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New coincidence and common fixed point theorems

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Abstract. In this paper, we obtain some extensions and a generalization of a remarkable fixed point theorem of Proinov. Indeed, we obtain some coincidence and fixed point theorems for asymptotically regular non-self and self-maps without requiring continuity and relaxing the completeness of the space. Some useful examples and discussions are also given.

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1. Introduction

The well-known Banach fixed point theorem has been generalized and extended by many authors in various ways. Recently, Proinov [15] has obtained two types of generalizations of Banach's fixed point theorem. The first type involves Meir- Keeler type conditions (see, for instance, Cho *et al.* [3], Jachymski [6], Lim [10], Matkowski [11], Park and Rhoades [14]) and the second type involves contractive gauge functions (see, for instance, Boyd and Wong [1] and Kim *et al.* [9]). Proinov [15] obtained equivalence between these two types of contractive conditions and also obtained a new fixed point theorem. Inspired by Jungck [7], Naimpally *et al.* [13], Proinov [15] and Romaguera [19], we obtain coincidence theorems on a very general setting and derive various fixed point theorems. Some special cases are also discussed.

In all that follows Y is an arbitrary non-empty set, (X, d) a metric space and $\mathbb{N} := \{1, 2, 3, ..., \}$. For $T, f : Y \to X$, let $C(T, f)$ denote the set of coincidence points of T and f, that is $C(T, f) := \{z \in Y : Tz = fz\}$.

The following definition comes from Sastry *et al.* [20] and S. L. Singh *et al.* [21].

Definition 1.1. *Let S, T and f be maps on Y with values in a metric space (X,d). The pair (S,T) is asymptotically regular with respect to f at $x_0 \in Y$ if there exists a sequence $\{x_n\}$ in Y such that*

$$fx_{2n+1} = Sx_{2n},\ fx_{2n+2} = Tx_{2n+1},\ n = 0,1,2,...,\ \text{and}$$
$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0.$$

If $Y = X$ and $S = T$ then we get the definition of asymptotic regularity of T with respect to f due to Rhoades *et al.* [18]. Further if $Y = X$, $S = T$ and f is the identity map on X, then we get the usual definition of asymptotic regularity for a map T due to Browder and Peteryshyn [2].

Definition 1.2 ([16]). *Let (X,d) be a metric space and $T,f : X \to X$. Then the self-maps T and f are R-weakly commuting if there exists a positive real number R such that*

$$d(Tfx, fTx) \le Rd(Tx, fx) \text{ for all } x \in X.$$

Following Itoh and Takahashi [5] and Singh and Mishra [22], we have the following definition for a pair of self-maps on a metric space X.

Definition 1.3. *Let $T,f : X \to X$. Then the pair (T,f) is (IT)-commuting at $z \in X$ if $Tfz = fTz$. They are (IT)-commuting on X (also called weakly compatible, by Jungck and Rhoades* [8]*) if $Tfz = fTz$ for all $z \in X$ such that $Tz = fz$.*

Definition 1.4 ([15] Definition 2.1 (i)). *Let ϕ denote the class of all functions $\varphi : R_+ \to R_+$ satisfying: for any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon < t < \delta$ implies $\varphi(t) \le \varepsilon$.*

2. Main Results

Proinov [15] obtained the following result generalizing some fixed point theorems of Jachymski [6] and Matkowski [11].

Theorem 2.1 ([15, Th. 4.1]). *Let T be a continuous and asymptotically regular self-map on a complete metric space (X,d) satisfying the following conditions:*

(P1): *$d(Tx,Ty) \le \varphi(D(x,y))$, for all $x,y \in X$;*
(P2): *$d(Tx,Ty) < D(x,y)$, for all distinct $x,y \in X$,*
where $D(x,y) = d(x,y) + \gamma[d(x,Tx) + d(y,Ty)]$, $\gamma \ge 0$ and $\varphi \in \phi$.

Then T has a unique fixed point.

Moreover if $D(x,y) = d(x,y) + d(x,Tx) + d(y,Ty)$ and φ is continuous and satisfies $\varphi(t) < t$ for all $t > 0$, then continuity of T can be dropped.

For a self-map $T : X \to X$ the quasi-contraction due to Ćirić [4] is as follows

(C) $$d(Tx,Ty) \le qM(x,y),$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$, $0 \le q < 1$.

We remark that following the listing of conditions due to Rhoades [17] the condition (C) is the condition (24). According to Rhoades [17] the condition (25):

$$d(Tx, Ty) < M(x, y),$$

is the most general condition among the contractive conditions.

