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Semigroups and their topologies arising from Green's left quasiorder

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Abstract. Given a semigroup $(S, \cdot)$, Green's left quasiorder on S is given by $a \leq b$ if $a = u \cdot b$ for some $u \in S^1$. We determine which topological spaces with five or fewer elements arise as the specialization topology from Green's left quasiorder for an appropriate semigroup structure on the set. In the process, we exhibit semigroup structures that yield general classes of finite topological spaces, as well as general classes of topological spaces which cannot be derived from semigroup structures via Green's left quasiorder.

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1. Introduction

The classification of finite semigroups of small order began over 50 years ago, with much emphasis on enumerating them with or without the aid of computers ([4], [11], [10]). Asymptotic results on the number of semigroups are given in [8], and methods for computing finite semigroups are discussed in [6]. In [2], Almeida points out that the theory of finite semigroups only developed as an independent field with its own set of problems and techniques after theoretical computer science provided strong motivation for the development of the discipline. One link noted in [2] is that the finite semigroups are exactly the transition semigroups of finite-state automata.

Inherently, computers can only deal with finite spaces, so computer implementation of the ideas of nearness and convergence require finite, non-Hausdorff topologies. The study of finite topologies often utilizes the connections with quasiorders (preorders) established by P. Alexandroff [1], and quasiorders are now prevalent in computer science (see [5]). Erné and Stege [3] provide an excellent foundational account of the enumeration of topologies on small sets and give an extensive bibliography.

The links between principal topologies and quasiorders via Alexandroff specialization and between semigroups and quasiorders via Green's relations are well-known. While direct links between principal topologies and underlying semigroups have received less attention, this paper provides a foundational direct link between finite topologies and semigroups by identifying all topological spaces on five or fewer elements that arise from semigroup structures in this context. With the techniques used, we are able to give general classes of topological spaces which allow or forbid corresponding semigroup structures.

2. Preliminaries

A *quasiorder* on a set X is a reflexive transitive relation on X. A quasiorder $\leq$ on X is a *total quasiorder* if every pair of elements $x, y \in X$ is comparable, that is, for any $x, y \in X$, either $x \leq y$ or $y \leq x$. A *partial order* (respectively, *total order*) is an antisymmetric quasiorder (respectively, an antisymmetric total quasiorder).

A topology on a set X is called a *principal topology* if any arbitrary intersection of open sets is open. Observe that in a principal topology on X, every element $x \in X$ has a minimal open neighborhood $M_x = \bigcap\{U : x \in U, U \text{open}\}$, and the collection $\{M_x : x \in X\}$ of minimal open neighborhoods forms a basis for the topology. Note that any finite topology is a principal topology.

Alexandroff showed in [1] that the quasiorders on a set X are equivalent to the principal topologies on X: If $\leq$ is a quasiorder on X, then a corresponding topology called the *specialization topology* on X is obtained by taking the minimal open neighborhood of $x \in X$ to be $M_x = \{y : y \leq x\}$. Conversely, for a given principal topology on X, we define the *specialization quasiorder* $\leq$ on X by taking $y \leq x$ if y is contained in every open set containing x. In other words, $y \leq x$ if y is contained in the minimal open neighborhood of x, or $y \leq x$ if x is in the closure of $\{y\}$.

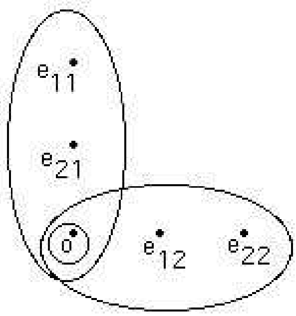
A *semigroup* $(S, \cdot)$ is a nonempty set S together with a binary associative operation $\cdot$. S is said to have an identity if there exists an element $e \in S$ such that $e \cdot s = s \cdot e = s$ for every $s \in S$. A semigroup is *commutative* if $a \cdot b = b \cdot a$ for every $a, b \in S$. A *subsemigroup* of a semigroup $(S, \cdot)$ is a subset A of S which is a semigroup under the operation inherited from S. If S is a semigroup, S^1 denotes the semigroup obtained from S by adjoining an identity, if necessary. That is, $S^1 = S$ if S has an identity and $S^1 = S \cup \{1\}$ with $1 \cdot s = s \cdot 1 = s$ for any $s \in S^1$ if S has no identity. For a semigroup $(S, \cdot)$, Green's left quasiorder $\leq_{\mathcal{L}}$ is given by $a \leq_{\mathcal{L}} b$ if $a = u \cdot b$ for some $u \in S^1$ (see [7]). This quasiorder in turn corresponds to a principal topology $\tau_{\mathcal{L}}$ on S. Note that the minimal open sets which form a basis for $\tau_{\mathcal{L}}$ are the principal left ideals $S^1 a = \{s \cdot a : s \in S^1\}$ $(a \in S)$ for the semigroup S. For a semigroup S and an element $x \in S$, we will denote the minimal open set of $\tau_{\mathcal{L}}$ containing x by M_x^S or, if the underlying semigroup is clear, by M_x. Hence, $M_x^S = M_x = S^1 x$, the principal left ideal of S generated by x, and $\{M_x^S : x \in S\}$ is a basis for $\tau_{\mathcal{L}}$.

3. General Remarks and Examples

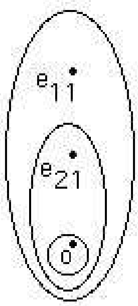
If $(S, \cdot)$ is a group, then every element is a left multiple of every other element, namely $b = (b \cdot a^{-1}) \cdot a$ for any $a, b \in S$ and hence the only minimal open set is S itself. It follows that every group structure on S yields the topology $\tau_{\mathcal{L}} = \tau_I$, the indiscrete topology on S. At the same time, a semigroup $(S, \cdot)$ defined by $a \cdot b = a$ for any $a, b \in S$ also yields the indiscrete topology. So, we see that many semigroup structures on S may yield the same topological structure on S. At the other extreme, for $\tau_{\mathcal{L}}$ to be discrete, the minimal left ideals of S all have to be singleton sets, forcing the only possible semigroup structure yielding the discrete topology to be $a \cdot b = b$ for any $a, b \in S$.

In view of these examples, it is natural to ask if all topological structures on a set X can be derived in this fashion from an appropriate semigroup structure on X, or if not all, which ones can. One of the difficulties in trying to answer this type question lies in the nature of the relationship between topological subspaces and subsemigroups as seen in the following example.

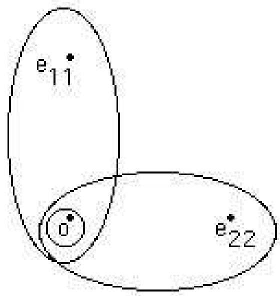
Example 3.1. Let $S = \{0, e_{11}, e_{12}, e_{21}, e_{22}\}$ be the subset of $M_2(\mathbb{Z}_2)$, the set of 2×2 matrices with entries in the integers modulo 2, consisting of the zero matrix 0 and the indicated basic matrices, where e_{ij} denotes the matrix which has entry 1 in the i-th row and j-th column and zero entries elsewhere. Then S is a semigroup under matrix multiplication. Furthermore, $V = \{0, e_{11}, e_{21}\}$ and $W = \{0, e_{11}, e_{22}\}$ are subsemigroups of S. The minimal open neighborhoods of $\tau_{\mathcal{L}}$ on S are easily seen to be $M_0^S = \{0\}$, $M_{e_{11}}^S = M_{e_{21}}^S = \{0, e_{11}, e_{21}\}$ and $M_{e_{12}}^S = M_{e_{22}}^S = \{0, e_{12}, e_{22}\}$ while the minimal open neighborhoods for $\tau_{\mathcal{L}}$ on V are $M_0^V = \{0\}$, $M_{e_{11}}^V = \{0, e_{11}, e_{21}\}$, and $M_{e_{21}}^V = \{0, e_{21}\}$. Finally, the minimal open neighborhoods of $\tau_{\mathcal{L}}$ on W are $M_0^W = \{0\}$, $M_{e_{11}}^W = \{0, e_{11}\}$, and $M_{e_{22}}^W = \{0, e_{22}\}$. The minimal open neighborhoods for S, V, and W are pictured below.



minimal open sets
for S



minimal open sets
for V



minimal open sets
for W

As V is an open set in S but the minimal open neighborhoods of V are not open sets in S, V is not a topological subspace of S. W is a topological subspace of S, but not an open subset of S.

Clearly, for any subsemigroup B of a semigroup A and any $x \in B$, we have $M_x^B \subseteq M_x^A$ as all left multiples of x in B are also left multiples of x in A. It is also easy to see that for a subset B of a semigroup A to be a subsemigroup of A it is sufficient that B be an open subspace of $(A, \tau_{\mathcal{L}})$. This condition is not necessary: W in the example above is not an open subspace of S but is a subsemigroup of S. Finally we observe that a subsemigroup B of a semigroup A is a subspace of A if and only if $Bb \cup \{b\} = (Ab \cup \{b\}) \cap B$ for each element $b \in B$, i.e., minimal open neighborhoods of B are the intersections of the minimal open neighborhoods of A with B.

4. Bases of Minimal Open Sets and the Quasiorder $\leq_E$

A principal topology on a set X is determined by the basis of minimal open neighborhoods $\{M_x : x \in X\}$. Using this basis we can put a total quasiorder on the elements of a finite topological space and will use this order to construct all possible such bases for sets having cardinality of less than or equal to five.

The following proposition completely describes when a collection of subsets of a set X constitutes a basis of minimal open sets for some principal topology on X.

Proposition 4.1. *Let X be a nonempty set and let $\mathcal{B} = \{B_i : i \in I\}$ be a collection of nonempty subsets of X. Then $\mathcal{B}$ is a basis consisting of all the minimal open neighborhoods for some principal topology on X if and only if*

(1) $\bigcup \mathcal{B} = X$.
(2) *For any subcollection $\mathcal{C} \subseteq \mathcal{B}$ and $x \in \bigcap \mathcal{C}$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq \bigcap \mathcal{C}$.*
(3) *For any $B \in \mathcal{B}$, $\mathcal{B}$ and $\mathcal{B} \setminus \{B\}$ are not equivalent bases; that is, $\mathcal{B} \setminus \{B\}$ is either not a basis for any topology on X or does not generate the same topology as $\mathcal{B}$.*

Proof. Since in a principal topology, arbitrary intersections of open sets are open, the first two conditions are equivalent to the collection $\mathcal{B}$ being a basis for a principal topology. If $\mathcal{B}$ is the collection of all minimal open neighborhoods then $B \in \mathcal{B}$ is a minimal neighborhood of some $x \in X$. So, there exists no open set A in $\mathcal{B} \setminus \{B\}$ such that $x \in A \subseteq B$, i.e., the open set B is not a union of sets in $\mathcal{B} \setminus \{B\}$ and hence $\mathcal{B} \setminus \{B\}$ cannot constitute a basis for a topology in which B is an open set.

