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Best proximity pair theorems for relatively nonexpansive mappings

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Abstract. Let A, B be nonempty closed bounded convex subsets of a uniformly convex Banach space and $T : A\cup B \to A\cup B$ be a map such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $\|Tx - Ty\| \leq \|x - y\|$, for x in A and y in B. The fixed point equation $Tx = x$ does not possess a solution when $A \cap B = \varnothing$. In such a situation it is natural to explore to find an element x_0 in A satisfying $\|x_0 - Tx_0\| = inf\{\|a - b\| : a \in A, b \in B\} = dist(A, B)$. Using Zorn's lemma, Eldred *et.al* proved that such a point x_0 exists in a uniformly convex Banach space settings under the conditions stated above. In this paper, by using a convergence theorem we attempt to prove the existence of such a point x_0 (called best proximity point) without invoking Zorn's lemma.

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1. Introduction

Let A, B be nonempty subsets of a normed linear space $(X, \|\cdot\|)$ and a map $T : A \cup B \to A \cup B$ is said to be a relatively nonexpansive map if it satisfies (i) $T(A) \subseteq B$, $T(B) \subseteq A$ and (ii) $\|Tx - Ty\| \leq \|x - y\|$, for all $x \in A$, $y \in B$. Note that a relatively nonexpansive map need not be continuous in general. But if $A \cap B$ is nonempty, then the map T restricted to $A \cap B$ is a nonexpansive self map. If the fixed point equation $Tx = x$ does not possess a solution it is natural to explore to find an $x_0 \in A$ satisfying $\|x_0 - Tx_0\| = dist(A, B) = inf\{\|a - b\| : a \in A, b \in B\}$. A point $x_0 \in A$ is said to be a best proximity point for T if it satisfies $\|x_0 - Tx_0\| = dist(A, B)$.

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In [2], Eldred *et.al* introduced a geometric concept called proximal normal structure which generalizes the concept of normal structure introduced by Milman and Brodskii [7].

Definition 1.1 (Proximal normal structure [2]). *A convex pair (K_1, K_2) in a Banach space is said to have proximal normal structure if for any closed, bounded, convex proximal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $dist(H_1, H_2) = dist(K_1, K_2)$ and $\delta(H_1, H_2) > dist(H_1, H_2)$, there exists $(x_1, x_2) \in H_1 \times H_2$ such that*

$$\delta(x_1, H_2) < \delta(H_1, H_2), \quad \delta(x_2, H_1) < \delta(H_1, H_2)$$

where $\delta(H_1, H_2) = sup\{ \|h_1 - h_2\| \ : \ h_1 \in H_1, h_2 \in H_2\}$.

Using the concept of proximal normal structure, Eldred *et.al* [2] proved the existence of best proximity points for relatively nonexpansive mappings.

Theorem 1.2 ([2]). *Let (A, B) be a nonempty, weakly compact convex pair in a Banach space $(X, \|\cdot\|)$, and suppose (A, B) has proximal normal structure. Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive map. Then there exists $(x, y) \in A \times B$ such that $\|x - Tx\| = \|Ty - y\| = dist(A, B)$.*

The proof of the above theorem invokes Zorn's lemma and the proximal normal structure idea. Also it has been proved that every closed bounded convex pair (A, B) of a uniformly convex Banach space has proximal normal structure and every compact convex pair has proximal normal structure.

In this paper, by using a convergence theorem we attempt to prove the existence of a best proximity point without invoking Zorn's lemma.

