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Pointwise convergence and Ascoli theorems for nearness spaces

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Abstract. We first study subspaces and product spaces in the context of nearness spaces and prove that *U-N* spaces, *C-N* spaces, *P-N* spaces and *totally bounded* nearness spaces are nearness hereditary; *T-N* spaces and *compact* nearness spaces are *N*-closed hereditary. We prove that N_2 plus compact implies *N*-closed subsets. We prove that *totally bounded*, *compact* and N_2 are productive. We generalize the concepts of neighborhood systems into the nearness spaces and prove that the nearness neighborhood systems are consistent with existing concepts of neighborhood systems in topological spaces, uniform spaces and proximity spaces respectively when considered in the respective sub-categories. We prove that a net of functions is convergent under the pointwise convergent nearness structure if and only if its cross-section at each point is convergent. We have also proved two Ascoli-Arzelà type of theorems.

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1. Introduction

As a natural extension of geometry, the concept of "near/apart" has been a center for topology and related studies. Topology characteries the "nearness" between a point and a set. Proximity [14] is an axiomatization of "nearness" between two sets. Contiguity [10] describes the concept of nearness among the elements of a finite family of sets. The concept of "nearness space" introduced by Herrlich [8] in 1974 attempts to characterize the nearness of an arbitrary collection of sets. The category of nearness spaces, the most general among the aforementioned structures, can be used as a unifying framework. The

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categories of several aforementioned structures can all be "nicely embedded" into the category of the nearness spaces as (either bireflective or bicoreflective) sub-categories ([8]).

In recent years, the notion of "nearness" in a number of different variations has found new applications in digital topology, image processing and pattern recognition areas, perhaps due to the fact that those structures are "richer" than classical topology. In 1995, Latecki and Prokop [12] used a weaker version of proximity spaces called semi-proximity spaces (sp-spaces). They talked about the possibility of describing all digital pictures used in computer vision and computer graphs as non-trivial semi-proximity spaces, which is not possible in classical topology. They also discussed the application of "semi-proximity continuous functions" in well-behaved operations such as thinning on digital images. In 1996, Chaudhuri [4] introduced a new definition for the neighbors of an arbitrary point P. This new definition used a "centroid criterion" to capture the idea that the neighbors of P should be as near to P and as symmetrically paced around P as possible. This new definition could be used for pattern classification, clustering and low-level description of dot patterns. In 2000, Ptak and Kropatsch [16] discussed the application of proximity spaces in studying of digital images. They showed by examples that the "proximate complexity" of a finite covering in a digital picture might be too high to be adequately depicted in a finite topological space, which might indicate another conceptual advantage of proximities over topologies. Most recently, in 2007, Wolski and Peters [18, 15] investigated approximation spaces in the context of topological structures which axiomatized certain notion of nearness. Peters [15] pointed out particularly that the concept of "nearness" was not confined to spatial nearness, or geometrical likeness. It was possible to introduce a nearness relation that could be used to determine the "nearness" among sets of objects that were spatially far away and, yet, "qualitatively" near to each other.

The main objectives of this paper are to establish a "pointwise convergent" nearness structure on a function space made of a family of functions from X to Y and to establish two versions of the Ascoli-Arzelà theorems for nearness spaces that relate the compactness of the underlining space Y with that of the function space. Since the function space is really a subspace of the product space Y^X, we begin with the nearness structures on a subspace and discuss the hereditary properties for a number of important concepts in nearness spaces. We then define the nearness structure on a product spaces and discuss its various properties. The nearness structure on a function space is then introduced as a subspace of the product space Y^X. We will end the discussion with two Ascoli-Arzelà type of theorems.

Some work in the past, such as [2] (1979), [7] (1979) and [1] (2006) have discussed a number of results with respect to subspaces and product spaces. Most of the results in that paper were dealing with topological nearness (T-N) spaces and do not duplicate what are to be presented in this paper. For the purpose of clarity and being self-contained, we will still give the definition of "subspace" and "product" space here and prove relevant results.

The classical Ascoli-Arzelà theorem was proved in the 19th century first by Ascoli and then independently by Arzelà. It characterizes compactness of sets of continuous real-valued functions on the interval [0,1] with respect to the topology of uniform convergence. It is commonly known that the issue came from the fact that a convergent sequence of continuous functions may not converge to a continuous function. So the natural question is: under what conditions the limit of a convergent sequence of continuous functions is still continuous. It turned out that the concept of *equicontinuous* was used to characterize the condition needed (see [11]) in topological spaces. In 1970, [13] discussed Ascoli's theorem for the spaces of multifunctions. In 1981, [6] discussed Ascoli's theorem for topological categories. In 1984, [3] discussed Ascoli's theorem for a class of merotopic spaces. In 1993, [5] studied a version of the Ascoli's theorem for set valued proximally continuous functions. In 2001, [17] proved a version of Ascoli's theorem for sequential spaces. As far as we know, no nearness space version of the Ascoli's theorem has been established yet at this time.

We have practical reason to be interested in this topic. In many cases, a digital image processing algorithm is essentially the application of a sequence of deformation functions to a digital plane. For example, [12] proved that a deletion of a simple point (a point that does not affect the connectness of the digital picture) can be regarded as a sp-continuous function. Hence a thinning algorithm that preserves connectness can be arranged as a sequence of sp-continuous functions. We may be able to use the tools of function spaces, and the results on convergence of function sequences to study the image processing algorithms, which opens a new set of doors.

The rest of this paper is organized as follows: Section 2 is a collection of the major definitions involving nearness spaces that are relevant to this paper. Section 3 studies the nearness structures on a subspaces. Section 4 is about the product spaces and the function spaces. The summary at the end concludes this paper.

2. Notation and Definitions

In this section, we define the basic concepts used throughout this paper.

We will use the language of Categories in some of our discussions. For readers who are not familiar with Category theory, a category is basically a family of objects with a particular type of structures. For example, we can talk about the category of all topological spaces, the category of all groups, etc. The so called "morphism" from one object to another is a function that preserves the structure on the objects. For example, a "morphism" in the category of topological spaces would be a continuous function. A "morphism" in the category of all groups would be a homomorphism. An embedding from one category into another category is a way to assign each object from one category to an object of the other category in some injective manner that also preserves the morphisms. For readers who are interested at further information about category theory, please see [20].

The readers can see Kelley [11] or any common general topology text book for terms in general topology.

2.1. **Basic Notations.** Let X be a set, $\mathcal{P}(X)$ represents the power set of X. $\mathcal{P}^0(X) = X, \mathcal{P}^1(X) = \mathcal{P}(X), \ldots, \mathcal{P}^n(X) = \mathcal{P}(\mathcal{P}^{n-1}(X))$

$A, B, \ldots$ represent elements in $\mathcal{P}(X)$, i.e. subsets in X

$\mathscr{A}, \mathscr{B}, \ldots$ represent elements in $\mathcal{P}^2(X)$, i.e. subsets in $\mathcal{P}(X)$

$\xi, \eta, \ldots$ represent elements in $\mathcal{P}^3(X)$, i.e. subsets in $\mathcal{P}^2(X)$

$\mathscr{A}^{\mathrm{C}} = \{X - A : A \in \mathscr{A}\}$

For each $B \subseteq X$, $\mathscr{A}(B) = \bigcup\{A : A \cap B \neq \phi, A \in \mathscr{A}\}$

$\xi\mathscr{A}$ denotes $\mathscr{A} \in \xi$, $\overline{\xi}\mathscr{A}$ denotes $\mathscr{A} \notin \xi$

$A\xi\, B$ denotes $\{A, B\} \in \xi$. $A\overline{\xi}\, B$ denotes $\{A, B\} \notin \xi$

$cl_\xi A = \{x : \{x\}\, \xi\, A\}, int_\xi A = X - cl_\xi(X - A)$

$cl_\xi\mathscr{A} = \{cl_\xi A : A \in \mathscr{A}\}$, $int_\xi\mathscr{A} = \{int_\xi A : A \in \mathscr{A}\}$

$\mathscr{A} \vee \mathscr{B} = \{A \cup B : A \in \mathscr{A}, B \in \mathscr{B}\}$

$\mathscr{A} \wedge \mathscr{B} = \{A \cap B : A \in \mathscr{A}, B \in \mathscr{B}\}$

$\mathscr{A} \prec \mathscr{B}$ if and only if for any $B \in \mathscr{B}$, there is $A \in \mathscr{A}$ such that $A \subseteq B$. This is referred to as "$\mathscr{B}$ co-refines $\mathscr{A}$".