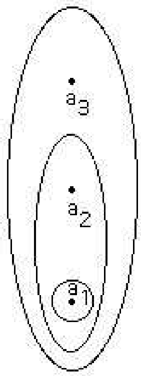
Conversely, suppose conditions 1)–3) hold, and hence $\mathcal{B}$ is a basis for some principal topology τ. Let $B \in \mathcal{B}$. If for every $x \in B$, $M_x \neq B$, where M_x denotes the minimal neighborhood of x in τ, then $B = \bigcup\{M_x : x \in B\}$, and each M_x in turn is a union of elements of $\mathcal{B}$. Hence $\mathcal{B} \setminus \{B\}$ and $\mathcal{B}$ are equivalent bases and condition 3) would be violated. □

It is obvious that two bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of minimal open neighborhoods for their respective topologies are equivalent if and only if $\mathcal{B}_1 = \mathcal{B}_2$.

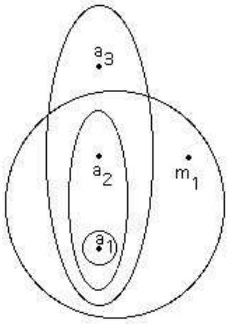
Let $E = \Pi_{i \in \mathbb{N}} \mathbb{N}_0$ be the set of all sequences with nonnegative integer terms. Then E is totally ordered by the lexicographic order and we can obtain a total quasiorder on a finite topological space X by associating a sequence $q(x)$ of E with every element $x \in X$ as follows: For a finite topological space X we define, for any $k \in \mathbb{N}$, $P_k = \{x \in X : |M_x| = k\}$ to be the set of all points in X whose minimal neighborhood has exactly k elements. The collection of nonempty P_k's forms a partition of X. Furthermore, for any $x \in P_t$ we can write $M_x = \{x = x_1, x_2, \ldots, x_t\}$ with $x_i \in P_{k_i}$ and $k_{i+1} \leq k_i$ for all i. We associate with x the sequence $q(x) = (k_1, k_2, \ldots, k_t, 0, 0, \ldots)$ in E, which we will abbreviate as $q(x) = (k_1, k_2, \ldots, k_t)$. Here $k_1 = t = |M_x|$. For $x, y \in X$, we define $x \leq_E y$ if and only if $q(x) \leq q(y)$ in the lexicographic order on E. This gives a total quasiorder $\leq_E$ on X.

Let X be a finite topological space with at least two elements and let $\mathcal{B}$ be the basis of minimal open neighborhoods for X. Then X contains a maximum element m with respect to $\leq_E$ and consequently its minimal open set M_m^X is maximal in $\mathcal{B}$ with respect to set inclusion. Let $Y = X \setminus \{m\}$ be a subspace of X. To derive a basis of minimal open neighborhoods for Y we have to consider two cases: If M_m^X is the minimal open neighborhood for m only, i.e., if $M_m^X \neq M_y^X$ for any $m \neq y$, then $\mathcal{B} \setminus \{M_m^X\}$ is a basis of minimal open neighborhoods for Y. If $M_m^X = M_y^X$ for some $y \neq m$, then $\{M_x^X \setminus \{m\} : x \in Y\} = \{B \in \mathcal{B} : m \notin B\} \cup \{M_m^X \setminus \{m\}\}$, the set of all open minimal neighborhoods in X with possibly m removed, forms a basis of minimal open neighborhoods for Y. It follows that X can be obtained from an appropriate subspace Y with $|X| - 1$ elements by adding one point m to Y whose minimal open neighborhood M_m will be maximal among minimal open sets in such a way that either the minimal open neighborhoods of $y \in Y \cap X$ have not been changed when passing to X or one minimal open neighborhood in Y that is maximal among open neighborhoods in Y with respect to set inclusion is expanded by the added point. For an appropriate Y the added point can be assumed to be maximal in X with respect to $\leq_E$.

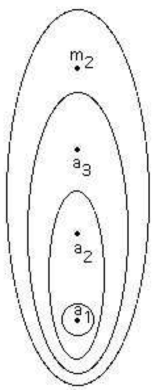
Example 4.2. Let $Y = \{a_1, a_2, a_3\}$ be the topological space whose basis of minimal open neighborhoods is shown below. Note that $q(a_1) = 1$ and $q(a_2) = (2, 1)$. The only maximal point with respect to $\leq_E$ is a_3, with $q(a_3) = (3, 2, 1)$. If $X = Y \cup \{m\}$ is a topological space which has subspace Y and in which m is maximal with respect to $\leq_E$, we must have $q(m) \geq q(a_3)$ for the added maximal point m. This gives three different choices for the minimal open neighborhood of the added point m. The basis for the three different resulting spaces X_i $(i = 1, 2, 3)$ are shown below.



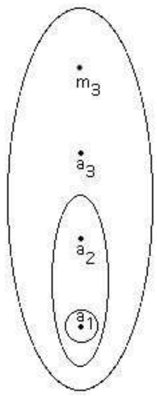
Y with maximal points a_3
where $q(a_3) = (3, 2, 1)$



$X_1 = Y \cup \{m_1\}$ with
maximal point m_1
$q(m_1) = (3, 2, 1)$



$X_2 = Y \cup \{m_2\}$ with
maximal point m_2
$q(m_2) = (4, 3, 2, 1)$



$X_3 = Y \cup \{m_3\}$ with
maximal point m_3
$q(m_3) = (4, 4, 2, 1)$

We can now inductively construct all possible topologies (up to isomorphism) on a set X of small cardinality, deriving the topologies on n points from the topologies on $n-1$ points by adding a newly-maximal point. We exhibit a basis of minimal open neighborhoods to determine the topology. In particular, the table at the end shows all 2- and 3-point topologies listed as #2–13 together with $q(m)$ for the maximal point m that was added.

Note that two topological spaces X_1 and X_2 are isomorphic if and only if there exists a bijection $f : X_1 \to X_2$ such that f preserves minimal open sets, i.e., $f(M_a^{X_1}) = M_{f(a)}^{X_2}$ for any $a \in X_1$. We easily see that the topologies #1–13 in the table at the end are not isomorphic.

5. Topologies $\tau_{\mathcal{L}}$

If a finite semigroup S yields a given topology $\tau_{\mathcal{L}}$ via Green's left quasiorder, the basis of minimal open neighborhoods of S gives some information on the semigroup structure map of S as seen in the following proposition.

Proposition 5.1. *Let S be a finite semigroup with topology $\tau_{\mathcal{L}}$. For elements $x, y \in S$ with $x \in P_k$ we have $xy \in P_t$ for some $t \leq k$. Moreover, for any $x, y \in S$, $q(xy) \leq q(x)$.*

Proof. Assume $M_x = Sx \cup \{x\} = \{x = x_1, x_2, \ldots, x_k\}$ with $x_i \in P_{a_i}$ such that $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k$ and hence $q(x) = (k = a_1, a_2, \ldots, a_k, 0, 0, \ldots)$. Then $M_{xy} = Sxy \cup \{xy\} = \{xy = x_1y = y_1, x_2y = y_2, \ldots, x_ky = y_k\}$, and hence $|M_{xy}| \leq |M_x|$. In general, it follows that $|M_{x_iy}| \leq |M_{x_i}|$ and hence $q(xy) \leq q(x)$. □

Corollary 5.2. *Let S be a finite semigroup with topology $\tau_{\mathcal{L}}$ and basis $\mathcal{B}$ of minimal open neighborhoods of $\tau_{\mathcal{L}}$. Then all sets in $\mathcal{B}$ that are minimal with respect to set inclusion have the same cardinality. In other words, if $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_t$ are elements in S with $M_{x_1} = M_{x_2} = M_{x_3} = \cdots = M_{x_k} = \{x_1, x_2, \ldots, x_k\}$ and $M_{y_1} = M_{y_2} = M_{y_3} = \cdots = M_{y_t} = \{y_1, y_2, \ldots, y_t\}$, then $k = t$. Consequently, if the least nonzero coordinate of $q(z)$ is j for some $z \in S$, then for every $x \in S$, $q(x)$ contains at least j entries equal to j as the final nonzero entries.*

Note that from a semigroup viewpoint this corollary simply states that all minimal left ideals of a finite semigroup have the same cardinality.

Proof. Under the hypotheses, note that $q(x_1) = q(x_2) = \cdots = q(x_k) = (k, k, \ldots, k, 0, 0, \ldots)$ and $q(y_1) = q(y_2) = \cdots = q(y_t) = (t, t, \ldots, t, 0, 0, \ldots)$. By the previous proposition, since $x_1y_1 \in M_{y_1}$, it follows that $t \leq k$. Analogously, $y_1x_1 \in M_{x_1}$ and hence $k \leq t$. Thus, $k = t$. □

The corollary shows that, in particular, the topology #7 cannot arise from a semigroup structure via Green's left quasiorder. As discussed earlier, the topologies #1, #4, and #13 are obtained by any groups of the appropriate order and the discrete topologies #2 and #5 are obtained from semigroups with $xy = y$ for all x, y.

Positive results about building semigroups yielding desired topologies are given in the following propositions.