The following example shows that (P1) is more general than condition (C).

Example 2.2. Let $X = \{1, 2, 3\}$ with the usual metric d and $T : X \to X$ such that

$$T1 = 1,\ T2 = 3,\ T3 = 1. \text{ Then } T \text{ satisfies (C) with } q > 1.$$

Clearly, the condition (P1) is satisfied with $\varphi(t) = \frac{t}{2}$ for all $t > 0$ and $\varphi(0) = 0$ and $\gamma \geq 1$.

Evidently T can not satisfy the conditions (24) and (25) listed by Rhoades [17].

First we extend the scope of Theorem 2.1 by introducing a dummy map f in Theorem 2.1. This idea comes essentially from Jungck [7].

We remark that the requirement "$\varphi(t) < t$ for all $t > 0$" in Theorem 2.1 is redundant as this is the consequence of Definition 1.4. We shall use this fact in the proof of the following theorem.

Theorem 2.3. *Let T and f be self-maps on a complete metric space (X, d) such that*

- **(A1):** *$T(X) \subseteq f(X)$;*
- **(A2):** *$d(Tx, Ty) \leq \varphi(g(x, y))$ for all $x, y \in X$,*
  where $g(x, y) = d(fx, fy) + \gamma[d(fx, Tx) + d(fy, Ty)]$, $\gamma \geq 0$ and $\varphi \in \phi$ is continuous;
- **(A3):** *$d(Tx, Ty) < g(x, y)$ for all distinct $x, y \in Y$;*
- **(A4):** *(T, f) is asymptotically regular at $x_0 \in X$.*

If T is continuous then T has a fixed point provided that T and f are R-weakly commuting. Further if f is continuous and $\gamma = 1$ then T and f have a unique common fixed point provided that T and f are R-weakly commuting.

Proof. Pick $x_0 \in X$. Define a sequence $\{y_n\}$ by $y_{n+1} = Tx_n = fx_{n+1}$, $n = 0, 1, 2, ...$ This can be done since the range of f contains the range of T. Let us fix $\varepsilon > 0$. Since $\varphi \in \phi$, there exists $\delta > \varepsilon$ such that for any $t \in (0, \infty)$,

$$\varepsilon < t < \delta \Rightarrow \varphi(t) \leq \varepsilon. \tag{2.1}$$

Without loss of generality we may assume that $\delta \leq 2\varepsilon$. Since the pair (T, f) is asymptotically regular, $\lim_{n\to\infty} d(y_n,\ y_{n+1}) = 0$. Hence, there exists an integer $N \geq 1$ such that

$$d(y_n, y_{n+1}) < \frac{\delta - \varepsilon}{1 + 2\gamma} \quad \text{for all } n \geq N. \tag{2.2}$$

By induction we shall show that

$$d(y_n, y_m) < \frac{\delta + 2\gamma\varepsilon}{1 + 2\gamma} \quad \text{for all } m, n \in \mathbb{N} \text{ with } m \geq n \geq N. \tag{2.3}$$

Let $n \geq N$ be fixed. Obviously, (2.3) holds for $m = n$. Assuming (2.3) to hold for an integer $m \geq n$, we shall prove it for $m+1$. By the triangle inequality, we get

$$d(y_n, y_{m+1}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{m+1})$$

or

$$d(y_n, y_{m+1}) \leq d(y_n, y_{n+1}) + d(Tx_n, Tx_m). \tag{2.4}$$

We claim that

$$d(Tx_n, Tx_m) \leq \varepsilon. \tag{2.5}$$

To prove (2.5), we consider two cases.

Case 1.: Let $g(x_n, x_m) \leq \varepsilon$. By (A2) and (A3),

$$d(Tx_n, Tx_m) \leq g(x_n, x_m) \leq \varepsilon, \text{ and (2.5) holds.}$$

Case 2.: Let $g(x_n, x_m) > \varepsilon$. By (A2),

$$d(Tx_n, Tx_m) \leq \varphi(g(x_n, x_m)). \tag{2.6}$$

By the definition of $g(x, y)$,

$$g(x_n, x_m) = d(y_n, y_m) + \gamma[d(y_n, y_{n+1}) + d(y_m, y_{m+1})].$$

From (2.2) and (2.3),

$$g(x_n, x_m) < \frac{\delta + 2\gamma\varepsilon}{1+2\gamma} + 2\gamma\frac{\delta - \varepsilon}{1+2\gamma} = \delta.$$

Now by (2.1),

$$\varepsilon < g(x_n, x_m) < \delta \Rightarrow \varphi(g(x_n, x_m)) \leq \varepsilon.$$