2.2. **Definitions Related to Nearness Structure.** We will restate some major definitions about nearness spaces here (due to [8]):

(i) Let X be a non-empty set. The ordered pair (X, ξ) is said to be a *nearness* space, or N-space, if the following are satisfied:
(N1) If $\bigcap\mathscr{A} \neq \phi$, then $\xi\mathscr{A}$.
(N2) If $\xi\mathscr{B}$, and for each $A \in \mathscr{A}$, there exists a $B \in \mathscr{B}$ such that $B \subseteq cl_\xi A$, then $\xi\mathscr{A}$, i.e. $\mathscr{B} \prec cl_\xi\mathscr{A}$.
(N3) If $\overline{\xi}\mathscr{A}$ and $\overline{\xi}\mathscr{B}$, then $\overline{\xi}(\mathscr{A} \vee \mathscr{B})$.
(N4) If $\phi \in A$, then $\overline{\xi}\mathscr{A}$.

(ii) Let X be a set. Let (X, ξ) and (Y, η) be two N-spaces. A function $f : X \to Y$ is said to be a (ξ, η) *N-preserving* map, or an *N-preserving* map, or simply an *N-mapping*, if one of the following two equivalent conditions is satisfied:
(M1) If $\xi\mathscr{A}$, then $\eta f(\mathscr{A})$, where $f(\mathscr{A}) = \{f(A) : A \in \mathscr{A}\}$
(M2) If $\overline{\eta}\mathscr{B}$, then $\overline{\xi}\, f^{-1}(\mathscr{B})$, where $f^{-1}(\mathscr{B}) = \{f^{-1}(B) : B \in \mathscr{B}\}$.
We will use the notation $\mathbb{NEAR}$ to represent the category of all nearness spaces with N-mappings.

(iii) An N-space is called an N_1-space, if the following is satisfied:
(N_0) If $\{x\}\xi\{y\}$, then $x = y$.

(iv) An N-space is called an N_2 space, if for any $x, y \in X$, $x \neq y$ implies the existence of $A \subseteq X$ and $B \subseteq X$ such that $A \cap B = \phi$, $\overline{\xi}\{\{x\}, X - A\}$ and $\overline{\xi}\{\{y\}, X - B\}$.

(v) An N-space is called a T-N space, if the following is satisfied:
(T) If $\xi\mathscr{A}$, then $\bigcap \mathrm{cl}_\xi\mathscr{A} \neq \phi$.
A $T - N_2$ space is an N_2 space that also satisfies the condition (T).

We will use the symbol $\mathbb{T}-\mathbb{NEAR}$ to represent the subcategory of $\mathbb{NEAR}$, consists of all T-N spaces with N-mappings.

(vi) An N-space is called a *U-N* space, if the following is satisfied:

(U) If $\overline{\xi}\mathscr{A}$, then there exists a $\mathscr{B}$ such that $\overline{\xi}\mathscr{B}$ and for each $B \in \mathscr{B}$, there is an $A \in \mathscr{A}$ such that $\mathscr{B}^{\mathrm{C}}(X-B) \subseteq X-A$.

We will use the notation $\mathbb{U}-\mathbb{NEAR}$ to represent the subcategory of $\mathbb{NEAR}$, consists of all U-N spaces with N-mappings.

(vii) An N-space is called a *C-N* space, if the following is satisfied:

(C) If $\overline{\xi}\mathscr{A}$, then there is a finite subcollection $\mathscr{B} \subseteq \mathscr{A}$ such that $\overline{\xi}\mathscr{B}$.

We will use the notation $\mathbb{C}-\mathbb{NEAR}$ to represent the subcategory of $\mathbb{NEAR}$, consists of all C-N spaces with N-mappings.

(viii) An N-space is called a *P-N* space, if it satisfies both of the conditions (U) and (C).

We will use the notation $\mathbb{P}-\mathbb{NEAR}$ to represent the subcategory of $\mathbb{NEAR}$, consists of all P-N spaces with N-mappings.

(ix) An N-space is called a *totally bounded* space, if one of the following equivalent conditions is satisfied:

(B1) If $\overline{\xi}\mathscr{A}$, then there is a finite subcollection $\mathscr{B} \subseteq \mathscr{A}$ such that $\bigcap \mathscr{B} = \phi$.

(B2) If $\mathscr{F}$ is a filter on X, then $\xi\mathscr{F}$.

(x) An N-space is called a *compact* space, if it satisfies both condition (T) and (C).

(xi) Let X be a set. $\{\xi_\alpha : \alpha \in \Lambda\}$ is a family of N-structures on X. The *least upper bound*, denoted by $\xi = \sup\{\xi_\alpha : \alpha \in \Lambda\}$, is defined as follows:

$\overline{\xi}\mathscr{A}$ if and only if there are finitely many $\mathscr{A}_i$'s such that for each $i = 1, 2, ...n$, $\overline{\xi_i}\mathscr{A}_i$, and $\mathscr{A} \prec \bigvee_{i=1}^{n} \mathscr{A}_{\mathrm{i}}$.

(xii) Let (X, ξ) be a nearness space. Then the topology induced by the closure operator $A \mapsto cl_\eta A$ is denoted by $\mathscr{T}_\xi$.

3. Nearness Structure on Subspaces

We will begin by giving the definition of a "nearness subspace", then proceed to show that subspaces as defined here are well-defined (Theorem 3.2), act "natural" (Lemma 3.3) and produces a topology that is consistent with the subspace topology (Theorem 3.8).

Definition 3.1. *Let (X, ξ) be a nearness space and $X_0 \subseteq X$. Define ξ_0 on X_0 as follows:*

$$\xi_0 = \{\mathscr{A} \subseteq \mathcal{P}(X_0) : \xi\mathscr{A}\}.$$

We will denote such ξ_0 as $\xi_0 = \xi|_{X_0}$ and refer to it as the "nearness structure on the subspace induced by the nearness structure ξ".

Theorem 3.2. *Let (X, ξ) be a nearness space, and $X_0 \subseteq X$. ξ_0, as defined in Definition 3.1, is a nearness structure on X_0.*

The proof is an easy deduction from the fact that ξ_0 is consists of the type of $\mathscr{A}$ that are in ξ already. We will skip the details. The following lemma is also easy to prove.

Lemma 3.3. *Let (X,ξ) be a nearness space and $X_0 \subseteq X$. Let $i : X_0 \to X$ be the inclusion map, then $\xi_0 = i^{-1}(\xi)$. Hence $i : X_0 \to X$ is N-preserving.*

Lemma 3.4. *Let X be a set, (Y,η) be a nearness space. Let $f : X \to Y$ be an injective map. Then*

$$\mathscr{T}_{f^{-1}(\eta)} = f^{-1}(\mathscr{T}_\eta).$$

Proof.

$$\begin{aligned} cl_{f^{-1}(\eta)}A &= \{x : \{x\}f^{-1}(\eta)A\} \\ &= \{x : f(x)\eta\, f(A)\} \\ &= \{x : f(x) \in cl_\eta(f(A))\} \\ &= \{x : x \in f^{-1}(cl_\eta(f(A))\} \\ &= f^{-1}(cl_\eta(f(A)). \end{aligned}$$

□

We will next exam whether some of the common properties are hereditary. i.e. whether a particular condition or property can be "inherited" by its subspaces from their "mother" spaces. It turns out that being a *T-N* space is not hereditary (Example 3.5). Neither was being a *compact* N-space. Those two properties can be inhered by N-closed subspaces. Many of the other properties are hereditary.