Proposition 5.3. *Let S be a finite semigroup with topology $\tau_{\mathcal{L}}$ and basis $\mathcal{B}$ of minimal open neighborhoods for $\tau_{\mathcal{L}}$. For $m \notin S$, let $S^{\uparrow} = S \cup \{m\}$ be a topological space with basis $\mathcal{B}^{\uparrow} = \mathcal{B} \cup \{S^{\uparrow}\}$ of minimal open sets. Then this topology arises from an appropriate semigroup structure on $S^{\uparrow}$.*

Proof. By adding an element m to S which acts as an identity, regardless of whether S already had an identity, we get the semigroup $S^{\uparrow} = S \cup \{m\}$ with the desired topology. □

Applying this proposition to the semigroups with topologies #1, #2, #3, and #4 will give, respectively, semigroup structures for topologies #3, #8, #10, and #12. In particular, note that a semigroup for topology #10 arises from a repeated application of the proposition to the trivial semigroup for topology #1.

Proposition 5.4. *Let S be a finite semigroup with topology $\tau_{\mathcal{L}}$ and basis $\mathcal{B}$ of minimal open neighborhoods for $\tau_{\mathcal{L}}$. For $m \notin S$, let $S^{\downarrow} = S \cup \{m\}$ be a topological space with basis $\mathcal{B}^{\downarrow} = \{B \cup \{m\} : B \in \mathcal{B}\} \cup \{\{m\}\}$ of minimal open sets. Then this topology arises from an appropriate semigroup structure on $S^{\downarrow}$.*

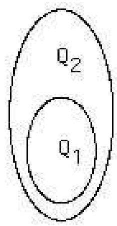
Proof. By adding an element m to S which acts as a zero, regardless of whether S already had a zero, we get the semigroup $S^{\downarrow} = S \cup \{m\}$ with the desired topology. □

Applied to the semigroups with topologies #2 and #4, respectively, this proposition yields a semigroup structure for topologies #9 and #11.

The two previous propositions can be generalized as follows.

Theorem 5.5. *Let $(Q_1, \cdot_1)$ and $(Q_2, \cdot_2)$ be disjoint finite semigroups with topologies $\tau^1_{\mathcal{L}}$ and $\tau^2_{\mathcal{L}}$ having bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of minimal open neighborhoods, respectively. Let $Q^* = Q_1 \cup Q_2$. Then Q^* is a semigroup with the following binary operation:*

$$xy = \begin{cases} x \cdot_1 y & \text{if } x, y \in Q_1 \\ x \cdot_2 y & \text{if } x, y \in Q_2 \\ x & \text{if } x \in Q_1 \text{ and } y \in Q_2 \\ y & \text{if } x \in Q_2 \text{ and } y \in Q_1. \end{cases}$$



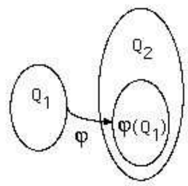
The corresponding topology $\tau^*_{\mathcal{L}}$ for Q^* has basis $\mathcal{B}^* = \mathcal{B}_1 \cup \{B \cup Q_1 : B \in \mathcal{B}_2\}$.

Proof. Observe that with this binary operation the elements of Q_1 act as zeroes when multiplied by elements of Q_2 or equivalently, the elements of Q_2 act as ones when multiplied by elements of Q_1. Since the product xyz takes place in Q_2 if all elements are in Q_2 and otherwise is simply the product of those elements that are in Q_1, the given binary operation does indeed give a semigroup structure on Q^* and the resulting topology $\tau^*_{\mathcal{L}}$ is easily seen to have basis $\mathcal{B}^*$. □

We will denote Q^* constructed in the previous proposition from Q_1 and Q_2 by $Q_1 \upharpoonleft Q_2$ and will make considerable use of this construction to verify the existence of semigroups structures yielding topologies on 4- and 5-point sets. While $Q_1 \upharpoonleft Q_2$ will always yield a connected topological space, the following theorem gives a construction of a disconnected topological space arising from semigroup structures.

Theorem 5.6. *Let* $(Q_1, \cdot_1)$ *and* $(Q_2, \cdot_2)$ *be disjoint finite semigroups with topologies* $\tau^1_{\mathcal{L}}$ *and* $\tau^2_{\mathcal{L}}$ *having bases* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *of minimal open neighborhoods, respectively. Assume there exists a semigroup embedding* $\varphi : Q_1 \to Q_2$ *that preserves minimal open neighborhoods, i.e.,* $\varphi(M^{Q_1}_{s_1}) = M^{Q_2}_{\varphi(s_1)}$ *for all* $s_1 \in Q_1$. *Let* $Q^* = Q_1 \cup Q_2$. *Then* Q^* *becomes a semigroup when given the following binary operation:*

$$xy = \begin{cases} x \cdot_1 y & \text{if } x, y \in Q_1 \\ x \cdot_2 y & \text{if } x, y \in Q_2 \\ \varphi(x) \cdot_2 y & \text{if } x \in Q_1 \text{ and } y \in Q_2 \\ \varphi^{-1}(x \cdot_2 \varphi(y)) & \text{if } x \in Q_2 \text{ and } y \in Q_1. \end{cases}$$

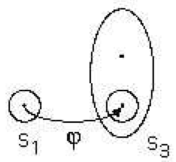


The corresponding topology $\tau^*_{\mathcal{L}}$ *for* Q^* *has basis* $\mathcal{B}_1 \cup \mathcal{B}_2$.

Proof. Observe that by assumption, for $y \in Q_1$, the minimal open neighborhood of $\varphi(y)$ lies in $\varphi(Q_1)$ and hence for $x \in Q_2$, $x \cdot_2 \varphi(y) \in \varphi(Q_1)$ and thus the product xy makes sense as defined. For $x, y, z \in Q^*$, by definition the product xyz can be thought of as taking place in Q_2 by identifying any element $s \in Q_1$ with its image $\varphi(s)$ in Q_2 and mapping the resulting element back into Q_1 via φ^{-1} if and only if $z \in Q_1$. This clearly makes the operation associative and this resulting semigroup structure is easily seen to yield the topology $\tau_{\mathcal{L}}$ with basis $\mathcal{B}_1 \cup \mathcal{B}_2$. □

We will denote Q^* constructed in the previous proposition from semigroups Q_1 and Q_2 by $Q_1 \cup_\varphi Q_2$. Observe that while $Q_1 \uparrow Q_2$ can be formed from any given semigroups Q_1 and Q_2, $Q_1 \cup_\varphi Q_2$ can only be defined if an appropriate semigroup embedding $\varphi : Q_1 \to Q_2$ exists.

If S_1 denotes the semigroup on one point and S_3 denotes the semigroup $S_1 \uparrow S_1$ giving topology #3 (as noted after Proposition 5.3), we see that the natural embedding



satisfies the conditions of Theorem 5.6 and the resulting semigroup $S_1 \cup_\varphi S_3$ yields the topology #6.

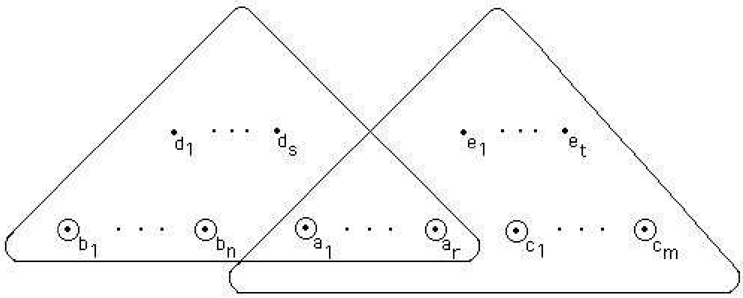
6. 4-Point Topologies and their Semigroups

We will now denote the earlier suggested semigroup structures yielding the topologies #1 through #13 by S_i for $i = 1, 2, \ldots, 13$ ($i \neq 7$), respectively. For example, S_1, S_4, and S_{13} are cyclic groups of appropriate order, S_2 and S_5 are semigroups of size 2 or 3 with operation $xy = y$ for any x, y, and $S_3 = S_1 \uparrow S_1$, $S_{11} = S_1 \uparrow S_4$, and $S_6 = S_1 \cup_\varphi S_3$ as given in the preceding paragraph.

The 4-point topologies, #14–46 in the table at the end, are constructed by adding a newly maximal element m to a 3-point topology. Most of them can be recognized as arising from some semigroup structure using Theorems 5.5 and 5.6.

We see that only topologies #22 and #35 could not readily be given a corresponding semigroup structure or be determined as not possibly arising from such by using Corollary 5.2. The following proposition on semigroup structures of a general class of topological spaces shows that, in particular, topologies #22 and #35 do arise from a semigroup structure.

Proposition 6.1. *Let* $X = \{a_i, b_j, c_k, d_p, e_q : 1 \leq i \leq r, 1 \leq j \leq n, 1 \leq k \leq m, 1 \leq p \leq s, 1 \leq q \leq t\}$. *Assume* $n + r > 0$, $m + r > 0$, $s > 0$, *and* $t > 0$. *Suppose* X *is a topological space with minimal open neighborhoods* $M_{a_i} = \{a_i\}$, $M_{b_j} = \{b_j\}$, $M_{c_k} = \{c_k\}$, $M_{d_p} = \{a_i, b_j, d_{p'} : 1 \leq i \leq r, 1 \leq j \leq n, 1 \leq p' \leq s\}$ *and* $M_{e_q} = \{a_i, c_k, e_{q'} : 1 \leq i \leq r, 1 \leq k \leq m, 1 \leq q' \leq t\}$ *as pictured below.*



If $r \leq 1$ *or* $r \geq 2$ *and* $(m, t) = (n, s)$, *then* X *arises from a semigroup structure on* X *via Green's left quasiorder.*

Observe that for $r = 1$, $m = s = t = 1$, and $n = 0$, we obtain topology #22, and topology #35 is the case with $r = 1, m = n = 0, t = 1$, and $s = 2$.

We will see in Theorem 7.2 that the conditions on r, m, n, s, and t are not only sufficient, but also necessary.

Proof. First assume $r = 0$ and define a binary operation on X as follows:

$$\begin{aligned}
xb_j = b_jd_p &= b_j \quad \text{all } j, p, \text{ any } x \in X \\
xc_k = c_ke_q &= c_k \quad \text{all } k, q, \text{ any } x \in X \\
d_pd_u &= d_p \quad \text{all } p, u \\
e_qe_u &= e_q \quad \text{all } q, u \\
e_qd_p = c_kd_p &= b_1 \quad \text{all } q, k, p \\
b_je_q = d_pe_q &= c_1 \quad \text{all } j, q, p.
\end{aligned}$$

Once the above definition has been shown to give a semigroup structure on X, it is easy to see that the topology arising via Green's left quasiorder has the desired basis of minimal open sets. Thus, we will only show that $(xy)z = x(yz)$ for any $x, y, z \in X$.

