha)$. A ballean is called bounded if its support X is bounded.

Given a ballean $\mathcal{B} = (X, P, B)$, we say that a subset $Y \subseteq X$ is *pseudodiscrete*, if for every $\alpha \in P$, there exists a bounded subset V of X such that $B(y, \alpha) \bigcap Y = \{y\}$ for every $y \in Y \setminus V$. A ballean $\mathcal{B}$ is called pseudodiscrete if its support X is pseudodiscrete.

For every subset $Y \subseteq X$, we put

$$\mid Y \mid_{\mathcal{B}} = \min\{|Y \backslash V| : V \text{ is a bounded subset of } X\},$$

and introduce a new cardinal invariant

$$spread(\mathcal{B}) = \sup\{|Y|_{\mathcal{B}} : Y \text{ is a pseudodiscrete subset of } X\}.$$

We note that $|Y|_{\mathcal{B}} = 0$ if and only if Y is bounded, so $spread(\mathcal{B}) = 0$ for every bounded ballean.

A subset L of X is called *large* if there exists $\alpha \in P$ such that $X = B(L, \alpha)$. The *density* of $\mathcal{B}$ is defined in [4] by

$$den(\mathcal{B}) = \min\{|L| : L \text{ is a large subset of } X\}.$$

Clearly, $den(\mathcal{B}) = 1$ for every bounded ballean $\mathcal{B}$, and $den(\mathcal{B})$ is an infinite cardinal for every unbounded ballean $\mathcal{B}$.

Proposition 2.1. *For every ballean $\mathcal{B}$, we have $spread(\mathcal{B}) \leq den(\mathcal{B})$.*

Proof. Let L be a large subset of X and Y be a pseudodiscrete subset of X. It suffices to show that $|L| \geq |Y \backslash V|$ for some bounded subset V of X. We may suppose that $B(x, \alpha) = B^*(x, \alpha)$ for all $x \in X$, $\alpha \in P$. We take $\beta \in P$ such that $X = B(L, \beta)$, and choose $\gamma \in P$ such that $B(B(x, \beta), \beta) \subseteq B(x, \gamma)$ for each $x \in X$. Since Y is pseudodiscrete, there exists a bounded subset V of X such that $B(y, \gamma) \bigcap Y = \{y\}$ for each $y \in Y \setminus V$. By the choice of γ, the family $\{B(y, \beta) : y \in Y \setminus V\}$ is disjoint. Since $B(x, \beta) \bigcap L \neq \varnothing$ for each $x \in X$, we have $L \bigcap B(y, \beta) \neq \varnothing$ for each $y \in Y \setminus V$. Hence $|L| \geq |Y \setminus V|$, as required. □

In the next example, for every infinite cardinal γ, we construct a ballean $\mathcal{B}$ such that $den(\mathcal{B}) = \gamma$ but $spread(\mathcal{B}) = 0$.

Example 2.2. Let X be a set of cardinality γ, κ be an infinite regular cardinal such that $\kappa \leq \gamma$. We denote by $\mathcal{F}$ the family of all subsets of X of cardinality $< \kappa$. Let P be the set of all mappings $f : X \longrightarrow \mathcal{F}$ such that, for every $x \in X$, we have $x \in f(x)$ and

$$|\{y \in X : x \in f(y)\}| < \kappa.$$

Given any $x \in X$ and $\alpha \in P$, we put $B(x, f) = f(x)$ and note that the ball structure $\mathcal{B} = (X, P, B)$ is a ballean. We need the regularity of κ to state that $|B(B(x, f), g)| < \kappa$ for all $x \in X$ and $f, g \in P$.

Note that a subset V of X is bounded if and only if $|V| < \kappa$; and a subset L of X is large if and only if $|L| = \gamma$ implying that $den(\mathcal{B}) = \gamma$.

Now we check that $spread(\mathcal{B}) = 0$. To this end, we take an arbitrary subset Y of X such that $|Y| \geq \kappa$, write it as $Y = \{y_\lambda : \lambda \in |Y|\}$, and define a mapping $f : X \longrightarrow \mathcal{F}$ by the rule: $f(y_\lambda) = \{y_\lambda, y_{\lambda+1}\}$ for each $\lambda < |Y|$, and $f(x) = \{x\}$ for each $x \in X \setminus V$. Then $f \in P$ and $\mid B(y, f) \bigcap Y \mid= 2$ for every $y \in Y$, so Y is not pseudodiscrete and $spread(\mathcal{B}) = 0$.