Example 3.5. A subspace of a *T-N* space may not be a *T-N* space.

Let $X = R$, the real line with an ordinary open interval topology $\mathscr{T}$. Then $(X,\mathscr{T})$ is a R_0- space, hence corresponding to a T-N space (X,ξ) ([8] Theorem 2.2). Now we let $X_0 = X - \{0\}$, $A = (-\infty, 0)$ and $B = (0,\infty)$. Then $cl_\xi A \cap cl_\xi B \neq \phi$, but $cl_{\xi_0} A \cap cl_{\xi_0} B = \phi$. This means that the nearness subspace (X_0,ξ_0) does not satisfy condition (T), hence not an *T-N* space.

Definition 3.6. *Let (X,ξ) be a nearness space and (X_0,ξ_0) be a subspace. (X_0,ξ_0) is said to be a N-closed subspace, if for any $A \subseteq X_0$, we have $cl_{\xi_0}A = cl_\xi A$.*

Theorem 3.7. *Let (X,ξ) be a nearness space and (X_0,ξ_0) be a subspace. Then (X_0,ξ_0) is a N-closed subspace if and only if $cl_{\xi_0}X_0 = cl_\xi X_0$.*

Proof. The necessity is obvious. We now will prove the sufficiency. Take any $A \subseteq X_0$, then

$$\begin{aligned} cl_{\xi_0}A &= cl_\xi A \cap X_0 \\ &= cl_\xi A \cap cl_\xi X_0 \\ &= cl_\xi(A \cap X_0) \\ &= cl_\xi A. \end{aligned}$$

□

Theorem 3.8. *Let (X,ξ) be a nearness space and (X_0,ξ_0) be a subspace. Then*
(a) $\mathscr{T}_{\xi_0} = \mathscr{T}_{\xi}|_{X_0}$.
(b) If (X,ξ) is a T-N space and (X_0,ξ_0) is an N-closed subspace, then (X_0,ξ_0) is also a T-N space.

Proof. (a) $\mathscr{T}_{\xi_0} = \mathscr{T}_{i^{-1}(\xi)} = i^{-1}(\mathscr{T}_{\xi}) = \mathscr{T}_{\xi}\big|_{X_0}$.
(b) Let $\xi_0\mathscr{A}_0$, then $\xi\mathscr{A}_0$. It follows from the assumption of (X_0,ξ_0) being an N-closed subspace that $cl_{\xi}A_0 = cl_{\xi_0}A_0$. Therefore $\bigcap\{cl_{\xi_0}A_0 : A_0 \in \mathscr{A}_0\} = \bigcap\{cl_{\xi}A_0 : A_0 \in \mathscr{A}_0\} \neq \phi$. So (X,ξ_0) satisfies the condition (T). □

Theorem 3.9. *Let (X,ξ) be a nearness space and (X_0,ξ_0) be a subspace. Then if (X,ξ) is a* U-N *space, so is (X_0,ξ_0). Moreover, $\mathscr{U}_{\xi_0} = \mathscr{U}_{\xi}|_{X_0}$, where $\mathscr{U}_{\xi_0}$ and $\mathscr{U}_{\xi}$ denotes the uniformity induced by the nearness structure ξ_0 and ξ respectively. $\mathscr{U}_{\xi}|_{X_0}$ is the uniformity $\mathscr{U}_{\xi}$ restricted to X_0.*

Proof. If $\overline{\xi}_0\mathscr{A}_0$, then $\overline{\xi}\mathscr{A}_0$. Since (X,ξ) is an *U-N* space, there exists a $\overline{\xi}\mathscr{B}$ that satisfies the condition (U) with respect to $\mathscr{A}_0$. Let $\mathscr{B}_0 = \{B\cap X_0 : B \in \mathscr{B}\}$. From (N2) we can see that $\overline{\xi}_0\mathscr{B}_0$. For each $B_0 \in \mathscr{B}_0 = \{B\cap X_0 : B \in \mathscr{B}\}$, there is a $B \in \mathscr{B}$ such that $B = B\cap X_0$. So by condition (U), there should be an $A_0 \in \mathscr{A}_0$ such that $A_0 \subseteq \bigcap\{C : B\cup C \neq X, C \in \mathscr{B}\}$. Also because $A_0 \subseteq X_0$, we have

$$\begin{aligned} A_0 &= A_0 \cap X_0 \\ &\subseteq \bigcap\{C : B\cup C \neq X, C\in\mathscr{B}\}\cap X_0 \\ &= \bigcap\{C\cap X_0 : B\cup C \neq X, C\in\mathscr{B}\} \\ &\subseteq \bigcap\{C_0 : B_0\cup C_0 \neq X_0, C_0\in\mathscr{B}_0\} \end{aligned}$$

Therefore, ξ_0 satisfies the condition (U). Furthermore, for any $\mathscr{A}_0 \in \mathcal{P}^2(X)$, we have the following equivalent deductions:

$$\begin{aligned} &\mathscr{A}_0 \in \mathscr{U}|_{i^{-1}(\xi)} \\ &\Leftrightarrow \mathscr{A}_0^C \notin i^{-1}(\xi) \\ &\Leftrightarrow \mathscr{A}_0^C \notin \xi_0 \\ &\Leftrightarrow \mathscr{A}_0 \in \mathscr{U}|_{\xi_0} \\ &\Leftrightarrow \mathscr{A}_0 \in i^{-1}(\mathscr{U}_{\xi}). \end{aligned}$$

Therefore,

$$\mathscr{U}_{\xi_0} = \mathscr{U}_{i^{-1}(\xi)} = i^{-1}(\mathscr{U}_{\xi}) = \mathscr{U}_{\xi}|_{X_0}.$$

□

Theorem 3.10. *Let (X,ξ) be a nearness space and (X_0,ξ_0) be a subspace. If (X,ξ) is a* C-N *space, so is (X_0,ξ_0). Moreover, $\mathscr{C}_{\xi_0} = \mathscr{C}_\xi|_{X_0}$, where $\mathscr{C}_{\xi_0}$ and $\mathscr{C}_\xi$ denote the contiguity induced by the nearness structure ξ_0 and ξ respectively. $\mathscr{C}_\xi|_{X_0}$ is the contiguity $\mathscr{C}_\xi$ restricted to X_0.*

Proof. For any $\mathscr{A}_0 \in \mathcal{P}^2(X)$ and $\overline{\xi}_0\mathscr{A}_0$, then $\overline{\xi}\mathscr{A}_0$. By condition (C), $\mathscr{A}_0$ has a finite subcollection $\mathscr{B}_0$ such that $\overline{\xi}\mathscr{B}_0$. This implies that $\overline{\xi}_0\mathscr{B}_0$. And furthermore,

$$\begin{aligned}
&\mathscr{A}_0 \in \mathscr{C}_{\xi_0}\\
&\Leftrightarrow \mathscr{A}_0 \in \xi_0 \text{ and } \mathscr{A}_0 \text{ is finite.}\\
&\Leftrightarrow \mathscr{A}_0 \in \xi,\ \mathscr{A}_0 \text{ is finite}\\
&\Leftrightarrow \mathscr{A}_0 \in \mathscr{C}_\xi\\
&\Leftrightarrow \mathscr{A}_0 \in \mathscr{C}_\xi|_{X_0}.
\end{aligned}$$

□

Since a *P-N* space is one that satisfies condition (U) and (C), the following Theorem is obvious from the Theorems 3.9 and 3.10:

Theorem 3.11. *Let (X,ξ) be a nearness space and (X_0,ξ_0) be a subspace. Then if (X,ξ) is a* P-N *space, so is (X_0,ξ_0).*

Since a *compact* nearness space is one that satisfies condition (T) and (C), the following theorem is obvious from Theorems 3.8 and 3.10:

Theorem 3.12. *Let (X,ξ) be a nearness space and (X_0,ξ_0) be a N-closed subspace. Then if (X,ξ) is a compact nearness space, so is (X_0,ξ_0).*

Lemma 3.13. *Let (X,ξ) be a T-N space. Then (X,cl_ξ) is topologically compact if and only if (X,ξ) is a compact nearness space.*

Proof. Recall that (X,ξ) is a compact nearness space if and only if condition (T) and (C) are satisfied.