If $z \in \{b_j, c_k : 1 \le i \le n, 1 \le k \le m\}$, then it follows from the definition that $(xy)z = z$ and that $x(yz) = xz = z$ for any $x, y \in X$. Now assume that $z \in \{d_p, e_q : 1 \le p \le s, 1 \le q \le t\}$. By symmetry of the operation defined, we may assume without loss of generality that $z = d_{p_0}$ for some $1 \le p_0 \le s$. We will consider three cases: 1) $y \in \{e_q, c_k : 1 \le q \le t, 1 \le k \le m\}$, 2) $y \in \{b_j : 1 \le j \le n\}$, and 3) $y \in \{d_p : 1 \le p \le s\}$. In case 1), we have $x(yd_{p_0}) = xb_1 = b_1$ as well as $(xy)d_{p_0} = wd_{p_0}$ for some $w \in \{e_q, c_k : 1 \le q \le t, 1 \le k \le m\}$ and hence $wd_{p_0} = b_1$. In case 2), $y = b_{j_0}$ for some $1 \le j_0 \le n$, so $(xb_{j_0})d_{p_0} = b_{j_0}d_{p_0} = b_{j_0}$ and $x(b_{j_0}d_{p_0}) = xb_{j_0} = b_{j_0}$. In case 3), $y = d_{p_1}$ for some $1 \le p_1 \le s$ and then

$$x(d_{p_1}d_{p_0}) = xd_{p_1} = \begin{cases} b_1 & \text{if } x \in \{e_q, c_k : 1 \le q \le t, 1 \le k \le m\} \\ x & \text{if } x \in \{b_j, d_p : 1 \le j \le n, 1 \le p \le s\} \end{cases}$$

and

$$(xd_{p_1})d_{p_0} = \begin{cases} b_1 d_{p_0} = b_1 & \text{if } x \in \{e_q, c_k : 1 \le q \le t, 1 \le k \le m\} \\ xd_{p_0} = x & \text{if } x \in \{b_j, d_p : 1 \le j \le n, 1 \le p \le s\}. \end{cases}$$

Thus, the operation is associative and X is a semigroup under the given operation.

For $r = 1$, in the above definition we may identify b_1 and c_1 and set $a_1 = b_1 = c_1$. This is easily seen to be a well-defined operation which gives the desired semigroup structure.

Now assume $r \ge 2$ and $(m, t) = (n, s)$ and define a binary operation on X as follows:

$$\begin{aligned} xb_j = b_j d_p = c_j d_p &= b_j && \text{all } j, p, \text{ any } x \in X \\ xa_i = a_i d_p = a_i e_q &= a_i && \text{all } i, q, \text{ any } x \in X \\ xc_k = c_k e_q = b_k e_q &= c_k && \text{all } k, q, \text{ any } x \in X \\ d_p d_u = e_p d_u &= d_p && \text{all } p, u \\ e_q e_u = d_q e_u &= e_q && \text{all } q, u. \end{aligned}$$

It is easy to see that the operation yields the desired topology once it has been shown to be associative. From the definition it is clear that $(xy)z = x(yz)$ for any $x, y \in X$ with $z \in \{a_i, b_j, c_k : 1 \le i \le r, 1 \le j \le n, 1 \le k \le m\}$. So, assume $z \in \{d_p, e_q : 1 \le p \le s, 1 \le q \le t\}$. By the symmetry of the operation defined, we may assume without loss of generality that $z = d_{p_0}$ for some $1 \le p_0 \le s$. We consider four cases for y: 1) $y \in \{b_j, c_k : 1 \le j \le n, 1 \le k \le m\}$, 2) $y \in \{a_i : 1 \le i \le r\}$, 3) $y \in \{d_p : 1 \le p \le s\}$ and 4) $y \in \{e_q : 1 \le q \le t\}$. In case 4), $y = e_{q_0}$ for some $1 \le q_0 \le t$, so

$$x(e_{q_o}d_{p_0}) = xd_{q_0} = \begin{cases} x & \text{if } x \in \{a_i, b_j, d_p : 1 \le i \le r, 1 \le j \le n, 1 \le p \le s\} \\ b_k & \text{if } x = c_k \text{ for some } k \\ d_q & \text{if } x = e_q \text{ for some } q \end{cases}$$

and

$$(xe_{q_o})d_{p_0} \quad = \begin{cases} xd_{p_0} & \text{if } x \in \{a_i, c_k, e_q : 1 \leq i \leq r, 1 \leq k \leq m, 1 \leq q \leq t\} \\ c_j d_{p_0} & \text{if } x = b_j \text{ for some } j \\ e_p d_{p_0} & \text{if } x = d_p \text{ for some } p \end{cases}$$

$$= \begin{cases} b_k & \text{if } x = c_k \text{ for some } k \\ d_q & \text{if } x = e_q \text{ for some } q \\ x & \text{if } x = a_i \text{ for some } i \\ x & \text{if } x = b_j \text{ for some } j \\ x & \text{if } x = d_p \text{ for some } p \end{cases}$$

and hence $(xy)z = x(yz)$ in this case. In the cases 1)–3), associativity can be verified similarly. So, X is a semigroup with the specified binary operation and has the desired basis of minimal open sets. □

7. 5-Point Topologies and their Semigroups

When determining which topologies arise from semigroups via Green's left quasiorder, the following proposition is useful for constructing semigroups yielding desired topologies from semigroup structures on appropriate open subspaces.

Proposition 7.1. *Let $X = X_1 \cup X_2$ be a finite topological space with open subspaces X_1 and X_2 such that $X_1 \cap X_2 = \{w\}$. Assume X_1 and X_2 arise from semigroup structures Q_1 and Q_2, respectively. Then X arises from a semigroup structure, denoted by $Q_1 \odot_w Q_2$.*

Proof. Let $(Q_1, \cdot_1)$ and $(Q_2, \cdot_2)$ be the semigroups yielding the open subspaces X_1 and X_2 respectively. Since $\{w\} = X_1 \cap X_2$ is open, we see that $M_w = \{w\}$ in both X_1 and X_2 so that $s_1 \cdot_1 w = s_2 \cdot_2 w = w$ for all $s_1 \in Q_1$, $s_2 \in Q_2$. Let $Q = Q_1 \cup Q_2$ and define a binary operation on Q as follows:

$$xy = \begin{cases} x \cdot_1 y & \text{if } x, y \in Q_1 \\ x \cdot_2 y & \text{if } x, y \in Q_2 \\ w \cdot_2 y & \text{if } x \in Q_1 \text{ and } y \in Q_2 \\ w \cdot_1 y & \text{if } x \in Q_2 \text{ and } y \in Q_1. \end{cases}$$

Observe that this operation is well-defined, i.e., we may consider $w \in Q_1$ or $w \in Q_2$ in the definition above and the product will be the same. For example, for $s_1 \in Q_1$, when considering $w \in Q_1$, we have $s_1 w = s_1 \cdot_1 w = w$ and when considering $w \in Q_2$, we have $s_1 w = w \cdot_2 w = w$.

To show that $(xy)z = x(yz)$, we consider 8 cases depending on whether the three factors are in Q_1 or Q_2. By associativity in Q_1 or Q_2, we need not consider the two cases in which all factors lie in Q_i ($i = 1$ or 2). Now by the symmetry of the definition, we need only verify the three cases:

$$(s_1 s_2)\bar{s_2} = (w \cdot_2 s_2)\bar{s_2} = (w \cdot_2 s_2) \cdot_2 \bar{s_2} = w \cdot_2 (s_2 \cdot_2 \bar{s_2}) = s_1(s_2 \bar{s_2})$$

$$\begin{aligned}(s_2 s_1)\bar{s_2} = (w \cdot_1 s_1)\bar{s_2} = w \cdot_2 \bar{s_2} = (s_2 \cdot_2 w) \cdot_2 \bar{s_2} \\ = s_2 \cdot_2 (w \cdot_2 \bar{s_2}) = s_2(w \cdot_2 \bar{s_2}) = s_2(s_1 \bar{s_2})\end{aligned}$$

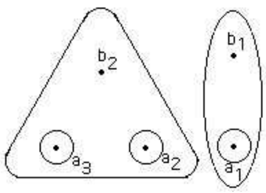
$$\begin{aligned}(s_2\bar{s_2})s_1 = (s_2 \cdot_2 \bar{s_2})s_1 = w \cdot_1 s_1 = (w \cdot_1 w) \cdot_1 s_2 \\ = w \cdot_1 (w \cdot_1 s_1) = s_2(w \cdot_1 s_1) = s_2(\bar{s_2}s_1)\end{aligned}$$

where $s_i, \bar{s_i} \in Q_i$ for $i = 1, 2$. This shows that Q is a semigroup under the given operation. It is easy to see that its corresponding topology has basis of minimal open neighborhoods being the union of the bases of minimal open neighborhoods for X_1 and X_2, as desired. □

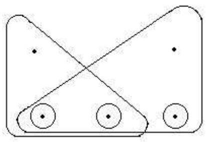
As before, for $i = 1, 2, \ldots, 46$, S_i will denote the suggested semigroup structure, if it exists, yielding the topology #i. For example, S_{22} and S_{35} will be the semigroups with operations defined in the proof of Proposition 6.1.

The table at the end lists all 139 5-point topologies (#47–185) as constructed from a 4-point topology by adding an element m which is newly maximal with respect to $\leq_E$ as described following Proposition 4.1. If for the added element m, $q(m) > q(x)$ for any $x \neq m$, then the resulting topology only arises from the 4-point topology used at this step and will not be duplicated later. If, on the other hand, $q(m) = q(\bar{m})$ for some $\bar{m} \neq m$, then the same topology may arise from a possibly different topology by adding $\bar{m}$, so we will make note of this and not relist the isomorphic topology as arising from a later considered 4-point topology. It turns out that only topology #78 could arise from two different 4-point topologies, namely #19 and #20; it will be listed in the table only among those arising from #19.

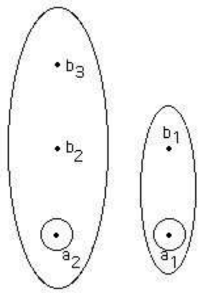
We see that only the 7 topologies below could not easily be derived from semigroup structures using Theorems 5.5 and 5.6 and Propositions 6.1 and 7.1.