For every ballean $\mathcal{B} = (X, P, B)$, we use the preodering $\leq$ on X defined by the rule: $\alpha \leq \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that $\alpha' \geq \alpha$. The *cofinality* $cf(\mathcal{B})$ is the minimal cardinality of the cofinal subsets of P.

A ballean $\mathcal{B}$ is called *ordinal* if P contains a cofinal subset of P' which is well-ordered by $\leq$. Replacing P' with its minimal cofinal subset, we get the same ballean. Hence, we can write $\mathcal{B}$ as (X, p, B), where p is a regular cardinal (considered as a set of ordinals).

Let (X, d) be a metric space. For all $x \in X$ and $r \in \mathbb{R}^+$, we put $B_d(x, r) = \{y \in X : d(x, y) \leq r\}$ and get the *metric ballean* $\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_d)$. Clearly, every metric ballean is ordinal.

We shall show in Theorem 2.3 that $spread(\mathcal{B}) = den(\mathcal{B})$ for every unbounded ordinal ballean $\mathcal{B}$. To this end we use another cardinal invariant of a ballean $\mathcal{B} = (X, P, B)$, the cellularity of $\mathcal{B}$, defined in [4]. A subset Y of X is called *thick* if, for every $\alpha \in P$, there exists $y \in Y$ such that $B(y, \alpha) \subseteq Y$. The *cellularity* of $\mathcal{B}$ is the cardinal

$$cell(\mathcal{B}) = \sup\{|\mathcal{F}| : \mathcal{F} \text{ is a disjoint family of thick subsets of } X\}.$$

By [4, Theorem 1], for every ordinal ballean $\mathcal{B}$, we have $cell(\mathcal{B}) = den(\mathcal{B})$ and there exists a disjoint family $\mathcal{F}$ of cardinality $den(\mathcal{B})$ consisting of thick subsets of X. For every infinite cardinal κ, there exists a metric ballean $\mathcal{B}$ with $den(\mathcal{B}) = \kappa$ (see [4, Example 1]).

Theorem 2.3. *For every unbounded ordinal ballean $\mathcal{B}$ with support X, we have $spread(\mathcal{B}) = den(\mathcal{B})$ and there exists a subset Y of X such that $|Y|_{\mathcal{B}} = |Y| = den(\mathcal{B})$.*

Proof. Let $\mathcal{B} = (X, \rho, B)$ where ρ is an infinite regular cardinal, $\kappa = den(\mathcal{B})$ and $cf(\kappa)$ be the cofinality of κ. Let $\{F_\lambda : \lambda \in \kappa\}$ be a disjoint family of thick subsets of X and put $F = \bigcup_{\lambda\in\kappa} F_\lambda$. We fix some element $x_0 \in X$ and consider four cases.

Case $\rho < cf(\kappa)$: We prove the following auxiliary statement. For every $\alpha \in \rho$, there exists a bounded subset Z of F such that $|Z| = \kappa$ and the family $\{B(z, \alpha) : z \in Z\}$ is disjoint.

For every $\lambda \in \kappa$, we take $y(\lambda) \in F_\lambda$ such that $B(y(\lambda), \lambda) \subseteq F_\lambda$, and pick $f(\lambda) \in \rho$ such that $y(\lambda) \in B(x_0, f(\lambda))$.

Since f maps κ to ρ, by assumption, there exist a subset $\bigwedge \subseteq \kappa$ and $\beta \in \rho$ such that $|\bigwedge| = \kappa$ and $f(\lambda) = \beta$ for every $\lambda \in \bigwedge$. Put $Z = \{y(\lambda) : \lambda \in \bigwedge\}$.

Using the auxiliary statement, we can construct inductively a family $\{Y_\alpha : \alpha \in \rho\}$ of bounded subsets of F such that the family $\{B(y, \alpha) : y \in Y_\alpha\}$ is disjoint for each $\alpha \in \rho$, $B(Y_\alpha, \alpha) \bigcap B(Y_{\alpha'}, \alpha') = \varnothing$ for all distinct $\alpha, \alpha' \in \rho$, and $|Y_\alpha| = \kappa$ for every $\alpha \in \rho$. Put $Y = \bigcup_{\alpha\in\rho} Y_\alpha$.