Necessity: If (X,cl_ξ) is topologically compact. Take an $\mathscr{A}$ such that $\overline{\xi}\mathscr{A}$. We want to show that condition (C) is met by showing that $\mathscr{A}$ has a finite subcollection $\mathscr{B}$ such that $\overline{\xi}\mathscr{B}$. We will first claim that $\bigcap cl_\xi\mathscr{A} = \phi$. If not, then by (N1), $\xi cl_\xi\mathscr{A}$ would be true. For each $A \in \mathscr{A}$, there would be a $cl_\xi A \in cl_\xi\mathscr{A}$ such that $cl_\xi A \subseteq cl_\xi A$. By (N2), $\xi\mathscr{A}$ would be true, and that would contradict to the assumption of $\overline{\xi}\mathscr{A}$. So $\bigcap cl_\xi\mathscr{A} = \phi$ must be true. $(cl_\xi\mathscr{A})^C$ is an open cover of X. Since (X,cl_ξ) is topologically compact, we will let $\mathscr{B}$ be the finite subcollection of $\mathscr{A}$ and $(cl_\xi\mathscr{B})^C$ is an open cover of X. This implies that $\bigcap cl_\xi\mathscr{B} = \phi$. By condition (T), $\overline{\xi}\mathscr{B}$ is true. Hence condition (C) has been met.
Sufficiency: If (X,ξ) is a compact nearness space, which means that it meets condition (T) and (C). Any open cover of (X,cl_ξ) can be expressed as the complement collection of a collection $cl_\xi\mathscr{A}$ and $\bigcap cl_\xi\mathscr{A} = \phi$. From condition (T), $\overline{\xi}cl_\xi\mathscr{A}$ is true. From condition (C), there must be a finite

subcollection of $cl_\xi\mathscr{A}$, say $cl_\xi\mathscr{B}$, such that $\overline{\xi}cl_\xi\mathscr{B}$ is true. It follows that $\bigcap cl_\xi\mathscr{B} = \phi$. Then $(cl_\xi\mathscr{B})^C$ is the finite subcover of the original open cover. Hence (X, cl_ξ) is topologically compact.

□

From Lemma 3.13, one can easily see the following is true:

Lemma 3.14. *Let (X,ξ) be a compact nearness space, (Y,η) be a T-N space and $f:(X,\xi)\to(Y,\eta)$ be N-preserving, then (Y,η) is a compact nearness space.*

Definition 3.15. *Let X be a set. Define a partial order among all possible nearness structures on X as follows: If ξ_1 and ξ_2 are two nearness structures on a set X, then $\xi_1 \leq \xi_2$ if and only if $\xi_1 \supseteq \xi_2$.*

Lemma 3.16. (i) *Let X be a set and ξ_1 and ξ_2 be two nearness structures on a set X. $\xi_2 \leq \xi_1$ if and only if $i:(X,\xi_1)\to(X,\xi_2)$ is N-preserving.*
(ii) *Let (X,ξ) and (Y,η) be two nearness spaces. Then $f:(X,\xi)\to(Y,\eta)$ is N-preserving if and only if $\xi \geq f^{-1}(\eta)$.*

Proof. (i)

$$\begin{aligned}&\xi_2 \leq \xi_1\\ &\Leftrightarrow \xi_2 \supseteq \xi_1\\ &\Leftrightarrow i(\xi_1)\subseteq \xi_2\\ &\Leftrightarrow i:(X,\xi_1)\to(X,\xi_2) \text{ is N} - \text{preserving.}\end{aligned}$$

(ii)

$$\begin{aligned}&f:(X,\xi)\to(Y,\eta) \text{ is N} - \text{preserving}\\ &\Leftrightarrow \xi\subseteq f^{-1}(\eta)\\ &\Leftrightarrow \xi \geq f^{-1}(\eta).\end{aligned}$$

□

Theorem 3.17. *Let (X,ξ) be a $T-N_2$ space and (X_0,ξ_0) be a N-compact subspace. Then (X_0,ξ_0) is N-closed.*

Theorem 3.18. *If (X,ξ) is totally bounded, and $X_0\subseteq X$, then (X_0,ξ_0) is totally bounded.*

Proof. $\overline{\xi}_0\mathscr{A}_0$ implies that $\overline{\xi}\mathscr{A}_0$. Hence, by condition (B1) for (X,ξ), there is a $\mathscr{B}_0$, a finite subcollection of $\mathscr{A}_0$, such that $\bigcap\mathscr{B}_0=\phi$. So (X_0,ξ_0) satisfies (B1). □

The following lemma proved by Hunsaker and Sharma as Corollary (2.5) in their 1974 paper [9] is used to prove the next theorem.

Lemma 3.19. *Let $f:(X,\xi_\alpha)\to(Y,\eta_\alpha)$ be an N-preserving map for each $\alpha\in\Lambda$. Then $f:(X,\sup\{\xi_\alpha\})\to(\mathrm{Y},\sup\{\eta_\alpha\})$ is an N-preserving map.*

The following theorem is needed to ensure that the concept of nearness structure on the function space, which will be introduced in next section, is well defined.

Theorem 3.20. *If* $\{\xi_\alpha : \alpha \in \Lambda\}$ *is a family of nearness structures on* X *and* $X_0 \subseteq X$. *Then*

$$\sup_{\alpha\in\Lambda}\{\xi_\alpha\}\Big|_{X_0} = \sup_{\alpha\in\Lambda}\{\xi_\alpha|_{X_0}\}.$$

Proof. Let $i : X_0 \to X$ be the inclusion map. For each $\alpha \in \Lambda$, By Lemma 3.3,

$$i : (X_0, \xi_\alpha|_{X_0}) \to (X, \xi_\alpha)$$

is N-preserving. From Lemma 3.19,

$$i : (X_0, \sup_{\alpha\in\Lambda}\{\xi_\alpha\Big|_{X_0}\}) \to (X, \sup_{\alpha\in\Lambda}\{\xi_\alpha\})$$

is still N-preserving. By Lemma 3.16,

$$\sup_{\alpha\in\Lambda}\{\xi_\alpha\Big|_{X_0}\} \geq i^{-1}(\sup_{\alpha\in\Lambda}\{\xi_\alpha\}) = \sup_{\alpha\in\Lambda}\{\xi_\alpha\}\Big|_{X_0}.$$

Moreover, for each $\alpha \in \Lambda$,

$$\sup_{\alpha\in\Lambda}\{\xi_\alpha\} \subseteq \xi_\alpha,$$

hence

$$\sup_{\alpha\in\Lambda}\{\xi_\alpha\}\Big|_{X_0} \subseteq \xi_\alpha|_{X_0}.$$

It follows that

$$\sup_{\alpha\in\Lambda}\{\xi_\alpha\}\Big|_{X_0} \subseteq \sup_{\alpha\in\Lambda}\{\xi_\alpha\Big|_{X_0}\}.$$

i.e.

$$\sup_{\alpha\in\Lambda}\{\xi_\alpha\}\Big|_{X_0} \geq \sup_{\alpha\in\Lambda}\{\xi_\alpha|_{X_0}\}.$$

Therefore,

$$\sup_{\alpha\in\Lambda}\{\xi_\alpha\}\Big|_{X_0} = \sup_{\alpha\in\Lambda}\{\xi_\alpha|_{X_0}\}.$$

□

4. The Pointwise Convergent Nearness Structure on Function Space

The following theorem shows that the least upperbound nearness structure is a generalization of the respective least upperbound structure when considered in each of the subcategory of $\mathbb{T} - \mathbb{NEAR}$, $\mathbb{U} - \mathbb{NEAR}$ and $\mathbb{P} - \mathbb{NEAR}$ respectively.