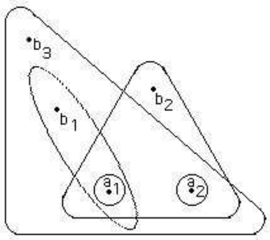
#56



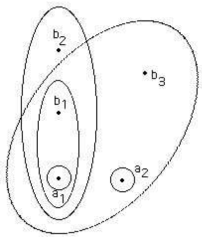
#71



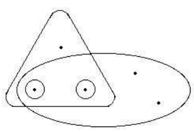
#81



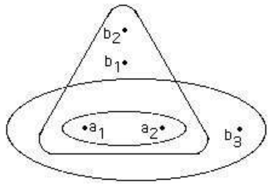
#101



#106



#132



#174

We now give semigroup structures yielding some of these topologies. The associativity of the indicated operations are easily checked by hand or computer.

Topology #56: Let S_3 denote the semigroup $\{a_1, b_1\}$ yielding topology #3 with $a_1 \in P_1$ and let S_8 be the semigroup $\{a_2, a_3, b_2\}$ yielding topology #8 with $a_2, a_3 \in P_1$. Let $S_{56} = S_8 \cup S_3$ with the product defined as in S_3 or S_8 between pairs of elements in S_3 or S_8, respectively, and taking $xy = a_1$, $ya_2 = yb_2 = a_2$, and $ya_3 = a_3$ for all $x \in S_8, y \in S_3$.

Topology #81: Let S_3 be as above and let $S_{11} = \{a_2, b_2, b_3\}$ be the semigroup yielding topology #11 with $a_2 \in P_1$. Let $S_{81} = S_3 \cup S_{11}$ with the product defined as in S_3 or S_{11} between pairs of elements in S_3 or S_{11}, respectively, and taking $xy = a_1$ and $yx = a_2$ for all $x \in S_{11}, y \in S_3$.

Topology #106: Let $S_{25} = \{a_1, a_2, b_1, b_3\}$ be the semigroup yielding topology #25 with $a_1, a_2 \in P_1, b_1 \in P_2$, and $b_3 \in P_4$. Here S_{25} denotes the semigroup that was earlier shown to generate topology #25, so $S_{25} = (S_1'' \cup_\varphi (S_1 \uparrow S_1')) \uparrow S_1'''$ where $S_1 = \{a_1\}, S_1' = \{b_1\}, S_1'' = \{a_2\}$, and $S_1''' = \{b_3\}$ are the trivial semigroups on one element. Let $S_{106} = S_{25} \cup \{b_2\}$ with product defined as in S_{25} between elements in S_{25} and $b_2b_2 = b_2$, $b_2b_3 = b_2b_1 = b_2a_1 = a_1$, $b_2a_2 = a_2$, $a_2b_2 = b_3b_2 = a_1b_2 = a_1$, and $b_1b_2 = b_1$.

Topology #174: Let $S_4 = \{a_1, a_2\}$ and $S_4' = \{b_1, b_2\}$ be cyclic groups of order two yielding topology #4. Let $S_{44} = S_4 \uparrow S_4'$ be the semigroup associated with topology #44. Let $S_{174} = S_{44} \cup \{b_3\}$ with product defined as in S_{44} for elements in S_{44} and $b_3b_3 = b_3$, $b_3x = a_1$ and $xb_3 = a_1$ for all $x \in S_{44}$.

Topologies #71 and #132 are of the type discussed in Proposition 6.1 for $r \geq 2$. The following theorem will show that these cannot be derived from any semigroup.

Theorem 7.2. *Let* $X = \{a_i, b_j, c_k, d_p, e_q : 1 \leq i \leq r, 1 \leq j \leq n, 1 \leq k \leq m, 1 \leq p \leq s, 1 \leq q \leq t\}$. *Assume* $n + r > 0$, $m + r > 0$, $s > 0$, *and* $t > 0$. *Suppose* X *is a topological space with minimal open neighborhoods* $M_{a_i} = \{a_i\}$, $M_{b_j} = \{b_j\}$, $M_{c_k} = \{c_k\}$, $M_{d_p} = \{a_i, b_j, d_{p'} : 1 \leq i \leq r, 1 \leq j \leq n, 1 \leq p' \leq s\}$ *and* $M_{e_q} = \{a_i, c_k, e_{q'} : 1 \leq i \leq r, 1 \leq k \leq m, 1 \leq q' \leq t\}$ *as shown in Proposition 6.1.*

X arises from a semigroup structure on X via Green's left quasiorder if and only if $r \leq 1$ *or* $r \geq 2$ *and* $(m, t) = (n, s)$.

Proof. In Proposition 6.1 it was shown that a semigroup structure exists if $r \leq 1$ or $r \geq 2$ and $(m, t) = (n, s)$. So assume $r \geq 2$ and $(m, t) \neq (n, s)$. We will assume X has an appropriate semigroup structure and when considering the two cases 1) $m+t \neq n+s$ and 2) $m+t = n+s$ we will arrive at a contradiction.

Case 1). By the symmetry of the topological space given we may assume $m+t < n+s$. Observe that $d_p \in P_{n+r+s}$ and $e_q \in P_{m+r+t}$ for any $1 \leq p \leq s, 1 \leq q \leq t$ with $m+r+t < n+r+s$. By Proposition 5.1 it follows that $e_q d_p \in P_u$ for some $u \leq m+r+t < n+r+s$ and from $e_q d_p \in M_{d_p}$ we conclude that $e_q d_p \in P_1$ for any p and q. Hence $x(e_q d_p) = e_q d_p$ for any $x \in X$. In particular, for $q = 1 = p$ we have $e_1 d_1 = x(e_1 d_1) = (xe_1)d_1$ for any x and since xe_1 takes on any value in $\{a_i, c_k, e_q : 1 \leq i \leq r, 1 \leq k \leq m, 2 \leq q \leq t\} \subseteq M_{e_1}$, it follows that $a_i d_1 = c_k d_1 = e_q d_1 = e_1 d_1$ for any i, k, q. Hence

$$M_{d_1} = \{e_1 d_1, d_p d_1, b_j d_1 : 1 \leq p \leq s, 1 \leq j \leq n\} \cup \{d_1\}$$

contains at most $s+n+2$ distinct elements. On the other hand, $d_1 \in P_{n+s+r}$, so $s+n+2 \geq n+s+r$ and we need only consider the case $r=2$. Since $r=2$, M_{d_1} contains $n+2$ elements of P_1 and hence $d_p d_1 \in P_1$ for some $1 \leq p \leq s$. Let us say $d_{p_0} d_1 \in P_1$. But then $(xd_{p_0})d_1 = x(d_{p_0} d_1) = d_{p_0} d_1$. Since, for $x \in X$, xd_{p_0} takes on all values b_j, a_i, and d_p with $p \neq p_0$, it follows that $d_{p_0} d_1 = b_j d_1 = a_i d_1 = d_p d_1$ for all j, i, p. So M_{d_1} contains at most 3 distinct elements. Since $r=2$ it follows that $s=1$ and $n=0$, which is a contradiction to $0 < m+t < n+s = 0+1$.

Case 2). By symmetry of the topological space given with $m+t = n+s$ and $(m,t) \neq (n,s)$ we may assume that $n < m$ and hence $s > t \geq 1$, which, in particular, implies $s \geq 2$. If $e_1 d_1 \notin P_1$ then $e_1 d_1 = d_{p_0}$ for some $p_0 \in \{1, 2, \ldots, s\}$. Observe that by Proposition 5.1, $yd_1 \in P_1$ for any $y \in P_1$ and thus $\{(xe_1)d_1 : x \in X\}$ contains at most t elements not in P_1 while $\{xd_{p_0} = x(e_1 d_1) : x \in X\}$ contains s elements not in P_1, since $s \geq 2$ and thus also d_{p_0} is necessarily a left multiple of itself. It follows that $t \geq s$, which contradicts $t < s$. Hence $e_1 d_1 \in P_1$. Thus $(xe_1)d_1 = x(e_1 d_1) = e_1 d_1 \in P_1$ for all $x \in X$. Since for $x \in X$, xe_1 takes on at least all values a_i, c_k, e_q with $q \neq 1$, it follows that $e_1 d_1 = (xe_1)d_1 = a_i d_1 = c_k d_1 = e_q d_1$ for all i, k, q. So all the left multiples of d_1 are of the form $b_j d_1, d_p d_1$, and $e_1 d_1$ and thus there are at most $n+s+1$ distinct ones. Since $s \geq 2$ all $n+s+r$ elements in M_{d_1} are indeed left multiples of d_1 and since $r \geq 2$ we have a contradiction. □

The only remaining 5-point topology that has not been resolved is topology #101. The following proposition will show that no semigroup structure exists for this topology.

Proposition 7.3. *Let $X = \{a_1, a_2, b_1, b_2, b_3\}$ be the topological space #101 with basis of minimal open sets as pictured on page 156. There exists no semigroup structure on X which yields the topological space X via Green's left quasiorders.*

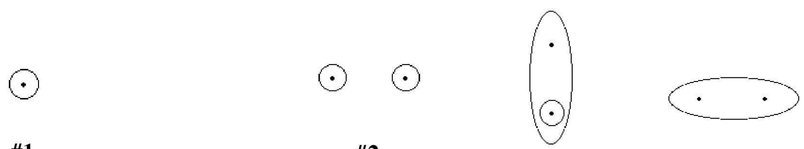
Proof. If X has a semigroup structure yielding the desired topology, we must have $xa_1 = a_1, xa_2 = a_2$ for any $x \in X$ as well as, by Proposition 5.1, $a_1 b_1 = a_2 b_1 = a_1$. By Proposition 5.1, $b_2 b_3 \in \{a_1, a_2, b_1\}$. If $b_2 b_3 \in \{a_1, a_2\} \subseteq P_1$, it follows that $xb_2 b_3 = b_2 b_3$ for all $x \in X$, and since a_1 and a_2 are left multiples of b_2 we have $a_1 b_3 = a_2 b_3 = b_2 b_3$. Since both a_1 and a_2 are left multiples of b_3

it follows that either $b_3b_3 \in \{a_1, a_2\}$ or $b_1b_3 \in \{a_1, a_2\}$ and it is different from b_2b_3. So $xb_3b_3 = b_3b_3$ or $xb_1b_3 = b_1b_3$ for all $x \in X$. In either case since a_1 is a left multiple of b_3 as well as b_1, it follows that $a_1b_3 = b_3b_3$ or $a_1b_3 = b_1b_3$ which equals b_2b_3. So b_3 would have at most two distinct left multiples which contradicts that $b_3 \in P_4$. So $b_2b_3 \in \{a_1, a_2\}$ is not possible and we only need to consider the case $b_2b_3 = b_1$. Since a_1 and a_2 are left multiples of b_2 there exist x_1, x_2 in X such that $x_1b_2 = a_1$ and $x_2b_2 = a_2$. Thus $x_1b_1 = x_1b_2b_3 = a_1b_3 \in P_1$ and hence $a_1b_3 = a_1$ as well as $x_2b_1 = x_2b_2b_3 = a_2b_3 \in P_1$ and hence $a_2b_3 = a_1$.