So (2.6) implies (2.5). From (2.5), (2.4) and (2.2), it follows that

$$d(y_n, y_{m+1}) \leq \frac{\delta - \varepsilon}{1+2\gamma} + \varepsilon = \frac{\delta + 2\gamma\varepsilon}{1+2\gamma}. \text{ This proves(2.3).}$$

Since $\delta \leq 2\varepsilon$, (2.3) implies that $d(y_n,\ y_m) < 2\varepsilon$ for all integers m and n with $m \geq n \geq N$. So $\{y_n\}$ is a Cauchy sequence. Since the space X is complete the sequence $\{y_n\}$ has a limit. Call it z.

Suppose T is continuous. Then $TTx_n \to Tz$ and $Tfx_n \to Tz$. Since T and f are R-weakly commuting,

$$d(Tfx_n, fTx_n) \leq Rd(Tx_n, fx_n).$$

Making $n \to \infty$,

$$fTx_n \to Tz. \text{ If } z \neq Tz, \text{ then by (A2),}$$

$$\begin{aligned} d(Tx_n, TTx_n) &\leq \varphi(g(x_n, Tx_n) \\ &= \varphi(d(fx_n, fTx_n) + \gamma[d(fx_n, Tx_n) + d(fTx_n, TTx_n)]). \end{aligned}$$

Making $n \to \infty$,

$d(z, Tz) \leq \varphi(d(z, Tz) < d(z, Tz)$, a contradiction. It follows that $z = Tz$.

If f continuous and $\gamma = 1$. Then $ffx_n \to fz$ and $fTx_n \to fz$. Since T and f are R-weakly commuting,

$$d(Tfx_n, fTx_n) \leq Rd(Tx_n, fx_n).$$

Making $n \to \infty$,

$$Tfx_n \to fz. \text{ If } z \neq fz, \text{ then by (A2),}$$

$$\begin{aligned} d(Tx_n, Tfx_n) &\leq \varphi(g(x_n, fx_n) \\ &= \varphi(d(fx_n, ffx_n) + \gamma[d(fx_n, Tx_n) + d(ffx_n, Tfx_n)]). \end{aligned}$$

Making $n \to \infty$,

$$d(z, fz) \leq \varphi(d(z, fz) < d(z, fz), \text{ a contradiction. It follows that } z = fz.$$

Now if $z \neq Tz$, then by (A2),

$$\begin{aligned} d(Tz, Tfx_n) &\leq \varphi(g(z, fx_n) \\ &= \varphi(d(fz, ffx_n) + [d(fz, Tz) + d(ffx_n, Tfx_n)]). \end{aligned}$$

Making $n \to \infty$,

$$d(Tz, fz) \leq \varphi(d(Tz, fz) < d(Tz, fz), \text{ a contradiction.}$$

It follows that $Tz = fz = z$, and z is a common fixed point of f and T. Uniqueness follows easily. □

We remark that Theorem 2.1 is obtained from Theorem 2.3 as a corollary. Notice that conditions (P1) and (P2) come respectively from (A2) and (A3) when f is the identity map on X. Further, the continuity of only one map is needed. The following example shows the superiority of Theorem 2.3 over Theorem 2.1.

Example 2.4. Let $X = [0, \infty)$ with usual metric d. Let $T : X \to X$ such that

$$Tx = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Theorem 2.1 is not applicable to this map T as it is not continuous. However, if we take a (dummy) map $f : X \to X$ such that $fx = 2x$ for all $x \in X$ then T and f satisfy all the hypotheses of Theorem 2.3. Notice that f is continuous and $T0 = f0 = 0$.

Now we modify certain requirements of Theorem 2.3 a slightly to obtain a new result.

Theorem 2.5. *Let T and f be maps on an arbitrary non-empty set Y with values in a metric space (X, d) such that*

(B1): *$T(Y) \subseteq f(Y)$;*

(B2): *$d(Tx, Ty) \leq \varphi(g(x, y))$ for all $x, y \in Y$,*
where $g(x, y) = d(fx, fy) + \gamma[d(fx, Tx) + d(fy, Ty)]$, $0 \leq \gamma \leq 1$, and $\varphi : R_+ \to R_+$ continuous;

(B3): (T,f) *is asymptotically regular at* $x_0 \in Y$.

If $T(Y)$ *or* $f(Y)$ *is a complete subspace of* X *then*

(i): $C(T,f)$ *is non-empty.*

Further, if $Y = X$, *then*

(ii): T *and* f *have a unique common fixed point provided that* T *and* f *are* (IT)*-commuting at a point* $u \in C(T,f)$.