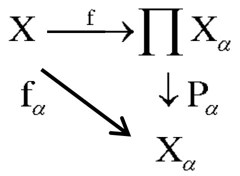

FIGURE 1. Commutative diagram of the natural projections

Theorem 4.1. *If* $\{\xi_\alpha : \alpha \in \Lambda\}$ *is a family of nearness structures on* X *and let* $\xi = \sup_{\alpha\in\Lambda}\{\xi_\alpha\}$*. Then, when considered in* $\mathbb{T}-\mathbb{NEAR}$*,* $\mathbb{U}-\mathbb{NEAR}$*, or* $\mathbb{P}-\mathbb{NEAR}$*,* ξ *will induce the least upper bound of the respective structures induced by* $\{\xi_\alpha : \alpha \in \Lambda\}$ *in the respective type of spaces.*

Proof. Let $\boldsymbol{F}$ be the isomorphic functor from $\mathbb{T}-\mathbb{NEAR}$ to $\mathbb{R}_0-\mathbb{TOP}$. We just need to prove that $\boldsymbol{F}$ is order preserving. Assume $\xi_1 \le \xi_2$. By Lemma 3.16, $\xi_1 \le \xi_2 \Leftrightarrow i : (X,\xi_2) \to (X,\xi_1)$ is N-preserving $\Leftrightarrow F(i)$ is a morphism from $F[(X,\xi_2)]$ to $F[(X,\xi_1)]_1) \Leftrightarrow \mathscr{T}_{\xi_2} \supseteq \mathscr{T}_{\xi_1}$. This shows that $\boldsymbol{F}$ does preserve the order.
The proofs for other two types are parallel and therefore omitted.

□

The following theorem ensures that the product nearness structure in the categorical sense is the largest nearness structure on the product space that makes all natural projections N-preserving.

Theorem 4.2. *If* $\{(X_\alpha,\xi_\alpha) : \alpha \in \Lambda\}$ *is a family of nearness spaces and let* $P_\alpha : \prod_{\alpha\in\Lambda} X_\alpha \to X_\alpha$ *be the natural projection map. Then the nearness structure* $\xi^* = \sup_{\alpha\in\Lambda}\{P_\alpha^{-1}(\xi_\alpha)\}$ *is exactly the categorical product of* $\{(X_\alpha,\xi_\alpha) : \alpha \in \Lambda\}$*.*

Proof. It would suffice to show that for any N-space (X,ξ), and any family of N-preserving maps $\{f_\alpha : X \to X_\alpha : \alpha \in \Lambda\}$, there is an unique N-preserving map $f : X \to \prod_{\alpha\in\Lambda} X_\alpha$ such that $\forall\alpha \in \Lambda$, $P_\alpha \circ f = f_\alpha$. i.e. the diagram in Figure 1 is commutative.

We will first make a claim that for any map $f : X \to \prod X_\alpha$, f is N-preserving if and only if for each $\alpha \in \Lambda$, $P_\alpha \circ f$ is N-preserving. In fact, the necessity is obvious. Let us assume that for each $\alpha \in \Lambda$, $P_\alpha \circ f$ is N-preserving. From Lemma 3.19, f is $(\xi, \sup_{\alpha\in\Lambda}\{P_\alpha^{-1}\})$ N-preserving. i.e. it is (ξ,ξ^*) N-preserving.

Now we consider the family of N-preserving maps $\{f_\alpha : X \to X_\alpha : \alpha \in \Lambda\}$. Define f in the natural way (usually known as the "*evaluation map*"): $\forall x \in X, f(x) = (f_\alpha(x))_{\alpha\in\Lambda}$. Then $P_\alpha \circ f = f_\alpha$. Since each f_α is N-perserving, each $P_\alpha \circ f$ is N-preserving. By earlier proof, f is N-preserving. We also know that such an f is unique from its definition. □

The following purely categorical lemma should be obvious:

Lemma 4.3. *If $\mathbb{C}$ is a category, $\{A_\alpha : \alpha \in \Lambda\}$ is a family of objects. $\prod_{\alpha\in\Lambda} A_\alpha$ is the categorical product in $\mathbb{C}$. $\mathbb{D}$ is another category isomorphic to $\mathbb{C}$ with* **F** *as the isomorphic functor from $\mathbb{C}$ to $\mathbb{D}$. Then $F(\prod_{\alpha\in\Lambda} A_\alpha) = \prod_{\alpha\in\Lambda} F(A_\alpha)$.*

We now officially define the nearness structure on the function space:

Definition 4.4. *If (Y, ξ) is a* N*-space, $X$is a non-empty set. $\mathscr{F} \subseteq Y^X$. If we consider Y^X as a product and let ξ^* be the product nearness structure as defined in Theorem 4.2. Let $\xi_\rho = \xi^*|_{\mathscr{F}}$. Then ξ_ρ is said to be the* pointwise convergent *nearness structure on $\mathscr{F} \subseteq Y^X$.*

Notice that if $\{e_x : Y^X \to Y : x \in X\}$ is the family of natural projections, then $\xi^* = \sup_{x\in X}\{e_x^{-1}(\xi)\}$. So by Theorem 3.20, $\xi_\rho = \xi^*|_{\mathscr{F}} = \sup_{x\in X}\{e_x^{-1}(\xi)\}\Big|_{\mathscr{F}} = \sup_{x\in X}\{\, e_x^{-1}(\xi)\big|_{\mathscr{F}}\}$.

The readers may refer to [11] for the concept of product topology, product uniformity, pointwise convergent topology and pointwise convergent uniformity. Refer to [14] for the concepts of product proximity and pointwise convergent proximity.

We would like to make sure that the product nearness structure and the pointwise convergent nearness structure is a generalization of the respective structures in topological spaces, uniform spaces and proximity spaces respectively.

Theorem 4.5. *When considered in each of the subcategory $\mathbb{T}-\mathbb{NEAR}$, $\mathbb{U}-\mathbb{NEAR}$, or $\mathbb{P}-\mathbb{NEAR}$,*

(i) *ξ^* will induce the Tychonoff product topology, product uniformity or the product proximity respectively.*
(ii) *ξ_ρ will induce the pointwise convergent topology, the pointwise convergent uniformity or the pointwise convergent proximity respectively*

Proof. The first conclusion can be obtained from Theorems 4.1 and 4.2. The second conclusion can be obtained from Theorems 3.8, 3.9 and 3.10. □

Next we will try to generalize the concept of "neighborhood", which is essential when characterizing "convergence".

Definition 4.6. *If (X, ξ) is a* N*-space, and $A \subseteq X$. A subset U is called a* nearness neighborhood *of A if there is a $\overline{\xi}\mathscr{A}$ such that $A \subseteq \mathscr{A}^{\mathrm{C}}(A) \subseteq U$. The notation $NearN(A)$ represents the collection of all nearness neighborhood of a subset A. If the set A contains only one point x, then we simplify the notation from $NearN(\{x\})$ to $NearN(x)$.*

If there are two types of neighborhood system on a space that characterize the same convergence, i.e. being convergent under one neighborhood system is equivalent to being convergent under the other neighborhood system, then we consider them as "equivalent" neighborhood systems. This is typically characterized by the condition that any neighborhood under one system always contains a neighborhood in the other system. For example, consider the two dimensional Cartesian plane. The neighborhood of circular disks centered at a point is equivalent to the neighborhood of squares centered at the same point.