Thus $a_1b_3 = a_2b_3 = a_1$ and $b_2b_3 = b_1$. Since a_2 is a left multiple of b_3 it follows that $b_1b_3 = a_2$ or $b_3b_3 = a_2$. If $b_1b_3 = a_2$, then $(a_1b_1)b_3 = a_1b_3 = a_1$ as well as $a_1(b_1b_3) = a_1a_2 = a_2$, which is impossible. If $b_3b_3 = a_2$, then $a_1(b_3b_3) = a_1a_2 = a_2$ as well as $(a_1b_3)b_3 = a_1b_3 = a_1$ which also gives a contradiction. □

TABLE

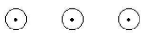
The nonisomorphic topologies on n points for $n = 1, 2, 3, 4, 5$ are given by showing the basis of minimal open neighborhoods of each. Each n-point topology ($n > 1$) is derived from the $(n-1)$-point topology indicated by adding a newly maximal point m with respect to $\leq_E$. The truncated sequence for $q(m)$ and, when possible, one possible semigroup structure determining the topology are given below each basis.



1-point topology	**2 point topologies**		
	from #1	from #1	from #1
#1 (1) cyclic group	**#2** (1) semigroup with $xy = y \quad \forall x, y$	**#3** (2,1) $S_1 \uparrow S_1$	**#4** (2,2) cyclic group

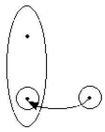
3-point topologies

from #2



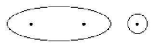
#5
(1)
semigroup with
$xy = y \quad \forall x, y$

from #2



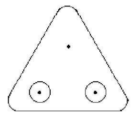
#6
(2,1)
$S_1 \cup_\varphi S_3$

from #2



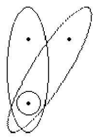
#7
(2,2)
impossible
by Cor. 5.2

from #2



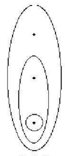
#8
(3,1,1)
$S_2 \uparrow S_1$

from #3



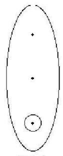
#9
(2,1)
$S_1 \uparrow S_2$

from #3



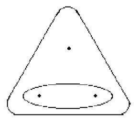
#10
(3,2,1)
$S_3 \uparrow S_1$

from #3



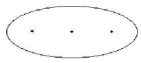
#11
(3,3,1)
$S_1 \uparrow S_4$

from #4



#12
(3,2,2)
$S_4 \uparrow S_1$

from #4



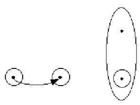
#13
(3,3,3)
cyclic group

4-point topologies

from #5

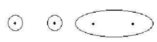
#14
(1)
semigroup with
$xy = y \quad \forall x, y$

from #5



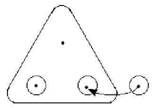
#15
(2,1)
$S_1 \cup_\varphi S_6$

from #5



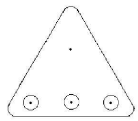
#16
(2,2)
impossible
by Cor. 5.2

from #5



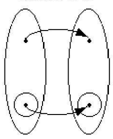
#17
(3,1,1)
$S_1 \cup_\varphi S_8$

from #5



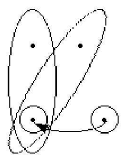
#18
(4,1,1,1)
$S_5 \uparrow S_1$

from #6



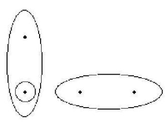
#19
(2,1)
$S_3 \cup_\varphi S_3$

from #6



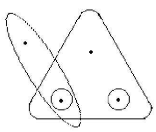
#20
(2,1)
$S_1 \cup_\varphi S_9$

from #6



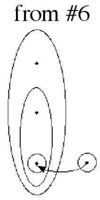
#21
(2,2)
impossible
by Cor. 5.2

from #6



#22
(3,1,1)
Prop. 6.1

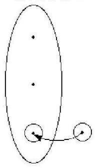
from #6



#23
(3,2,1)
$S_1 \cup_\varphi S_{10}$

4-point topologies

from #6

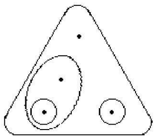


#24

(3,3,1)

$S_1 \cup_\varphi S_{11}$

from #6

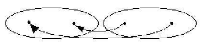


#25

(4,2,1,1)

$S_6 \uparrow S_1$

from #7

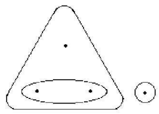


#26

(2,2)

$S_4 \cup_\varphi S_4$

from #7

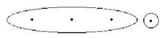


#27

(3,2,2)

impossible by Cor. 5.2

from #7

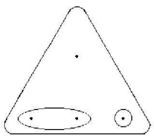


#28

(3,3,3)

impossible by Cor. 5.2

from #7

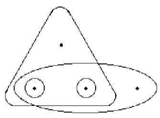


#29

(4,2,2,1)

impossible by Cor. 5.2

from #8

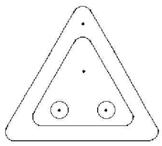


#30

(3,1,1)

$S_2 \uparrow S_2$

from #8

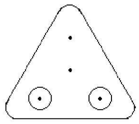


#31

(4,3,1,1)

$S_8 \uparrow S_1$

from #8

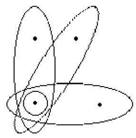


#32

(4,4,1,1)

$S_2 \uparrow S_4$

from #9

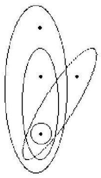


#33

(2,1)

$S_1 \uparrow S_5$

from #9

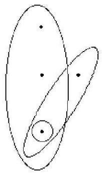


#34

(3,2,1)

$S_1 \uparrow S_6$

from #9

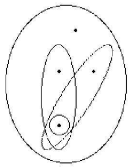


#35

(3,3,1)

Prop. 6.1

from #9

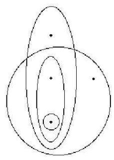


#36

(4,2,2,1)

$S_9 \uparrow S_1$

from #10

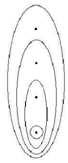


#37

(3,2,1)

$S_3 \uparrow S_2$

from #10

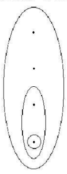


#38

(4,3,2,1)

$S_{10} \uparrow S_1$

from #10

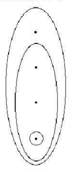


#39

(4,4,2,1)

$S_3 \uparrow S_4$

from #11

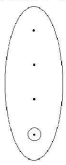


#40

(4,3,3,1)

$S_{11} \uparrow S_1$

from #11

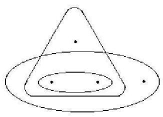


#41

(4,4,4,1)

$S_1 \uparrow S_{13}$

from #12

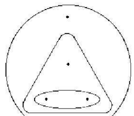


#42

(3,2,2)

$S_4 \uparrow S_2$

from #12



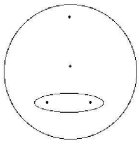
#43

(4,3,2,2)

$S_{12} \uparrow S_1$

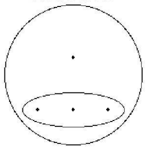
4-point topologies

from #12



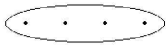
#44
(4,4,2,2)
$S_4 \uparrow S_4$

from #13



#45
(4,3,3,3)
$S_{13} \uparrow S_1$

from #13



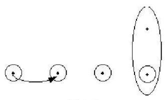
#46
(4,4,4,4)
cyclic group

5-point topologies

from #14

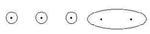
#47
(1)
semigroup with
$xy = y \quad \forall x, y$