Case $cf(\kappa) \leq \rho < \kappa$: Using the assumption and repeating the arguments from the previous case, we get the following auxiliary statement. For every $\alpha \in \rho$ and every cardinal $\kappa' < \kappa$, there exists a bounded subset Z of F such that $|Z| > \kappa'$ and the family $\{B(z, \alpha) : z \in Z\}$ is disjoint.

Using the auxiliary statement, we can construct inductively a family $\{Y_\alpha : \alpha \in \rho\}$ of bounded subsets of F and an increasing sequence $(\kappa_\alpha)_{\alpha\in\rho}$ of cardinals such that $\kappa = \sup\{\kappa_\alpha : \alpha \in \rho\}$, the family $\{B(y, \alpha) : y \in Y_\alpha\}$ is disjoint for each $\alpha \in \rho$, $B(Y_\alpha, \alpha) \bigcap B(Y_{\alpha'}, \alpha') = \varnothing$ for all distinct $\alpha, \alpha' \in \rho$, and $|Y_\alpha| \geq \kappa_\alpha$ for every $\alpha \in \rho$. We put $Y = \bigcup_{\alpha\in\rho} Y_\alpha$.

Case $\rho = \kappa$: For every $\alpha \in \rho$, we take $y_\alpha \in F_\alpha$ such that $B(y_\alpha, \alpha) \subseteq F_\alpha$. Put $Y = \{y_\alpha : \alpha \in \rho\}$.

Case $\rho > \kappa$: This variant is impossible, see *Case* $\rho > \kappa$ in the proof of Theorem 1 from [4]. □

By definition, a $G-$*space* is a set X endowed with a (left) action

$$G \times X \longrightarrow X,\ (g, x) \longmapsto g(x)$$

of a group G with identity e such that $e(x) = x$ and $g(h(x)) = (gh)(x)$ for all $x \in X$ and $g, h \in G$.

Now let X be an infinite transitive $G-$space, i.e., for any $x, y \in X$, there exists $g \in G$ such that $g(x) = y$.

Let κ be an infinite cardinal such that $\kappa \leq |X|$, and consider

$$\mathcal{F}_\kappa = \{F \subseteq G : |F| < \kappa,\ e \in A\}.$$

For any $x \in X$ and $F \in \mathcal{F}_\kappa$, we put

$$B(x, F) = F(x) = \{f(x) : f \in F\},$$

and get the ballean $\mathcal{B}(X, \kappa) = (X, \mathcal{F}_\kappa, B)$. Let L be a large subset of X. We take $F \in \mathcal{F}_\kappa$ such that $B(L, F) = X$. Since $|F| < |X|$ and $|B(L, F)| \leq |L||F|$, we have $|L| = |X|$, so $den(\mathcal{B}(X, \kappa)) = |X|$.

Theorem 2.4. *Let X be an infinite transitive G-space, κ be an infinite cardinal such that $\kappa \leq \mid X \mid$. Then the following statements hold.*

(1) *If $G = X$ and $g(x) = gx$, then spread$(\mathcal{B}(X, \kappa)) = |X|$.*
(2) *If G is the group of all permutations of X, then spread$(\mathcal{B}(X, \kappa)) = 0$.*
(3) *Let ρ be an infinite cardinal such that $\rho \leq |X|$, and let G be the group of all permutations of X with $\operatorname{supp}(g) < \rho$, $g \in G$ where $\operatorname{supp}(g) = \{x \in X : g(x) \neq x\}$. If either $\rho < \kappa$, or $\rho = \kappa$ and κ is regular, then spread$(\mathcal{B}(X, \kappa)) = |X|$. If either $\rho > \kappa$, or $\rho = \kappa$ and κ is singular, then spread$(\mathcal{B}(X, \kappa)) = 0$.*

Proof. (1) By [2, Proposition 4.1], there exists a subset Y of G such that $|Y| = |G|$ and $|gY \bigcap Y| \leq 3$ for every $g \in G$, $g \neq e$. We take an arbitrary $F \in \mathcal{F}_\kappa$, and let $Z = FY \cap Y$. Since $|F| < \kappa$, the subset Z of Y satisfies $|Z| \leq 3|F| < \kappa$, i.e., Z is bounded, and

$$B(y, F) \bigcap Y = Fy \bigcap Y = \{y\}$$

for every $y \in Y \backslash Z$. It follows that Y is pseudodiscrete. Since $|Y|_{\mathcal{B}} = |Y|$, we have *spread*$(\mathcal{B}(X, \kappa)) = |X|$.