We will try to show that the *nearness neighborhood* generalizes the topological neighborhood([11]), uniform neighborhood ([11]), and proximal neighborhood ([14]) by showing that the nearness neighborhood system is really equivalent to the respective neighborhood system in the respective subcategories.

Theorem 4.7. *Let (X,ξ) be a N-space, $x \in X$ and $A \subseteq X$.*

(i) *In $\mathbb{T}-\mathbb{NEAR}, NearN(x)$ is equivalent to a topological neighborhood system at x.*
(ii) *In $\mathbb{U}-\mathbb{NEAR}, NearN(x)$ is equivalent to a uniform neighborhood system at x.*
(iii) *In $\mathbb{P}-\mathbb{NEAR}, NearN(A)$ is equivalent to a proximal neighborhood system at A.*

Proof. (i) Take an $U \in NearN(x)$, there will be an $\mathscr{A} \notin \xi$ such that $x \in \mathscr{A}^{\mathrm{C}}(x) \subseteq U$. For this $\mathscr{A}$, by conditions (N_1) and (N_2), $\bigcap cl_\xi \mathscr{A} = \varnothing$. So $(cl_\xi \mathscr{A})^{\mathrm{C}}$ is a cover of X. There will be an $A \in \mathscr{A}$ such that $x_0 \in X - cl_\xi A \subseteq X - A \subseteq \mathscr{A}^{\mathrm{C}}(x_0)$. The set $X - cl_\xi A$ is an open neighborhood of x.

On the other hand, if we take an open neighborhood V of x under the topology $\mathscr{T}_\xi$, then $x \notin X - V$ and $X - V$ is closed, i.e. $cl_\xi(X - V) = X - V$. Let $\mathscr{A} = \{\{x\}, X - V\}$, then $\mathscr{A} \notin \xi$. And $\mathscr{A}^{\mathrm{C}}(x) = X - (X - V) = V \subseteq V$. This shows that V is a nearness neighborhood also.

(ii) Take an $U \in NearN(x)$, there will be an $\mathscr{A} \notin \xi$ such that $x \in \mathscr{A}^{\mathrm{C}}(x) \subseteq U$. For this $\mathscr{A}$, since $\mathscr{A} \notin \xi$, $\mathscr{A}^{\mathrm{C}}$ is a cover of X, which means that there has to be an $A \in \mathscr{A}$ such that $x \in X - A$. Let

$$U_{\mathscr{A}} = \{ \bigcup_{A \in \mathscr{A}} ((X - A) \times (X - A)) : \mathscr{A} \notin \xi\},$$

then

$$\begin{aligned} U_{\mathscr{A}}[x] &= \{y : (x,y) \in U_{\mathscr{A}}\} \\ &= \{y : \exists A \in \mathscr{A} \ni (x,y) \in (X - A) \times (X - A)\} \\ &= \{y : \exists A \in \mathscr{A} \ni x \in (X - A), y \in (X - A)\} \\ &= \bigcup \{X - A : x \in X - A\} \\ &= \mathscr{A}^{\mathrm{C}}(x) \end{aligned}$$

This “equal” relation shows the equivalency between the uniform neighborhood system and the nearness neighborhood system.

(iii) [14] stated that a set B is a proximal ($\delta-$) neighborhood of a set A if $A \ll B$. In [19], the proximity $\ll_\xi$ induced by a nearness structure is defined as $A \ll_\xi B$ if and only if there is a $\mathscr{A} \notin \xi$ such that $\mathscr{A}^{\mathrm{C}}(A) \subseteq B$. So the equivalency between the nearness neighborhood system and the proximal neighborhood systems is obvious.

□

The following theorem demonstrates the consistency of the N-converges with those previously established concepts of convergent nets. With the establishment of the previous theorem, the proof should be obvious.

Theorem 4.8. *The* N*-convergence, as it is defined in Definition 4.9, when considered in* $\mathbb{T}-\mathbb{NEAR}, \mathbb{U}-\mathbb{NEAR}$ *and* $\mathbb{P}-\mathbb{NEAR}$*, is equivalent to the convergence with respect to the corresponding types of structures, respectively.*

A set D is said to be a *directed* set, if it is endowed with a reflexive and transitive binary relation $\geq$ such that $\forall m, n \in D, \exists p \in D$ s.t. $p \geq m$ and $p \geq n$. i.e. for any two elements of D, there is always another element that *precedes* them. (see [11])

As a generalization of sequences, a *net* in a set X is a function $x : D \to X$, where D is a directed set. We typically write a net as $\{x_d : d \in D\}$.

We would like to exam the relation of the convergency of a net of functions and that of the nets obtained by fixing the net of functions at any arbitrary point x of X. Of course, we expect the two convergences are to be equivalent. Theorem 4.7 shows exactly that.

Definition 4.9. *If* (X, ξ) *is a* N*-space,* $\{x_d : d \in D\}$ *is a net in* X*. We say* $\{x_n : n \in D\}$ N-converges *to a point* $x_0 \in X$*, if for any* $\overline{\xi}\mathscr{A}$*, there is an* $N \in D$ *such that for each* $n \geq N, n \in D$*, we have* $x_n \in \mathscr{A}^{\mathrm{C}}(x_0)$*.*

Theorem 4.10. *If* (Y, ξ) *is a N-space,* X*is a non-empty set.* $\{f_n : n \in D\}$ *is a net in* $\mathscr{F} \subseteq Y^X$*. Then* $\{f_n : n \in D\}$ *N-converges to a function* f *in* $(\mathscr{F}, \xi_\rho)$ *if and only if for any* $x \in X$*,* $\{f_n(x) : n \in D\}$*, as a net in* (Y, ξ)*, N-converges to the point* $f(x)$*.*

Proof. **Necessity**: Take an arbitrary point $x \in X$, take an $\overline{\xi}\mathscr{A}$, since the natural projection map

$$e_x : Y^X \to X$$

is N-preserving, we have $\overline{\xi_\rho} e_x^{-1}(\mathscr{A})$, so $\overline{\xi_\rho}(e_x^{-1}(\mathscr{A})\big|_{\mathscr{F}})$. Since $\{f_n : n \in D\}$ is N-convergent to f in $\mathscr{F}$. There is an $n \in D$, such that for each $m > n, m \in D$, we have

$$f_m \in (e_x^{-1}(\mathscr{A})\big|_{\mathscr{F}})^C(f).$$

i.e. There is an $A \in \mathscr{A}$, such that

$$\{f_m, f\} \subseteq \mathscr{F} - e_x^{-1}(A).$$

So

$$f_m(x) = e_x(f_m) \notin A$$

and

$$f(x) = e_x(f) \notin A.$$

i.e.

$$f_m(x) \in X - A,$$

and

$$f(x) \in X - A$$

Therefore,

$$f_m(x) \in \mathscr{A}^C(f(x)).$$

Sufficiency: Assume that $\{f_n : n \in D\}$ is a net in $\mathscr{F} \subseteq Y^X$. Furthermore, for any $x \in X$, assume that $\{f_n(x) : n \in D\}$, as a net in X, N-converges to the point $f(x)$. We now arbitrarily take a $\mathscr{B}$ such that $\overline{\xi_\rho}\mathscr{B}$. By the definition of ξ^* as the least upper bound, there should be finitely many $\mathscr{B}_i \subseteq \mathcal{P}(X),\ i = 1, 2, ..., n$, such that for each i, we have $\mathscr{B}_i \notin e_x^{-1}(\xi)\big|_{\mathscr{F}}$, and $\mathscr{B} \prec \vee\mathscr{B}_i$. $\overline{\xi}e_{x_i}(\mathscr{B}_i)$, so there is an m_i such that for any $n \geq m_i$, there should be an $A_i \in e_{x_i}(\mathscr{B}_i)$, $\{f_n(x_i), f(x_i)\} \subseteq X - A_i$. Since $A_i \in e_{x_i}(\mathscr{B}_i)$, there exists a $B_i \in \mathscr{B}_i$ and $A_i = e_{x_i}(B_i)$. So $f_n(x_i) \notin e_{x_i}(B_i)$, and $f(x_i) \notin e_{x_i}(B_i)$. i.e. $e_{x_i}(f_n) \notin e_{x_i}(B_i)$, and $e_{x_i}(f) \notin e_{x_i}(B_i)$. Therefore, $\{f_n, f\} \subseteq \mathscr{F} - B_i$, or we can say that $f_n \in \mathscr{B}_i^C(f)$. Since there are only finitely many m_i's. We will let $N = \max\{m_1, ..., m_n\}$. Then for any $n \geq N$, $f_n \in \wedge(\mathscr{B}_i^C)(f) = (\vee\mathscr{B}_i)^C(f)$. Hence $f_n \in \mathscr{B}^C(f)$.