2. Preliminaries

In this section we give some basic definitions and concepts which are useful and related to the context of our results. We shall say that a pair (A, B) of sets in a Banach space satisfies a property if each of the sets A and B has that property. Thus (A, B) is said to be convex if both A and B are convex. $(C, D) \subseteq (A, B) \Leftrightarrow C \subseteq A, D \subseteq B$ etc.

$$dist(A, B) = \inf\{\|x - y\| \ : \ x \in A, \ y \in B\}$$

$$A_0 = \{x \in A \ : \ \|x - y\| = dist(A, B) \ for \ some \ y \in B\}$$

$$B_0 = \{y \in B \ : \ \|x - y\| = dist(A, B) \ for \ some \ x \in A\}$$

Let X be a normed linear space and C be a nonempty subset of X. Then the metric projection operator $P_C : X \to 2^C$ is defined as

$$P_C(x) = \{y \in C \ : \ \|x - y\| = dist(x, C)\}, \ for \ each \ x \in X.$$

It is well known that the metric projection operator P_C on a strictly convex Banach space X is a single valued map from X to C, where C is a nonempty weakly compact convex subset of X.

In [6], Kirk *et.al* proved the following lemma which guarantees the nonemptiness of A_0 and B_0.

Lemma 2.1 ([6]). *Let X be a reflexive Banach space and A be a nonempty closed bounded convex subset of X, and B be a nonempty closed convex subset of X. Then A_0 and B_0 are nonempty and satisfy $P_B(A_0) \subseteq B_0$, $P_A(B_0) \subseteq A_0$.*

In [8], Sadiq Basha and Veeramani proved the following result.

Lemma 2.2 ([8]). *If A and B are nonempty subsets of a normed linear space X such that $dist(A, B) > 0$, then $A_0 \subseteq \partial(A)$ and $B_0 \subseteq \partial(B)$ where $\partial(C)$ denotes the boundary of C in X for any $C \subseteq X$.*

Suppose (A, B) is a nonempty weakly compact convex pair of subsets in a Banach space X. Consider the map $P : A \cup B \to A \cup B$ defined as

$$P(x) = \begin{cases} P_B(x), & if \quad x \in A \\ P_A(x), & if \quad x \in B \end{cases} \tag{2.1}$$

If X is a strictly convex Banach space, then the map P is a single valued map and satisfies $P(A) \subseteq B, P(B) \subseteq A$.

Proposition 2.3. *Let A, B be nonempty weakly compact convex subsets of a strictly convex Banach space X. Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive map and $P : A \cup B \to A \cup B$ be a map defined as in (2.1). Then $TP(x) = P(Tx)$, for all $x \in A_0 \cup B_0$.*

Proof. Let $x \in A_0$. Then there exists $y \in B$ such that $\|x - y\| = dist(A, B)$. By the uniqueness of the metric projection on a strictly convex Banach space, we have $P_B(x) = y, P_A(y) = x$. Since T is relatively nonexpansive, we have $\|Tx - Ty\| \leq \|x - y\| = dist(A, B)$. *ie* $P_A(Tx) = Ty$. This implies that $P_A(Tx) = TP_B(x)$ □

This observation will play an important role in this article. In [3], Eldred and Veeramani introduced a notion of cyclic contraction and studied the existence of best proximity point for such maps. We make use of the main results proved in [3] to obtain best proximity pair theorems for relatively nonexpansive mappings.

Definition 2.4 ([3]). *Let A and B be nonempty subsets of a metric space X. A map $T : A \cup B \to A \cup B$ is said to be a cyclic contraction map if it satisfies :*

1. *$T(A) \subseteq B$, $T(B) \subseteq A$*
2. *there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y) + (1 - k)dist(A, B)$ for each $x \in A, y \in B$*

We can easily see that every cyclic contraction map satisfies $d(Tx, Ty) \leq d(x, y)$, for all $x \in A, y \in B$. In [3], a simple existence result for a best proximity point of a cyclic contraction map has been given. It states as follows:

Theorem 2.5 ([3]). *Let A and B be nonempty closed subsets of a complete metric space X. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map, let $x_0 \in A$ and define $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$. Suppose $\{x_{2n}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that $d(x, Tx) = dist(A, B)$.*

In uniformly convex Banach space settings, the following result proved in [3] ensures the existence, uniqueness and convergence of a best proximity point for a cyclic contraction map. We use this result to prove our main results.