(2) It suffices to show that every subset $Y = \{y_\alpha : \alpha \in \lambda\}$ of X of cardinality $\lambda \geq \kappa$ is not pseudodiscrete. We say that an ordinal $\alpha \in \lambda$ is even if either α is a limit ordinal or $\alpha = \beta + n$ for some limit ordinal β and some even natural number n. Otherwise, we say that α is odd. Then we define a permutation f of X by the rule:

$$f(x) = \begin{cases} y_{\alpha+1} & \text{if } x = y_\alpha \text{ and } \alpha \text{ is even,} \\ y_{\alpha-1} & \text{if } x = y_\alpha \text{ and } \alpha \text{ is odd,} \\ x & \text{for each } x \in X \backslash Y. \end{cases}$$

We put $F = \{f, e\}$. Clearly, $|B(y_\alpha, F) \bigcap Y| = 2$ for every $\alpha \in \lambda$. Since Y is not bounded, Y is not pseudodiscrete.

(3) If either $\rho < \kappa$, or $\rho = \kappa$ and κ is regular, we take an arbitrary subset $F \in \mathcal{F}_\kappa$ and put

$$Z = \bigcup \{\operatorname{supp}(g) : g \in F\}.$$

Then $|Z| < \kappa$, so it is bounded, and clearly $B(x, F) \bigcap X = \{x\}$ for every $x \in X \setminus Z$. It follows that X is pseudodiscrete. Since $|X|_{\mathcal{B}} = |X|$ we conclude that *spread*$(\mathcal{B}(X, \kappa)) = |X|$.

If $\rho > \kappa$ we can use the arguments proving (2) to show that every subset of X of cardinality $\geq \kappa$ is not pseudodiscrete.

If $\rho = \kappa$ and κ is singular, we fix an arbitrary subset Y of X with $|Y| = \kappa$ and partition

$$Y = \bigcup \{Y_\beta \ : \ \beta \in cf(\kappa)\}$$

so that $|Y_\beta| < \kappa$ for every $\beta \in cf(\kappa)$. For every $\beta \in cf(\kappa)$, we fix a permutation f_β of X such that $\operatorname{supp}(f_\beta) = Y_\beta$, $f(Y_\beta) = Y_\beta$. Then we put

$$F = \{e\} \bigcup \{f_\beta : \beta \in cf(\kappa)\}$$

and note that $F \in \mathcal{F}_\kappa$ and $|B(y,F)\bigcap Y| \geq 2$ for every $y \in Y$. It follows that Y is not pseudodiscrete and $spread(\mathcal{B}(X,\kappa)) = 0$. □

Theorem 2.5. *For every unbounded pseudodiscrete ballean $\mathcal{B}$, we have*

$$den(\mathcal{B}) = spread(\mathcal{B}) \quad and \quad cell(\mathcal{B}) = 1.$$

Proof. Let X be the support of $\mathcal{B}$. By [5, Theorem 3.6], there exists a filter φ on X such that $\bigcap \varphi = \varnothing$ and $\mathcal{B} = (X, \varphi, B)$ where

$$B(x,F) = \begin{cases} \{x\}, & \text{if } \ x \in F; \\ X \setminus F, & \text{if } \ x \notin F \end{cases}$$

for all $x \in X$ and $F \in \varphi$. We put $\kappa = \min\{|F| : F \in \varphi\}$ and note that a subset L of X is large if and only if $L \in \varphi$ so $den(\mathcal{B}) = \kappa$. On the other hand, a subset V of X is bounded if and only if $X \backslash V \in \varphi$. Hence, $|X|_{\mathcal{B}} = \kappa$, and so $spread(\mathcal{B}) = \kappa$. We note also that every unbounded subset of X is thick. Hence, if φ is an ultrafilter then $cell(\mathcal{B}) = 1$. □

For every pair γ, λ of infinite cardinals with $\gamma < \lambda$, we construct next a ballean $\mathcal{B}$ such that $den(\mathcal{B}) = \gamma$ and $spread(\mathcal{B}) = \lambda$.