□

The following corollary is associated with the concept of "accumulation points" in classical topology.

Corollary 4.11. *If (X, ξ) is a N-space, B is a subset of X, $\{x_d : d \in D\}$ is a net in B. Then*

(i) *If $\{x_d : d \in D\}$ is N-convergent to $x_0 \in X$, then $x_0 \in cl_\xi B$.*
(ii) *In $\mathbb{T} - \mathbb{NEAR}$, for any $x_0 \in cl_\xi B$, there is a $\{x_d : d \in D\}$ in B and $\{x_d : d \in D\}$ is N-convergent to x_0.*

Proof. (i) If, to the contrary, $x_0 \notin cl_\xi B$. Then $\mathscr{A} = \{\{x_0\}, B\} \notin \xi$. Since $\{x_d : d \in D\}$, and $\mathscr{A}^{\mathrm{C}} = \{X - \{x_0\}, X - B\}$, there is an $N \in D$ such that for any $n \geq N$, $x_n \in \mathscr{A}^{\mathrm{C}}(x_0) = X - B$. But this contradicts to the assumption that $\{x_d : d \in D\} \subseteq B$. Hence $x_0 \in cl_\xi B$ must be true.

(ii) In $\mathbb{T} - \mathbb{NEAR}$, from Theorem 4.8(i), N-convergence is equivalent to topological convergence and $cl_\xi B$ is the topological closure of the set B. The conclusion must be true due to classical topology.

□

The following Corollary is a natural consequence of the Corollary 4.11:

Corollary 4.12. *Let (X, ξ) be a N-space and $B \subseteq X$.*

(i) *If B is N-closed, then any convergent net in B must converge to a point in B.*
(ii) *In $\mathbb{T} - \mathbb{NEAR}$, if the limit of any convergent net in B always remains in B, then B is N-closed.*

The next several theorems show that the properties of being *totally bounded*, *compact* or N_2 are productive respectively.

Theorem 4.13. *Let X be a set, (Y, η) be a N-space, and $f : X \to Y$ be a N-preserving map. Then $f^{-1}(\eta)$ is totally bounded if and only if η is totally bounded.*

Proof. Assume that $f^{-1}(\eta)$ is totally bounded. We would like to show that η is totally bounded by showing that it meets condition (B1). Arbitrarily take a $\mathscr{B}$ such that $\overline{\eta}\mathscr{B}$. Then $f^{-1}(\mathscr{B}) \notin f^{-1}(\eta)$. There should be a finite subcollection of $\mathscr{B}$, say $\mathscr{B}_0 \subseteq \mathscr{B}$, such that $\bigcap f^{-1}(\mathscr{B}_0) = \phi$. Hence $\bigcap \mathscr{B}_0 = \phi$.

Now we assume that η is totally bounded. Arbitrarily take $\mathscr{A} \notin f^{-1}(\eta)$, then $\overline{\eta} f(\mathscr{A})$. So there should be a finite subcollection of $\mathscr{A}$, say $\mathscr{A}_0 \subseteq \mathscr{A}$, such that $\bigcap f(\mathscr{A}_0) = \phi$. Now we can easily see that $\bigcap \mathscr{A}_0 = \phi$. □

Theorem 4.14. *Let $\{\xi_\alpha : \alpha \in \Lambda\}$ be a family of nearness structures on X. Let $\xi = \sup\{\xi_\alpha : \alpha \in \Lambda\}$. Then ξ is totally bounded if and only if for each $\alpha \in \Lambda, \xi_\alpha$ is totally bounded.*

Proof. First we assume that for each $\alpha \in \Lambda$, ξ_α is totally bounded. Then for any $\overline{\xi}\mathscr{A}$, there should be finitely many $\xi_{\alpha_i}, i = 1, 2, ...n$ as well as $\mathscr{A}_{\alpha_i}, i = 1, 2, ...n$ such that $\mathscr{A}_{\alpha_i} \notin \xi_{\alpha_i}$ and $\mathscr{A} \prec \bigvee_{i=1}^{n} \mathscr{A}_{\alpha_i}$. Since each ξ_{α_i} is totally bounded, each $\mathscr{A}_{\alpha_i}$ contains a finite subcollection $\mathscr{B}_{\alpha_i}$ and $\bigcap \mathscr{B}_{\alpha_i} = \phi$. $\bigvee_{i=1}^{n} \mathscr{B}_{\alpha_i}$ is a finite subcollection of $\bigvee_{i=1}^{n} \mathscr{A}_{\alpha_i}$. We will claim that $\bigcap \bigvee_{i=1}^{n} \mathscr{B}_{\alpha_i} = \varnothing$. Take an arbitrary point $x \in X$, then for each $i = 1, 2, ..., n$, there is a $B_i^x \in \mathscr{B}_{\alpha_i}$ such that $x \notin B_{\alpha_i}^x$. So $x \notin \bigcup_{i=1}^{n} B_{\alpha_i}^x$. Hence $x \notin \bigcap \bigvee_{i=1}^{n} \mathscr{B}_{\alpha_i}$. This shows that $\bigcap \bigvee_{i=1}^{n} \mathscr{B}_{\alpha_i} = \phi$. So ξ is totally bounded.

Now we assume that ξ is totally bounded. Arbitrarily take a ξ_α. Then for any $\overline{\xi_\alpha}\mathscr{A}_\alpha$, we have $\overline{\xi}\mathscr{A}_\alpha$. Since ξ is assumed to be totally bounded, $\mathscr{A}_\alpha$ must have finite subcollection with empty intersection. □

Theorem 4.15. *If $\{\xi_\alpha : \alpha \in \Lambda\}$ is a family of nearness structures on X . Its product $(\prod_{\alpha\in\Lambda} X_\alpha, \prod_{\alpha\in\Lambda} \xi_\alpha)$ is totally bounded if and only if each (X_α, ξ_α) is totally bounded. Particularly, if (Y, η) is totally bounded and $\mathscr{F} \subseteq Y^X$ where X is a non-empty set, then $(\mathscr{F}, \xi_\rho)$ is totally bounded.*

Proof. The first conclusion can be deduced from Theorem 4.13 and Theorem 4.14. The second conclusion can be deduced from the Theorem 3.18. □

The next Lemma, due to Herrlich ([8], 4.5 Proposition, Part (2)), will be used in the proof of the following theorem.