Theorem 2.6 ([3]). *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction map, then there exists a unique best proximity point $x \in A$(that is with $\|x - Tx\| = dist(A, B)$). Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

We need the notion of "approximatively compact set" to prove a convergence result in the next section.

Definition 2.7 ([9]). *Let X be a metric space. A subset C of X is said to be approximatively compact if for any $y \in X$, and for any sequence $\{x_n\}$ in C such that $d(x_n, y) \to dist(y, C)$ as $n \to \infty$, then $\{x_n\}$ has a subsequence which converges to a point in C.*

In a metric space, every approximatively compact set is closed and every compact set is approximatively compact. Also a closed convex subset of a uniformly convex Banach space is approximatively compact.

3. Main Results

The following convergence theorem will play an important role in our main results.

Theorem 3.1. *Let X be a strictly convex Banach space and A be a nonempty approximatively compact convex subset of X and B be a nonempty closed subset of X. Let $\{x_n\}$ be a sequence in A and $y \in B$. Suppose $\|x_n - y\| \to dist(A, B)$, then $x_n \to P_A(y)$.*

Proof. Suppose that $\{x_n\}$ does not converges to $P_A(y)$, then there exists $\varepsilon > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\|x_{n_k} - P_A(y)\| \geq \varepsilon \tag{3.1}$$

Since $\{x_{n_k}\}$ is a sequence in A such that $\|x_{n_k} - y\| \to dist(A, B)$, and A is approximatively compact, $\{x_{n_k}\}$ has a convergent subsequence $\{x_{n'_k}\}$ such that $x_{n'_k} \to x$ for some $x \in A$. Then

$$\|x_{n'_k} - y\| \to \|x - y\|$$

$$\text{also, } \|x_{n'_k} - y\| \to dist(A, B) \text{ implies } \|x - y\| = dist(A, B).$$

By the uniqueness of P_A we have $x = P_A(y)$. But from (3.1) we have

$$\varepsilon \leq \|x_{n'_k} - P_A(y)\| \implies 0 < \varepsilon \leq \|x - P_A(y)\| \implies x \neq P_A(y)$$

which is a contradiction. Hence $x_n \to P_A(y)$. □

The above theorem generalizes the following convergence result proved in [3] ([3],Corollary 3.9) for a strictly convex Banach space.

Corollary 3.2 ([3]). *Let A be a nonempty closed convex subset and B be nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ be a sequence in A and $y_0 \in B$ such that $\|x_n - y_0\| \to dist(A,B)$. Then x_n converges to $P_A(y_0)$.*

Remark 3.3. Let X be a normed linear space, let A be a nonempty closed convex subset of X, and B be a nonempty approximatively compact convex subset of X. If A_0 is compact, then B_0 is also compact.

Proof. If B_0 is empty, then nothing to prove. Assume B_0 is nonempty. Let $\{y_n\}$ be a sequence in B_0. Then for each $n \in \mathbb{N}$, there exists $x_n \in A_0$ such that $\|x_n - y_n\| = dist(A,B)$. Since A_0 is compact, there exists a convergent subsequence $\{x_{n_k}\}$ which converges to some $x \in A_0$. Consider the inequality,

$$\|y_{n_k} - x\| \leq \|y_{n_k} - x_{n_k}\| + \|x_{n_k} - x\| \to dist(A,B).$$

Since B is approximatively compact, $\{y_{n_k}\}$ has a convergent subsequence $\{y_{n'_k}\}$ converges to some $y \in B$. Since B_0 is closed, it implies that $y \in B_0$. Hence B_0 is compact. □

Now we prove our main results.