Proof. Pick $x_0 \in Y$. Define a sequence $\{y_n\}$ by $y_{n+1} = Tx_n = fx_{n+1}$, $n = 0,1,2...$, this can be done since the range of f contains the range of T. Since the pair (f,T) is asymptotically regular, $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

First we shall show that $\{y_n\}$ is a Cauchy sequence. Suppose $\{y_n\}$ is not Cauchy. Then there exists $\mu > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that for all $n \le m_k < n_k$,

$$d(y_{m_k}, y_{n_k}) \ge \mu \textit{ and } d(y_{m_k}, y_{n_k-1}) < \mu.$$

By the triangle inequality,

$$d(y_{m_k}, y_{n_k}) \le d(y_{m_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}).$$

Making $k \to \infty$,

$$d(y_{m_k}, y_{n_k}) < \mu.$$

Thus, $d(y_{m_k}, y_{n_k}) \to \mu$ *as* $k \to \infty$. Now by (B2),

$$\begin{aligned} d(y_{m_k+1}, y_{n_k+1}) &= d(Tx_{m_k}, Tx_{n_k}) \\ &\le \varphi(g(x_{m_k}, x_{n_k})) \\ &= \varphi(d(fx_{m_k}, fx_{n_k}) + \gamma[d(fx_{m_k}, Tx_{m_k}) + d(fx_{n_k}, Tx_{n_k})]). \end{aligned}$$

Making $k \to \infty$,

$$\mu \le \varphi(\mu) < \mu,$$

a contradiction. Therefore $\{y_n\}$ is Cauchy. Suppose $f(Y)$ is complete. Then $\{y_n\}$ being contained in $f(Y)$ has a limit in $f(Y)$. Call it z. Let $u \in f^{-1}z$. Then $fu = z$. Using (B2),

$$d(Tu, Tx_n) \le \varphi(d(fu, fx_n) + \gamma[d(Tu, fu) + d(Tx_n, fx_n)]).$$

Making $n \to \infty$,

$$d(Tu, z) \le \varphi(\gamma d(Tu, z)) < d(Tu, z),$$

a contradiction. Therefore $Tu = z = fu$. This proves (i). Now if $Y = X$ and the pair(T,f) is (IT)-commuting at u then $Tfu = fTu$ and $TTu = Tfu = fTu = ffu$. In view of (B2), it follows that

$$\begin{aligned} d(Tu, TTu) &< \varphi(g(u, Tu)) \\ &= \varphi(d(fu, fTu) + \gamma[d(Tu, fu) + d(TTu, fTu)]) < d(Tu, TTu), \end{aligned}$$

a contradiction. Therefore $TTu = Tu$ and $fTu = TTu = Tu = z$. This proves (ii).

In the case $T(Y)$ is a complete subspace of X, the condition (B1) implies that sequence $\{y_n\}$ converges in $f(Y)$, and the previous proof works. The uniqueness of common fixed point follows easily. □

The following result generalizes an important result of Proinov [15, Cor. 4.3]

Corollary 2.6. *Let T and f be maps on an arbitrary non-empty set Y with values in metric space (X,d) such that*

(C1): $T(Y) \subseteq f(Y)$;

(C2): $d(Tx,Ty) \leq \varphi(M(x,y))$, *for all* $x,y \in Y$, *where* $M(x,y) = max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}[d(fx,Ty)+d(fy,Tx)]\}$ *and* $\varphi : R_+ \to R_+$ *continuous.*

If $T(Y)$ or $f(Y)$ is a complete subspace of X then conditions (i) *and* (ii) *of above Theorem* 2.5 *hold.*

Now we obtain a new common fixed point theorem for three non self-maps.

Theorem 2.7. *Let S,T and f be maps on an arbitrary non-empty set Y with values in a metric space (X,d). Let (S,T) be asymptotically regular with respect to f at $x_0 \in Y$ and the following conditions are satisfied:*

(D1): $S(Y) \cup T(Y) \subseteq f(Y)$;

(D2): $d(Sx,Ty) \leq \varphi(h(x,y))$, *for all* $x,y \in X$, *where* $h(x,y) = d(fx,fy) + \gamma[d(Sx,fx) + d(Ty,fy)]$, $0 \leq \gamma \leq 1$, *and* $\varphi : R_+ \to R_+$ *continuous.*