Example 2.6. We take a ballean $\mathcal{B}_1 = (X_1, P_1, B_1)$ such that $spread(\mathcal{B}_1) = 0$, $|X_1| = den(\mathcal{B}_1) = \gamma$ and each ball $\mathcal{B}_1(x_1, \alpha_1)$ is finite (see Example 2.2). Let $\mathcal{B}_2 = (X_2, P_2, B_2)$ be a pseudodiscrete ballean such that $spread(\mathcal{B}_2) = |X_2| = \lambda$. We consider the ballean $\mathcal{B} = (X, P, B)$ with $X = X_1 \times X_2$, $P = P_1 \times P_2$ and

$$B((x_1, x_2), (\alpha_1, \alpha_2)) = B(x_1, \alpha_1) \times B(x_2, \alpha_2).$$

Since $den(\mathcal{B}_1) = \gamma$ and $|X| = \gamma$, we see that $den(\mathcal{B}) = \gamma$. Since $spread(\mathcal{B}_2) = \lambda$, we see that $spread(\mathcal{B}) \geq \lambda$. Let now Z be any subset of X with $|Z| > \lambda$. Then there exist an infinite subset Y of X_1 and $a \in X_2$ such that $Y \times \{a\} \subseteq Z$. Since Y is not pseudodiscrete in $\mathcal{B}_1$, Z is not pseudodiscrete in $\mathcal{B}$. Hence, $spread(\mathcal{B}) = \lambda$.

We conclude the exposition with the following open questions.

Problem 2.7. *Given a ballean $\mathcal{B}$ with support X, does there exist a pseudodiscrete subset $Y \subseteq X$ such that $|Y|_{\mathcal{B}} = spread(\mathcal{B})$?*

Problem 2.8. *Let $\mathcal{B} = (X, P, B)$ be a ballean, $|X| = \kappa$ and let $|P| \leq \kappa$. Assume that there exists $\kappa' < \kappa$ such that $\mid B(x, \alpha) \mid \leq \kappa'$ for all $x \in X$, $\alpha \in P$. Is $spread(\mathcal{B}) = \kappa$? By* [4, Theorem 2(i)], *$den(\mathcal{B}) = cell(\mathcal{B}) = \kappa$.*

Problem 2.9. *Let $\mathcal{B} = (X, P, B)$ be a ballean, $|X| = \kappa$ and let $|P| \leq \kappa$. Assume that κ is regular and $|B(x, \alpha)| < \kappa$ for all $x \in X$, $\alpha \in P$. Is $spread(\mathcal{B}) = \kappa$? By* [4, Theorem 2(ii)], *$den(\mathcal{B}) = cell(\mathcal{B}) = \kappa$.*

References

[1] A. Dranishnikov, *Asymptotic topology*, Russian Math. Surveys **55** (2000), 1085–1129.
[2] C. Chou, *On the size of the set of left invariant means on a semigroup*, Proc. Amer. Math. Soc. **23** (1969), 199–205.
[3] I. V. Protasov, *Normal ball structures*, Math. Stud. **20** (2003), 3–16.
[4] I. V. Protasov, *Cellularity and density of balleans*, Appl. Gen. Topology **8** (2007), 283–291.
[5] I. V. Protasov and M. Zarichnyi, *General Asymptology*, Math. Stud. Monogr. Ser., VNTL, Lviv (2006).
[6] J. Roe, *Lectures on Coarse Geometry*, AMS University Lecture Ser., **31** (2003).

Received October 2006

Accepted February 2007

Mahmoud Filali (Mahmoud.Filali@oulu.fi)
Dept of Math. Sciences, University of Oulu, FIN - 90014, Oulu, Finland

Igor Protasov (islab@unicyb.kiev.ua)
Department of Cybernetics, Kyiv University, Volodimirska 64, Kyiv 01033, Ukraine