Theorem 3.4. *Let X be a uniformly convex Banach space. Let A be a nonempty closed bounded convex subset of X and B be a nonempty closed convex subset of X. Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive map. Then there exist a sequence $\{x_n\}$ in A_0 and $x^* \in A_0$ such that*

(1) $x_n \xrightarrow{w} x^*$

(2) $\|x^* - Tx^*\| \leq dist(A,B) + \liminf_n \|Tx_n - Tx^*\|$.

Proof. By *Lemma 2.1,* A_0 is nonempty, hence there exist $x_0 \in A_0$ and $y_0 \in B_0$ such that $\|x_0 - y_0\| = dist(A,B)$. For each $n \in \mathbb{N}$, define a map $T_n : A \cup B \to A \cup B$ by

$$T_n(x) = \begin{cases} \dfrac{1}{n}y_0 + \left(1 - \dfrac{1}{n}\right) Tx, \; if \;\; x \in A \\ \dfrac{1}{n}x_0 + \left(1 - \dfrac{1}{n}\right) Tx, \; if \;\; x \in B \end{cases} \tag{3.2}$$

Since A and B are convex and T is a relatively nonexpansive map, for each $n \in \mathbb{N}$, $T_n(A) \subseteq B, T_n(B) \subseteq A$. Also for each $x \in A$, $y \in B$,

$$\begin{aligned} \|T_n(x) - T_n(y)\| &\leq \frac{1}{n}\|x_0 - y_0\| + \left(1 - \frac{1}{n}\right)\|Tx - Ty\| \\ &\leq \left(1 - \frac{1}{n}\right)\|x - y\| + \frac{1}{n}dist(A,B). \end{aligned} \tag{3.3}$$

This implies that for each $n \in \mathbb{N}$, T_n is a cyclic contraction on $A \cup B$. Hence by *Theorem 2.6,* for each $n \in \mathbb{N}$ there exists $x_n \in A$ such that

$$\|x_n - T_n x_n\| = dist(A,B). \tag{3.4}$$

Hence $x_n \in A_0$, for each $n \in \mathbb{N}$. Since A_0 is bounded, B_0 is also bounded, and $T(A_0) \subseteq B_0$, $T_n(A_0) \subseteq B_0$. Also observe that for any $x \in A_0$,

$$\|T_n x - Tx\| \leq \frac{1}{n}\|y_0 - Tx\| \leq \frac{1}{n}\delta(B_0) \to 0 \text{ as } n \to \infty. \tag{3.5}$$

Since A_0 is a closed bounded convex set, $\{x_n\}$ has a weakly convergent subsequence. Without loss of generality, let us assume that $\{x_n\}$ itself weakly converges to x^*, for some $x^* \in A_0$. Then $x_n - Tx^* \xrightarrow{w} x^* - Tx^*$. Since $\|\cdot\|$ is weakly lower semi continuous, and by (3.4), (3.5) we have

$$\begin{aligned}\|x^* - Tx^*\| &\leq \liminf_n \|x_n - Tx^*\| \\ &\leq \liminf_n \{\|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| + \|Tx_n - Tx^*\|\} \\ &\leq \liminf_n \left\{dist(A,B) + \frac{1}{n}\delta(B_0) + \|Tx_n - Tx^*\|\right\} \\ &\leq dist(A,B) + \liminf_n \|Tx_n - Tx^*\|\end{aligned}$$

Hence the theorem. □

We use the above theorem to prove :

Theorem 3.5. *Let X be a uniformly convex Banach space. Let A be a nonempty closed bounded convex subset of X such that A_0 is compact, and B be a nonempty closed convex subset of X. Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive map. Then there exist $x^* \in A$ such that $\|x^* - Tx^*\| = dist(A,B)$.*

Proof. By *Theorem 3.4*, there exist a sequence $\{x_n\}$ in A_0 and $x^* \in A_0$ such that $x_n \xrightarrow{w} x^*$ and satisfies the inequality

$$\|x^* - Tx^*\| \leq dist(A,B) + \liminf_n \|Tx_n - Tx^*\|.$$

Since A_0 is compact, x_n converges to x^* strongly. The proof will be complete if we show that $\|Tx_n - Tx^*\| \to 0$.