If $S(Y)$ or $T(Y)$ or $f(Y)$ is a complete subspace of X then

(I): *$C(S,f)$ is non-empty;*

(II): *$C(T,f)$ is non-empty.*

Further, if Y=X then

(III): *S and f have a common fixed point provided that S and f are* (IT)*-commuting at a point $u \in C(S,f)$.*

(IV): *T and f have a common fixed point provided that T and f are* (IT)*-commuting at a point $v \in C(T,f)$.*

(V): *S,T and f have a unique common fixed point provided that* (III) *and* (IV) *both are true.*

Proof. Let x_0 be an arbitrary point in Y. Since (S,T) is asymptotically regular with respect to f, then there exists a sequence $\{x_n\}$ in Y such that

$$fx_{2n+1} = Sx_{2n},\ fx_{2n+2} = Tx_{2n+1},\ n = 0,1,2,...,\ \text{and}$$

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0.$$

Now we shall show that $\{fx_n\}$ is Cauchy sequence. Suppose $\{fx_n\}$ is not Cauchy. Then there exists $\mu > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers, such that for all $n \leq m_k < n_k$,

$$d(fx_{m_k}, fx_{n_k}) \geq \mu \text{ and } d(fx_{m_k}, fx_{n_k-1}) < \mu.$$

By the triangle inequality,

$$d(fx_{m_k}, fx_{n_k}) \leq d(fx_{m_k}, fx_{n_k-1}) + d(fx_{n_k-1}, fx_{n_k}).$$

Making $k \to \infty$, we get

$$d(fx_{m_k}, fx_{n_k}) < \mu.$$

Thus

$$d(fx_{m_k}, fx_{n_k}) \to \mu \ as \ k \to \infty.$$

By (D2) we have

$$\begin{aligned} d(fx_{m_k+1}, fx_{n_k+1}) &= d(Sx_{m_k}, Tx_{n_k}) \\ &\leq \varphi(h(x_{m_k}, x_{n_k})) \\ &= \varphi(d(fx_{m_k}, fx_{n_k}) + \gamma[d(Sx_{m_k}, fx_{m_k}) + d(Tx_{n_k}, fx_{n_k})]). \end{aligned}$$

Making $k \to \infty$

$$\mu \leq \varphi(\mu) < \mu, \text{ a contradiction.}$$

Thus $\{fx_n\}$ is Cauchy sequence. Suppose $f(Y)$ is a complete subspace of X. Then $\{y_n\}$ being contained in $f(Y)$ has a limit in $f(Y)$. Call it z. Let $u = f^{-1}z$. Thus $fu = z$ for some $u \in Y$. Note that the subsequences $\{fx_{2n+1}\}$ and $\{fx_{2n+2}\}$ also converge to z. Now by (D2),

$$d(Su, T_{2n+1}) \leq \varphi(d(fu, f_{2n+1}) + \gamma[d(Su, fu) + d(T_{2n+1}, f_{2n+1})]).$$

Making $n \to \infty$,

$$d(Su, fu) \leq \varphi(\gamma d(Su, fu)) < d(Su, fu) \text{ a contradiction.}$$

Therefore $Su = fu = z$. This proves (I). Since $S(Y) \cup T(Y) \subseteq f(Y)$. Therefore there exists $v \in Y$ such that $Su = fv$. We claim that $fv = Tv$. Using (D2),

$$\begin{aligned} d(fv, Tv) &= d(Su, Tv) \\ &\leq \varphi(d(fu, fv) + \gamma[d(Su, fu) + d(Tv, fv)]) \\ &= \varphi(\gamma d(fv, Tv)) < d(fv, Tv), \end{aligned}$$

which is a contradiction. Therefore $Tv = fv = Su = fu$. This proves (II). Now if $Y = X$, (S, f) and (T, f) are (IT)-commuting then $Sfu = fSu$ and $SSu = Sfu = fSu = ffu$, $Tfv = fTv$ and $TTv = Tfv = fTv = ffv$. In view of (D2), it follows that

$$\begin{aligned} d(SSu, Su) &= d(SSu, Tv) \\ &\leq \varphi(d(fSu, fv) + \gamma[d(SSu, fSu) + d(Tv, fv)]) \\ &= \varphi(\gamma d(SSu, Su)) < d(SSu, Su). \end{aligned}$$

Therefore $SSu = Su = fSu$, Su is a common fixed point of S and f. Similarly, Tv is a common fixed point of T and f. Since $Su = Tv$, we conclude that Su is a common fixed point of S, T and f. The proof is similar when $S(Y)$ or $T(Y)$ are complete subspaces of X since, $S(Y) \cup T(Y) \subseteq f(Y)$. Uniqueness of the common fixed point follows easily. □

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