Lemma 4.16. *For a* T-N *space, the following conditions are equivalent:*

(1) (X, ξ) *is contigual;*
(2) (X, ξ) *is totally bounded;*
(3) (X, ξ) *is compact.*

Theorem 4.17. *If $\{\xi_\alpha : \alpha \in \Lambda\}$ is a family of nearness structures on X. If each (X_α, ξ_α) is compact, then its product $(\prod_{\alpha\in\Lambda} X_\alpha, \prod_{\alpha\in\Lambda} \xi_\alpha)$ is also compact.*

Proof. Recall that a compact nearness space satisfies condition (T) and (C). It is easy to verify that the product is a T-N space, if each (X_α, ξ_α) is a T-N space. According to Lemma 4.16, a T-N space is *C-N* if and only it it is totally bounded. So the conclusion of this theorem follows from Theorem 4.15. □

Theorem 4.18. *If $\{\xi_\alpha : \alpha \in \Lambda\}$ is a family of nearness structures on X . If each (X_α, ξ_α) is N_2, then the product $(X, \xi) = (\prod_{\alpha\in\Lambda} X_\alpha, \prod_{\alpha\in\Lambda} \xi_\alpha)$ is also N_2.*

Proof. By the definition of product of nearness structures, $\xi = \sup\{P_\alpha^{-1}(\xi_\alpha) : \alpha \in \Lambda\}$, where $P_\alpha : \prod_{\alpha\in\Lambda} X_\alpha \to X_\alpha$ are natural projection maps. Let $x, y \in X$ and $x \neq y$, there should be at least one $\alpha \in \Lambda$ such that $P_\alpha(x) \neq P_\alpha(y)$. Since ξ_α is N_2, there are $A_\alpha \subseteq X$ and $B_\alpha \subseteq X$ such that $A_\alpha \cap B_\alpha = \varnothing$ and

$$P_\alpha(x) \in X_\alpha - cl_{\xi_\alpha}(X_\alpha - A_\alpha), P_\alpha(y) \in X_\alpha - cl_{\xi_\alpha}(X_\alpha - B_\alpha).$$

It is easy to see that $P_\alpha^{-1}(A_\alpha) \cap P_\alpha^{-1}({}_\alpha B) = \varnothing$. We will try to show that

$$x \in X - cl_\xi(X - P_\alpha^{-1}(A_\alpha)),$$

and

$$y \in X - cl_\xi(X - P_\alpha^{-1}(B_\alpha)).$$

Take an arbitrary point

$$z \in P_\alpha^{-1}(X_\alpha - cl_{\xi_\alpha}(X_\alpha - A_\alpha)),$$

then

$$P_\alpha(z) \in X_\alpha - cl_{\xi_\alpha}(X_\alpha - A_\alpha).$$

and

$$\{\{P_\alpha(z)\}, X_\alpha - A_\alpha\} \notin \xi_\alpha.$$
$$\{\{P_\alpha^{-1}(P_\alpha(z))\}, P_\alpha^{-1}(X_\alpha - A_\alpha)\} \notin P_\alpha^{-1}(\xi_\alpha).$$
$$\{\{z\}, X - P_\alpha^{-1}(A_\alpha)\} \notin P_\alpha^{-1}(\xi_\alpha).$$

The last statement is true since

$$z \in P_\alpha^{-1}(P_\alpha(z))$$

and

$$P_\alpha^{-1}(X_\alpha - A_\alpha) \subseteq X - P_\alpha^{-1}(A_\alpha).$$

Therefore,

$$z \in X - cl_{P_\alpha^{-1}(\xi_\alpha)}(X - P_\alpha^{-1}(A_\alpha)).$$

This shows that

$$P_\alpha^{-1}(X_\alpha - cl_{\xi_\alpha}(X_\alpha - A_\alpha)) \subseteq X - cl_{P_\alpha^{-1}(\xi_\alpha)}(X - P_\alpha^{-1}(A_\alpha)).$$

Hence

$$\begin{aligned} x &\in P_\alpha^{-1}(X_\alpha - cl_{\xi_\alpha}(X_\alpha - A_\alpha)) \\ &\subseteq X - cl_{P_\alpha^{-1}(\xi_\alpha)}(X - P_\alpha^{-1}(A_\alpha)) \\ &\subseteq X - cl_\xi(X - P_\alpha^{-1}(A_\alpha)). \end{aligned}$$

By similar argument,

$$\begin{aligned} y &\in P_\alpha^{-1}(X_\alpha - cl_{\xi_\alpha}(X_\alpha - B_\alpha)) \\ &\subseteq X - cl_{P_\alpha^{-1}(\xi_\alpha)}(X - P_\alpha^{-1}(B_\alpha)) \\ &\subseteq X - cl_\xi(X - P_\alpha^{-1}(B_\alpha)). \end{aligned}$$

Therefore, (X, ξ) is a N_2- space. □

Now we will try to establish the relation between the compactness of the underderlining set Y and a function space $\mathscr{F} \subseteq Y^X$.

Theorem 4.19. *Let X be a set, and (Y, η) be a compact N-space. $\mathscr{F} \subseteq Y^X$. Then*

(i) *The condition (a) is sufficient for $(\mathscr{F}, \xi_\rho)$ to be compact.*
(ii) *If (Y, η) is also N_2, then the condition (a) is also necessary for $(\mathscr{F}, \xi_\rho)$ to be compact.*

(a) $\mathscr{F}$ is N-closed in (Y^X, ξ^*).

Proof. (i) Since (Y, η) is a compact N-space. By Theorem 4.17, (Y^X, ξ^*) is compact. By Theorem 3.12, $\mathscr{F} \subseteq Y^X$, as an N-closed subset of a compact space, is also compact under the subspace nearness structure.
(ii) If (Y, η) is a N_2-space. By Theorem 4.18, (Y^X, ξ^*) is an N_2- space. Then by Theorem 3.17, $(\mathscr{F}, \xi_\rho)$, as a compact subspace of an N_2- space, is N-closed.

□

Theorem 4.20. *If X is a set, and (Y, η) is an N-space. $\mathscr{F} \subseteq Y^X$. Then*

(i) *the conditions (a) and (b) are sufficient for $(\mathscr{F}, \xi_\rho)$ to be compact.*
(ii) *If (Y, η) is also N_2, then the conditions (a) and (b) are also necessary for $(\mathscr{F}, \xi_\rho)$ to be compact.*

(a) $\mathscr{F}$ is N-closed in (Y^X, ξ^*)
(b) For any $x \in X$, $\mathscr{F}[x] = \{f(x) : f \in \mathscr{F}\}$ is contained in a compact subspace of (Y, η).

Proof. (i) Assume that (Y_x, η_x) is a compact nearness subspace of (Y, η) with $\eta_x = \eta|_{Y_x}$ and $\mathscr{F}[x] \subseteq Y_x \subseteq Y$. Then $\mathscr{F} \subseteq (\prod_{x \in X} Y_x, \prod_{x \in X} \eta_x)$ and the later space is compact, according to Theorem 4.17. It is easy to see that $\xi_\rho = \prod_{x \in X} \eta_x|_{\mathscr{F}}$. Since $\mathscr{F}$ is N-closed also, it is N-compact due to Theorem 3.12.
(ii) If $(Y, \eta)$is a N_2-space, and $(\mathscr{F}, \xi_\rho)$ is compact. It follows from Theorem 3.17 that (a) is true. And since the evaluation map $e_x : (\mathscr{F}, \xi_\rho) \to (Y, \eta)$ is N-preserving. By Lemma 3.14, $\mathscr{F}[x] = \{f(x) : f \in \mathscr{F}\} = \{e_x(f) : f \in \mathscr{F}\}$ is also compact.

□

Summary. *This paper essentially lays the foundation for some possible applications of the theory of nearness function spaces in digital topology. The main results is the introduction of the pointwise convergent nearness spaces in the function spaces in such a way that is consistent with the existing and established structures. Two Ascoli's type of theorems on nearness spaces are established also.*

Any deformation of a digital image (such as thinning) can be considered as a function from the digital plane to itself. Of course, we would prefer those functions to preserve some properties of the digital image, such as "nearness". When a sequence of deformations are applied to a digital image, we would like to be able to make some type of projection or prediction about the final images based on the type of deformations involved in the sequence. We believe that the line of work presented in this paper will be helpful to address those issues.

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