Claim : $\|Tx_n - Tx^*\| \to 0$ as $n \to \infty$.

It is enough to show that $\|Tx_n - P_A(Tx^*)\| \to dist(A,B)$ as $n \to \infty$. Then by *Theorem 3.1,* we have $Tx_n \to P_B(P_A(Tx^*)) = Tx^*$. Consider

$$\|x_n - P_B x^*\| \leq \|x_n - x^*\| + \|x^* - P_B x^*\| \to dist(A,B).$$

Since T is relatively nonexpansive we have,

$$\|Tx_n - P_A Tx^*\| = \|Tx_n - T(P_B x^*)\| \leq \|x_n - P_B x^*\| \to dist(A,B).$$

This ends the claim and hence the theorem. □

Theorem 3.6. *Let X be a strictly convex Banach space, let A be a nonempty closed convex subset of X such that A_0 is a nonempty compact set and B be a nonempty closed convex subset of X. Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive map. Then there exists $x^* \in A$ such that $\|x^* - Tx^*\| = dist(A,B)$.*

Proof. Since A_0 is nonempty and compact, we can construct a sequence of cyclic contraction maps $T_n : A \cup B \to A \cup B$ as in *Theorem 3.4.* We use *Theorem 2.5* for an existence of best proximity point $x_n \in A_0$ such that $\|x_n - T_n x_n\| = dist(A, B)$. Since A_0 is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x^*$ for some $x^* \in A_0$. As in the proof of *Theorem 3.5*, we can show $Tx_{n_k} \to Tx^*$. The proof ends by considering the following inequality,

$$\|x^* - Tx^*\| \leq \|x^* - x_{n_k}\| + \|x_{n_k} - T_{n_k}x_{n_k}\| + \|T_{n_k}x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tx^*\|$$

and by observing $\|T_{n_k}x_{n_k} - Tx_{n_k}\| \leq \frac{1}{n_k}\delta(B_0) \to 0$. □

We give below some situations where A_0 is a compact subset of A.

Example 3.7. Let A be a unit ball in a strictly convex Banach space X and B be a closed convex subset of X with $dist(A, B) > 0$. Then A_0 contains atmost one point.

Proof. Clearly A_0 is a bounded convex subset of A, moreover by *Lemma 2.2*, A_0 is contained in the boundary of A. *ie* $A_0 \subseteq \partial A$. Suppose $x_1, x_2 \in A_0$ with $x_1 \neq x_2$, then by strict convexity $\|\frac{x_1+x_2}{2}\| < 1$ which implies that $\frac{x_1+x_2}{2} \notin \partial A$, a contradiction to the convexity of A_0. Hence A_0 contains atmost one point. □

Example 3.8. Let A be a nonempty closed bounded convex subset of a uniformly convex Banach space X and B be a nonempty closed convex subset of X such that $span(B)$ is finite dimensional with $dist(A, B) > 0$. Then A_0 and B_0 are nonempty compact subsets of A, B respectively.

Proof. Let $\{y_n\}$ be a sequence in B_0 then there exists a sequence $\{x_n\}$ in A_0 such that $\|x_n - y_n\| = dist(A, B)$. Since $\{x_n\}$ is bounded, $\{y_n\}$ is also a bounded sequence in B_0. Since B is finite dimensional, $\{y_n\}$ has a convergent subsequence. Hence B_0 is compact. Then by *Remark 3.3*, A_0 is also a compact set. □

Corollary 3.9. *Let A be a nonempty closed bounded convex subset of a uniformly convex Banach space X and B be a nonempty closed convex subset of X such that $span(B)$ is finite dimensional with $dist(A, B) > 0$. Let $T : A \cup B \to A \cup B$ be a relatively nonexpansive map. Then there exists $x \in A$ such that $\|x - Tx\| = dist(A, B)$.*

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