from #14



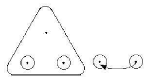
#48
(2,1)
$S_1 \cup_\varphi S_{15}$

from #14



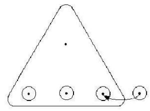
#49
(2,2)
impossible
by Cor. 5.2

from #14



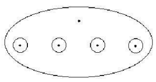
#50
(3,1,1)
$S_1 \cup_\varphi S_{17}$

from #14



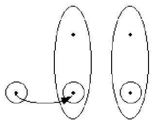
#51
(4,1,1,1)
$S_1 \cup_\varphi S_{18}$

from #14



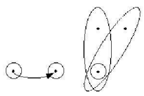
#52
(5,1,1,1,1)
$S_{14} \uparrow S_1$

from #15



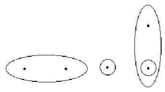
#53
(2,1)
$S_1 \cup_\varphi S_{19}$

from #15



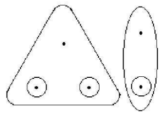
#54
(2,1)
$S_1 \cup_\varphi S_{20}$

from #15



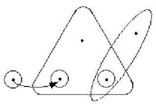
#55
(2,2)
impossible
by Cor. 5.2

from #15



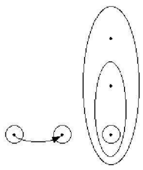
#56
(3,1,1)
see remarks
before Thm. 7.2

from #15



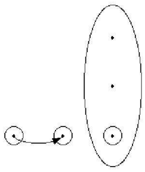
#57
(3,1,1)
$S_1 \cup_\varphi S_{22}$

from #15



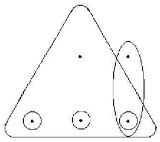
#58
(3,2,1)
$S_1 \cup_\varphi S_{23}$

from #15



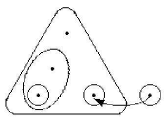
#59
(3,3,1)
$S_1 \cup_\varphi S_{24}$

from #15



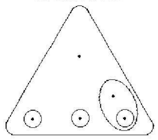
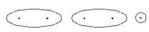
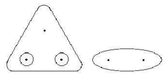
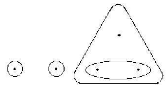
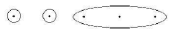
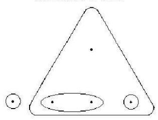
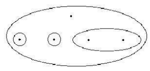
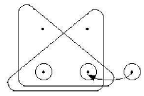
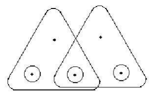
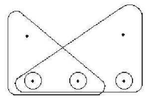
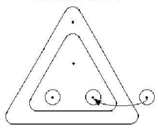
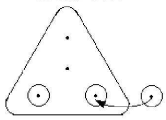
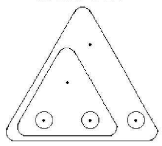
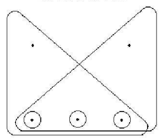
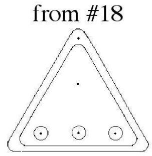
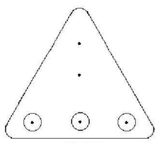
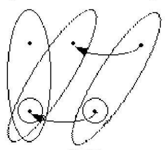
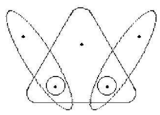
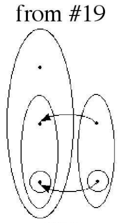
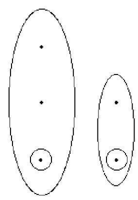
#60
(4,1,1,1)
$S_5 \odot_w S_3$

from #15



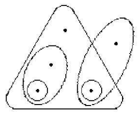
#61
(4,2,1,1)
$S_1 \cup_\varphi S_{25}$

5-point topologies

from #15  **#62** (5,2,1,1,1) $S_{15} \uparrow S_1$	from #16  **#63** (2,2) impossible by Cor. 5.2	from #16  **#64** (3,1,1) impossible by Cor. 5.2	from #16  **#65** (3,2,2) impossible by Cor. 5.2	from #16  **#66** (3,3,3) impossible by Cor. 5.2
from #16  **#67** (4,2,2,1) impossible by Cor. 5.2	from #16  **#68** (5,2,2,1,1) impossible by Cor. 5.2	from #17  **#69** (3,1,1) $S_1 \cup_{\varphi} S_{30}$	from #17  **#70** (3,1,1) $S_8 \odot_{w} S_8$	from #17  **#71** (4,1,1,1) impossible by Thm. 7.2
from #17  **#72** (4,3,1,1) $S_1 \cup_{\varphi} S_{31}$	from #17  **#73** (4,4,1,1) $S_1 \cup_{\varphi} S_{32}$	from #17  **#74** (5,3,1,1,1) $S_{17} \uparrow S_1$	from #18  **#75** (4,1,1,1) Prop. 6.1	from #18  **#76** (5,4,1,1,1) $S_{18} \uparrow S_1$
from #18  **#77** (5,5,1,1,1) $S_5 \uparrow S_4$	from #19 (also from #20)  **#78** (2,1) $S_3 \cup_{\varphi} S_9$	from #19  **#79** (3,1,1) $S_3 \odot_{w} S_{22}$	from #19  **#80** (3,2,1) $S_3 \cup_{\varphi} S_{10}$	from #19  **#81** (3,3,1) see remarks before Thm. 7.2

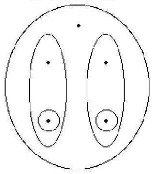
5-point topologies

from #19



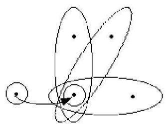
#82
(4,2,1,1)
$S_3 \odot_w S_{25}$

from #19



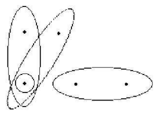
#83
(5,2,2,1,1)
$S_{19} \uparrow S_1$

from #20



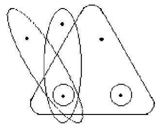
#84
(2,1)
$S_1 \cup_\varphi S_{33}$

from #20



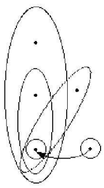
#85
(2,2)
impossible
by Cor. 5.2

from #20



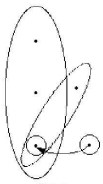
#86
(3,1,1)
$S_3 \odot_w S_{22}$

from #20



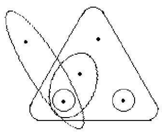
#87
(3,2,1)
$S_1 \cup_\varphi S_{34}$

from #20



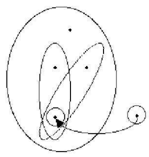
#88
(3,3,1)
$S_1 \cup_\varphi S_{35}$

from #20



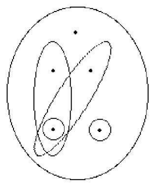
#89
(4,2,1,1)
$S_3 \odot_w S_{25}$

from #20



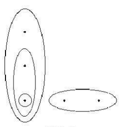
#90
(4,2,2,1)
$S_1 \cup_\varphi S_{36}$

from #20



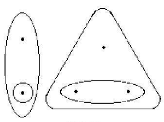
#91
(5,2,2,1,1)
$S_{20} \uparrow S_1$

from #21



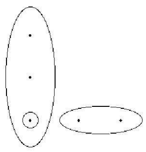
#92
(3,2,1)
impossible
by Cor. 5.2

from #21



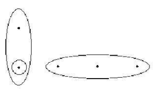
#93
(3,2,2)
impossible
by Cor. 5.2

from #21



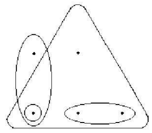
#94
(3,3,1)
impossible
by Cor. 5.2

from #21



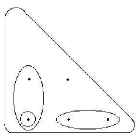
#95
(3,3,3)
impossible
by Cor. 5.2

from #21



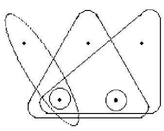
#96
(4,2,2,1)
impossible
by Cor. 5.2

from #21



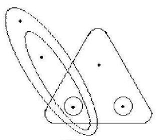
#97
(5,2,2,2,1)
impossible
by Cor. 5.2

from #22



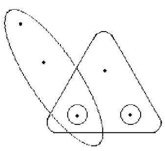
#98
(3,1,1)
$S_3 \odot_w S_{30}$

from #22



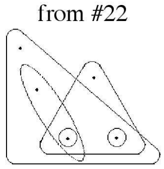
#99
(3,2,1)
$S_{10} \odot_w S_8$

from #22



#100
(3,3,1)
$S_8 \odot_w S_{11}$

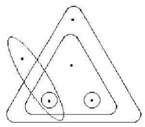
from #22



#101
(4,2,1,1)
impossible
by Prop. 7.3

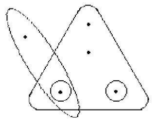
5-point topologies

from #22



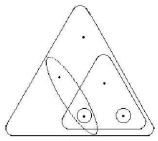
#102
(4,3,1,1)
$S_3 \odot_w S_{31}$

from #22



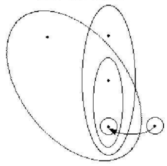
#103
(4,4,1,1)
$S_3 \odot_w S_{32}$

from #22



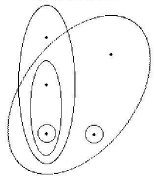
#104
(5,3,2,1,1)
$S_{22} \uparrow S_1$

from #23



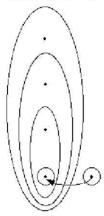
#105
(3,2,1)
$S_1 \cup_\varphi S_{37}$

from #23



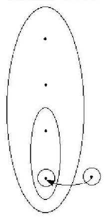
#106
(4,2,1,1)
see remarks
before Thm. 7.2

from #23



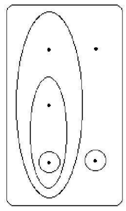
#107
(4,3,2,1)
$S_1 \cup_\varphi S_{38}$

from #23



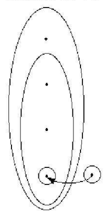
#108
(4,4,2,1)
$S_1 \cup_\varphi S_{39}$

from #23



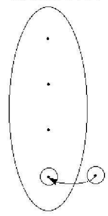
#109
(5,3,2,1,1)
$S_{23} \uparrow S_1$

from #24



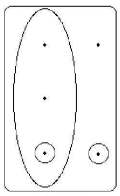
#110
(4,3,3,1)
$S_1 \cup_\varphi S_{40}$

from #24



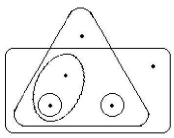
#111
(4,4,4,1)
$S_1 \cup_\varphi S_{41}$

from #24



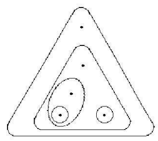
#112
(5,3,3,1,1)
$S_{24} \uparrow S_1$

from #25



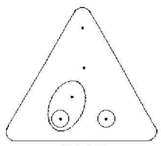
#113
(4,2,1,1)
$S_6 \uparrow S_2$

from #25



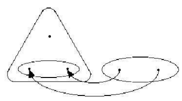
#114
(5,4,2,1,1)
$S_{25} \uparrow S_1$

from #25



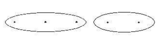
#115
(5,5,2,1,1)
$S_6 \uparrow S_4$

from #26



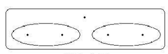
#116
(3,2,2)
$S_4 \cup_\varphi S_{12}$

from #26



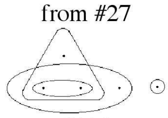
#117
(3,3,3)
impossible
by Cor. 5.2

from #26



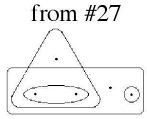
#118
(5,2,2,2,2)
$S_{26} \uparrow S_1$

from #27



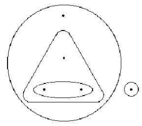
#119
(3,2,2)
impossible
by Cor. 5.2

from #27



#120
(4,2,2,1)
impossible
by Cor. 5.2

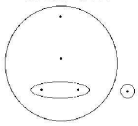
from #27



#121
(4,3,2,2)
impossible
by Cor. 5.2

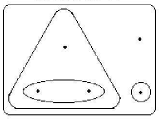
5-point topologies

from #27



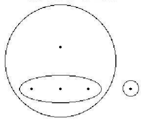
#122
(4,4,2,2)
impossible
by Cor. 5.2

from #27



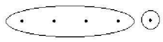
#123
(5,3,2,2,1)
impossible
by Cor. 5.2

from #28



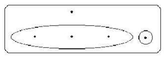
#124
(4,3,3,3)
impossible
by Cor. 5.2

from #28



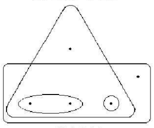
#125
(4,4,4,4)
impossible
by Cor. 5.2

from #28



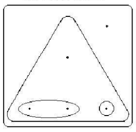
#126
(5,3,3,3,1)
impossible
by Cor. 5.2

from #29



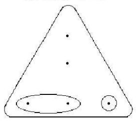
#127
(4,2,2,1)
impossible
by Cor. 5.2

from #29



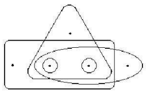
#128
(5,4,2,2,1)
impossible
by Cor. 5.2

from #29



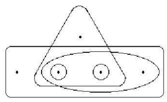
#129
(5,5,2,2,1)
impossible
by Cor. 5.2

from #30



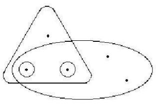
#130
(3,1,1)
$S_2 \uparrow S_5$

from #30



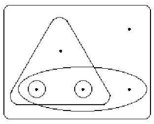
#131
(4,3,1,1)
$S_2 \uparrow S_6$

from #30



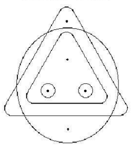
#132
(4,4,1,1)
impossible
by Thm. 7.2

from #30



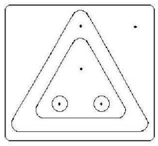
#133
(5,3,3,1,1)
$S_{30} \uparrow S_1$

from #31



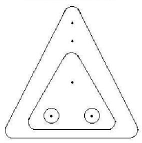
#134
(4,3,1,1)
$S_2 \uparrow S_9$

from #31



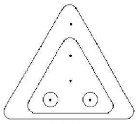
#135
(5,4,3,1,1)
$S_{31} \uparrow S_1$

from #31



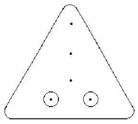
#136
(5,5,3,1,1)
$S_8 \uparrow S_4$

from #32



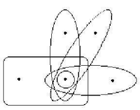
#137
(5,4,4,1,1)
$S_{32} \uparrow S_1$

from #32



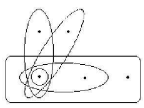
#138
(5,5,5,1,1)
$S_2 \uparrow S_{13}$

from #33



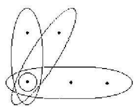
#139
(2,1)
$S_1 \uparrow S_{14}$

from #33



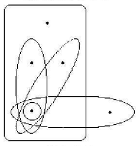
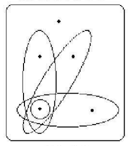
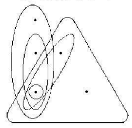
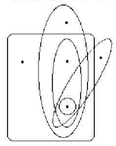
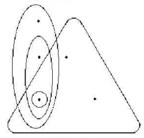
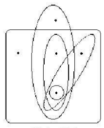
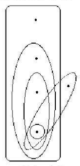
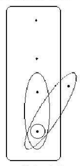
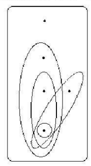
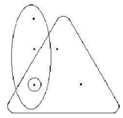
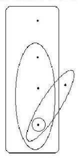
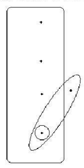
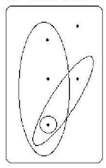
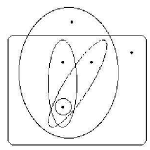
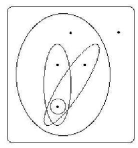
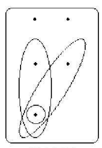
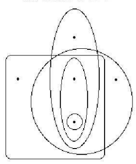
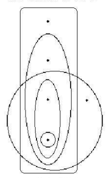
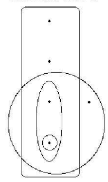
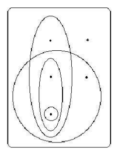
#140
(3,2,1)
$S_1 \uparrow S_{15}$

from #33



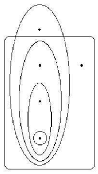
#141
(3,3,1)
$S_3 \odot_w S_{35}$

5-point topologies

from #33	from #33	from #34	from #34	from #34
				
#142	**#143**	**#144**	**#145**	**#146**
(3,2,2,1)	(4,2,2,2,1)	(3,2,1)	(3,2,1)	(3,3,1)
$S_1 \uparrow S_{17}$	$S_{33} \uparrow S_1$	$S_1 \uparrow S_{19}$	$S_1 \uparrow S_{20}$	$S_{10} \odot_{\mathrm{w}} S_{11}$
from #34	from #34	from #34	from #34	from #35
				
#147	**#148**	**#149**	**#150**	**#151**
(4,2,2,1)	(4,3,2,1)	(4,4,2,1)	(5,3,2,2,1)	(3,3,1)
$S_1 \uparrow S_{22}$	$S_1 \uparrow S_{23}$	$S_1 \uparrow S_{24}$	$S_{34} \uparrow S_1$	$S_1 \uparrow S_{26}$
from #35	from #35	from #35	from #36	from #36
				
#152	**#153**	**#154**	**#155**	**#156**
(4,3,3,1)	(4,4,4,1)	(5,3,3,2,1)	(4,2,2,1)	(5,4,2,2,1)
$S_{40} \odot_{\mathrm{w}} S_3$	$S_3 \odot_{\mathrm{w}} S_{41}$	$S_{35} \uparrow S_1$	$S_1 \uparrow S_{30}$	$S_{36} \uparrow S_1$
from #36	from #37	from #37	from #37	from #37
				
#157	**#158**	**#159**	**#160**	**#161**
(5,5,2,2,1)	(3,2,1)	(4,3,2,1)	(4,4,2,1)	(5,3,3,2,1)
$S_9 \uparrow S_4$	$S_1 \uparrow S_{14}$	$S_1 \uparrow S_{34}$	$S_1 \uparrow S_{35}$	$S_{37} \uparrow S_1$

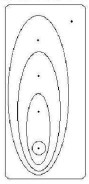
5-point topologies

from #38



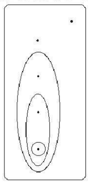
#162
(4,3,2,1)
$S_1 \uparrow S_{37}$

from #38



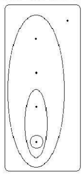
#163
(5,4,3,2,1)
$S_{38} \uparrow S_1$

from #38



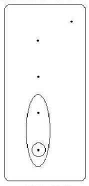
#164
(5,5,3,2,1)
$S_1 \uparrow S_{39}$

from #39



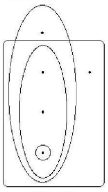
#165
(5,4,4,2,1)
$S_{39} \uparrow S_1$

from #39



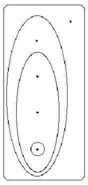
#166
(5,5,5,2,1)
$S_1 \uparrow S_{41}$

from #40



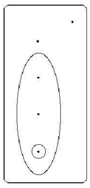
#167
(4,3,3,1)
$S_1 \uparrow S_{42}$

from #40



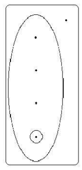
#168
(5,4,3,3,1)
$S_{40} \uparrow S_1$

from #40



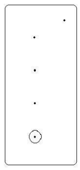
#169
(5,5,3,3,1)
$S_1 \uparrow S_{44}$

from #41



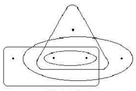
#170
(5,4,4,4,1)
$S_{41} \uparrow S_1$

from #41



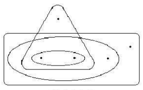
#171
(5,5,5,5,1)
$S_1 \uparrow S_{46}$

from #42



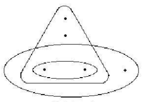
#172
(3,2,2)
$S_4 \uparrow S_5$

from #42



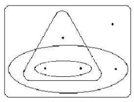
#173
(4,3,2,2)
$S_2 \uparrow S_6$

from #42



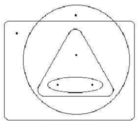
#174
(4,4,2,2)
see remarks
before Thm. 7.2

from #42



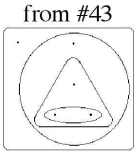
#175
(5,3,3,2,2)
$S_{42} \uparrow S_1$

from #43



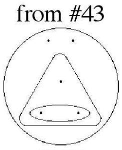
#176
(4,3,2,2)
$S_4 \uparrow S_9$

from #43



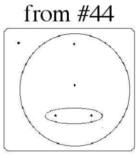
#177
(5,4,3,2,2)
$S_{43} \uparrow S_1$

from #43



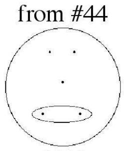
#178
(5,5,3,2,2)
$S_4 \uparrow S_{11}$

from #44



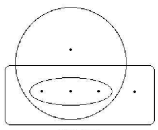
#179
(5,4,4,2,2)
$S_{44} \uparrow S_1$

from #44



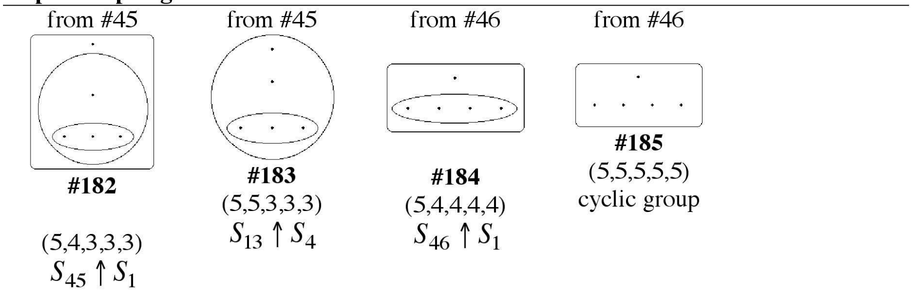
#180
(5,5,5,2,2)
$S_4 \uparrow S_{13}$

from #45



#181
(4,3,3,3)
$S_{13} \uparrow S_2$

5-point topologies



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